SCHOOL MATHEMATICS 1

TEACHERS' EDITION

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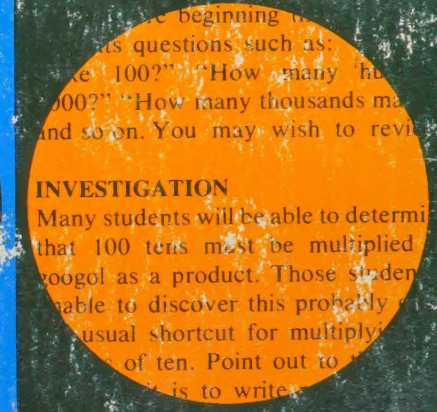

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Teachers' Edition to accompany

School Mathematics 1

1974 Edition

CHARLES R. FLEENOR

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FOREWORD

The *School Mathematics* series integrates a sound mathematics program with a teaching strategy that encourages the student to participate actively in the learning process. The essence of the *School Mathematics* series is reflected in some basic beliefs about the learning process to which we are committed:

1. Effective learning requires active participation on the part of the learner.
2. Understanding of mathematical concepts is a necessary prerequisite to adequate development and acquisition of computational skills.
3. Learning proceeds from concrete experiences to abstractions.
4. Learning is enhanced by motivation.
5. Each learner must experience earned success.

School Mathematics provides for the necessary mastery of basic mathematical concepts and computational skills while allowing for the wide range of abilities that are possessed by students at this level. Students are encouraged to discover patterns and relationships, reason logically, and develop their own generalizations.

Problem solving skills are carefully developed by means of diagrams, flow charts, formulas, and equations. Numerous sets of word problems give the student experience with applications of mathematics to the fields of physics, chemistry, geography, history, and economics as well as everyday applications of mathematics in simple business transactions such as sales tax, discounts, and interest.

Many other special features of this series such as the Think problems, Research Projects, and Mathematical Recreations are provided in the belief that the study of mathematics can be stimulating, enjoyable, and rewarding.

The authors wish to express their appreciation to numerous institutions and individuals in developing this series. Our teaching experiences at Ball State University, the Educational Research Council of Greater Cleveland, and Illinois State University have given us the opportunity to experiment, and to develop and test our ideas. We are indebted to many outstanding mathematicians and mathematics educators for their continual efforts to upgrade the teaching of mathematics in our schools. Special thanks go to Ms Jeanette Powelson for her contributions to the program, and to the editorial and production staffs of our publisher. Finally, we are deeply appreciative of the many teachers and students who have proved conclusively that the study of mathematics can be an exciting and gratifying experience.

Charles Fleenor
Robert Eicholz
Phares O'Daffer

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THE BOOK 1 PROGRAM

Mathematics of the Book 1 Program

The program for *School Mathematics*, Book 1, is organized into four learning units. Each learning unit contains three to five subunits, called *modules*. Each module is devoted to a specific mathematical topic.

Unit A: The Arithmetic of Whole Numbers
This learning unit reviews whole number concepts and operations. The primary objective of the unit is to improve the student's computational skills with whole numbers and to strengthen problem solving ability. Special attention is given to solving simple linear equations and additional emphasis is given to furthering the student's ability in solving word problems. Some elementary concepts of number theory are presented.

Unit B: The Arithmetic of Fractional Numbers
Only a minimal background in fractional numbers is assumed at the outset of this unit. The basic operations for the nonnegative rational numbers are carefully reviewed and considerable attention is given to problem solving and graphing. Work with decimals is closely related to the student's experiences with fractional numbers. Many of the applications of decimals are related to metric units of measure. A wide variety of applications of fractional numbers are presented in the module covering ratio, proportion, and percent.

Unit C: Geometry and Measurement
The four modules of this unit contain a broad spectrum of plane and space geometry concepts. Module 1 presents a nonmetric development of the basic concepts related to points, lines, and planes. Module 2 covers length, area, volume, and angle measure. Metric units of measure are used throughout the module. Module 3 introduces the student to some basic ideas of *transformational geometry*. Geoboards, mirrors, and co-ordinate grids are used to help the student study reflections, translations, and rotations in an informal manner with concrete materials. Little formal symbolism or language is developed at this point. The important concept of congruence of geometric figures grows out of the study of the rigid motions—reflections, translations, and rotations. Module 3 covers such standard topics as circumference and area of circles as well as volume and surface area of various space figures.

Unit D: Integers, Probability, and Statistics
Module 1, Integers, emphasizes the important role of the basic principles along with physical models in interpreting integers and the operations. Inequalities and graphs of integers in the co-ordinate plane are presented and related to similar work with whole numbers and fractional numbers.

Module 2, Experiments in Probability, introduces some elementary concepts of probability by providing a wide variety of simple probability experiments. The student compares empirical or experimental results with results expected from theoretical considerations.

The final module is devoted to basic ideas of statistics. The major emphasis is on the collection, organization, display, and interpretation of statistical data. Interpretation of graphical data is given special emphasis. Actual experience is provided in construction of graphs. Simple measures of central tendency—mean, median, and mode—are discussed and applied to relevant problem situations.

Features of Book 1

School Mathematics, Book 1, consists of four *learning units*. Each learning unit is comprised of three to five *modules*. The four learning units are published as a single case-bound book.

Each module opening presents a list of the major objectives for the module. These objectives serve as general guidelines for the student's work in the module. At the end of the module, a self-evaluation test, called *Test Yourself*, is presented so that the students themselves may determine their degree of mastery of the objectives. Answers to the self-evaluation test are included in the student text so that immediate reinforcement is available.

Each lesson, presented on two facing pages, normally constitutes enough material for a class session of 45 to 60 minutes.

The core lessons are designed in one of two ways: those lessons which include an Investigation activity begin on the left-hand page with the Investigation, followed by a set of Discussion Exercises and then by a set of exercises on the right-hand page for the student to work independently. (The titles of these sections are *Investigating the Ideas*, *Discussing the Ideas*, and *Using the Ideas*, respectively.) Other lessons are designed around a set of discussion exercises *(Discussing the Ideas)* and a set of independent exercises *(Using the Ideas)*.

Many lessons throughout the book contain starred exercises for enrichment and a *Think* problem for the more able or interested student. At the bottom of many right-hand pages, the student is referred to the special sets of Supplementary Exercises found in the back of the student text. These exercises provide additional practice and reinforcement of basic mathematical skills.

At the end of each module, a set of review exercises called *Reviewing the Ideas* is presented. This is followed by the self-evaluation test *(Test Yourself)* previously discussed. At the end of each learning unit, there are two pages of *Cumulative Review* exercises which cover the significant ideas of all the modules of the learning unit. Separate module achievement tests and unit achievement tests are also available as part of the evaluation program.

At the end of each module is a list of suggested Research Projects. These projects may encourage individual students to study some mathematical topics independently and in a greater depth than is possible in group work. Many of the suggested explorations could be developed into projects suitable for mathematics fairs and exhibits.

Each module concludes with a Mathematical Recreations page which will attract the attention of many students. Most of these recreations require little mathematical background. The puzzles, games, and activities that are suggested will serve as excellent motivation for many students.

The Teaching Strategy

The *School Mathematics* series is adaptable to a wide variety of classroom organizations and teaching strategies. Although teachers are encouraged to use their own teaching techniques and methods, we offer a broad general teaching strategy for this book. Each day's lesson in which the student is presented with a new concept can be divided into these stages:

1. *Preparation*
2. *Investigation*
3. *Discussion*
4. *Utilization*
5. *Extension*

Preparation: The Preparation stage is usually a "prebook" stage. Sometimes this stage consists in a review of concepts studied previously, in order to provide readiness for new ideas. At other times, it may consist in collecting and assembling material necessary for the lesson. In any event, the time devoted to the Preparation should be very short, and care should be taken that any activity undertaken at this stage does not preempt the Investigation which follows.

Investigation: Generally, the Investigation is a student-centred activity which presents the rudiment of the concept treated in the lesson. Each Investigation presents the students with a challenge question. To answer the challenge question, the students must speculate, experiment, probe, estimate, and explore. The Investigation stage enables students to develop nonroutine methods of problem solving. They are often asked to find as many ways as they can to do a certain thing. This allows for individual differences in students, since the open-ended questions permit less able students to experience success even when they have found only a few ways to accomplish a certain task.

In general, the Investigation should be carried out by students working independently or in small groups. The length of time needed to carry out an Investigation will vary, but should not be unduly prolonged. It is not necessary that every student complete all of the Investigation. During the Investigation stage, the teacher should remain somewhat in the background, acting as a moderator and advisor to those students needing help. Encourage students to do their own thinking and exploring. Do not give too much help at this stage.

Discussion: Following the Investigation, students are given an opportunity in the Discussion section to talk about what they did and to summarize the mathematical ideas of the lesson in preparation for working independently in the Using the Ideas section. Generally, the beginning discussion exercises are designed to stimulate students to talk about what they did in the Investigation. You should encourage them to discuss the various methods that they used to investigate and explore the concepts. Also, you should follow your teachers' guide carefully to be sure that whatever mathematical ideas are to be developed in the section are actually summarized and understood. You should provide additional examples as needed in order to strengthen understanding before proceeding.

Utilization: The material presented on the right-hand page of the student book, entitled *Using the Ideas*, gives students the opportunity to apply or utilize what they have learned in the preceding phases of the lesson.

You should provide for individual differences in assigning problems in this section. The teachers' commentary for each lesson provides suggestions for assignments on three levels: Minimum, Average, and Maximum. These suggested assignments should be regarded as guidelines, not prescriptions; you should feel free to vary assignments to meet the needs of your students.

Extension: The *School Mathematics* series provides opportunity for a wide variety of extension experiences. The Extension material can be thought of as including three principal types of activities: *enrichment, reinforcement,* and *remedial.*

The enrichment material is primarily, but not exclusively, intended for use by more able students. The reinforcement material should be used for all students who would benefit from further experiences in both the reteaching and practice areas. The remedial material, though primarily drill and practice, can be effectively used by any students who, for one reason or another, need to improve basic skills. Note that there is considerable overlapping in the categories of Extension materials, and no student should necessarily be excluded from participation in any one of the forms of Extension activities. Your choice of Extension suggestions for individual students constitutes one of your best opportunities for providing for individual differences and needs.

Materials that can be utilized in the Extension stage are offered in the student text, in the Teachers' Edition commentary, and in the supplements *(Workbooks, Duplicator Masters).*

Within the student text, the starred exercises represent either reinforcement exercises or enrichment. The Think problems are most often enrichment problems. The Research Projects and Mathematical Recreations at the end of each module also should be considered enrichment material. For remedial work as well as for reinforcement, the Supplementary Exercises at the back of the text constitute a source of more than 1600 problems.

In the Teachers' Edition, in the *Extension* section, you will often find additional suggestions either for enrichment or for remedial work. You should also check through the list of Resources for Active Learning in the introductory notes for each module for outside sources of material that might be used for Extension activities.

Other materials that may be used in the Extension stage are provided in the supplements designed for use with the student text. The *Duplicator Masters* provide additional practice for remediation and reinforcement. The *Workbooks* also can serve as a reteaching tool for remedial or reinforcement work.

The Teachers' Edition

General Format

The Teachers' Edition provides a wealth of suggestions and aids designed to assist you in providing an effective mathematics program for your students. On each Teachers' Edition page, the student page is reproduced in full size and includes annotated answers. The teaching commentary pertaining to each page is found in the margins beside the appropriate student page.

Preceding each module are two pages of introductory notes for teachers which contain general objectives for the module, performance objectives, comments concerning the mathematical content of the module, suggestions for teaching the module, a suggested lesson schedule, a discussion of evaluation procedures, and recommended resources for active learning. Teachers should find it very worthwhile to read these opening pages before teaching the material in the module.

At the top of each left-hand page of the Teachers' Edition, under the heading *Objective(s)*, the goal of the lesson is stated in terms of what the student should be able to do as a result of the lesson. This objective summarizes the key idea of the lesson in terms of the student's performance.

Suggestions for the Preparation, Investigation, and Discussion stages also follow in the left-hand margin of the Teachers' Edition.

In the right-hand margin of the Teachers' Edition appear suggestions for the Utilization stage, along with certain selected answers where space was not available for annotations on the student page. Also in this column will be found follow-up suggestions for the Extension stage, which may be either remedial or enrichment material. Included here also are suggestions for *Supplementary Exercises* in the back of the text, *Duplicator Master* sheets, and *Workbook* assignments. Complete solutions to most Think problems occur under the Extension section.

General Suggestions

We offer two general suggestions regarding use of the module and individual lesson notes:

(1) Read and consider each point as it applies to the immediate objectives for the lesson and the overall objectives of the module.
(2) Do not allow the Teachers' Edition notes to deter you from using your own effective teaching methods or to stifle creative efforts.

The Teachers' Edition does not attempt to dictate all of the activities in the day-to-day handling of your class and the individuals in it. You should use your Teachers' Edition as a guide to be co-ordinated with those methods which you have found to be most effective in teaching mathematics in the past.

Provision for Individual Differences

Each module introduction begins with a listing of both general objectives and performance objectives. The performance objectives are related in chart form to materials that would be helpful for background, reteaching, reinforcement, and related activities. These materials are given in the chart in abbreviated form. The Abbreviation Key follows (with Addison-Wesley's Order Code Numbers given in parentheses).

Abbreviation Key

WB	*Workbook* (1272)
DM	*Duplicator Masters* (1273)
ASC	*Arithmetic Skill Cards* for Development, Practice, and Applications: W – Whole Numbers; F – Fractions; D – Decimals and Percent; AP – Applications (1599)
SWM 1	*Success with Mathematics 1* (1580)
SWM 2	*Success with Mathematics 2* (1590)
SWM 3	*Success with Mathematics: A Course for Secondary School* (2411)
SP	*Skills and Patterns* (2159)
PD	*Patterns and Discovery* (2296)
DS	*Discovery and Structure* (2321)

Each performance objective listed in the chart is classified as red, yellow, or green. The color assignments are made on the following basis:

1. The red objectives present minimal performance standards. When only these are studied, the resulting course is the Basic Course, as outlined in the Long-Range Planning Chart (see page x).

2. The yellow objectives are within the capabilities of average students. When both the red and the yellow objectives are studied, the resulting course is the Average Course, as indicated in the Long-Range Planning Chart.

3. The green objectives represent a maximal level of achievement for the module. These green objectives fall into two categories:

(a) material that is considered to be more difficult or challenging, and
(b) material that is considered to be extension or supplementary to a basic program.

As an additional aid to individualization, minimum, average, and maximum assignments are suggested in the commentary accompanying each lesson other than review lessons.

Long-Range Planning Chart

The long-range planning chart on the next page provides the teacher with some general guidelines in planning *basic*, *average*, and *maximum* coverage of *School Mathematics*.

The *basic course* covers all the essential parts of the program necessary for adequate student preparation at the next level in mathematics. The *average course* covers all the material of the basic course but offers more extensive coverage of in-depth and extension material. The *maximum course* provides for nearly total coverage of the text material. The suggested time schedule based on 170 teaching days should be viewed only as an *aid* in helping you plan a time schedule that allows for individual differences within your class. You should not view it as a rigid schedule. Adjust the schedule according to the needs and abilities of your students.

Classroom Organization

The Book 1 program can be used in situations where the entire class works together on the same lesson, where small groups work together on the same lesson, or where individual students are allowed to proceed at their own rate

LONG-RANGE PLANNING CHART							
Learning Module	**Basic Course** (Red)		**Average Course** (Yellow)		**Maximum Course** (Green)		**Review and Evaluation** pages
	Lesson pages	Time (days)	Lesson pages	Time (days)	Lesson pages	Time (days)	
A-1 Numbers and Numerals	A 6-15	8	A 2-17	8	A 2-25	10	A 26, 27
A-2 Whole Numbers: Operations and Computation	A 30-37, 42-55	12	A 30-55	11	A 30-55	10	A 56, 57
A-3 Problem Solving	A 60-67	10	A 60-71	8	A 60-73	8	A 74, 75
A-4 Number Theory	A 78-85	10	A 78-91	9	A 78-91	9	A 92-95
B-1 Fractions and Fractional Numbers	B 2-7, 10-15	10	B 2-7, 10-17	10	B 2-17	8	B 18, 19
B-2 Adding and Subtracting Fractional Numbers	B 22-29	13	B 22-33	10	B 22-35	8	B 36, 37
B-3 Multiplying and Dividing Fractional Numbers	B 40-53	12	B 40-55, 58-59	10	B 40-61	10	B 62, 63
B-4 Decimals	B 66-83	18	B 66-89, 93	16	B 66-93	16	B 94, 95
B-5 Ratio, Proportion, and Percent	B 98-103, 108-113	13	B 98-103, 108-113, 115-121	12	B 98-123	12	B 124-127
C-1 Basic Concepts of Geometry	C 2-11, 14-25	12	C 2-25, 28-29	15	C 2-31	14	C 32, 33
C-2 Measurement	C 36-41, 44-49, 52-59	14	C 36-41, 44-61	14	C 36-63	14	C 64, 65
C-3 Rigid Motions and Constructions	C 68-81, 86-87	12	C 68-81, 86-87, 90-97	12	C 68-97	14	C 98, 99
C-4 Circles and Space Figures	C 102-103, 108, 112	10	C 102-112, 116-121	10	C 102-123	10	C 124-127
D-1 Integers	D 2-19	10	D 2-23	11	D 2-25	12	D 26, 27
D-2 Probability			D 30-37	6	D 30-41	7	D 42, 43
D-3 Statistics	D 46-55	6	D 46-59	8	D 46-59	8	D 60-63

of speed. You will want to employ the type of classroom organization that best suits the physical facilities of your particular situation. Teachers often find it stimulating to vary group sizes for different lessons and units of work. Smaller-size groups can often work more effectively together and allow greater opportunity to participate in the Investigation and the Discussion. Whatever class organization you choose, keep in mind that the key to the *School Mathematics* program is active student participation.

Evaluation of Progress

A student's attitude toward mathematics is often influenced by the methods used for evaluating his progress. All too often, evaluation procedures focus attention on what the student *did not* understand or master, rather than on what the student *did* accomplish. In evaluating progress, try to maintain a positive view, one which capitalizes on successes and develops confidence.

Pretests and achievement tests for Book 1 may be obtained from the publisher. Module reviews as well as cumulative reviews are provided in the text, to aid in evaluation of students' progress. However, you may find that a day-to-day evaluation of students, often involving personal interviews, will be a significant factor in determining how well they grasp the concepts and how well they are able to apply them.

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BIBLIOGRAPHY OF RESOURCES FOR ACTIVE LEARNING

Since implementing the investigative approach means more student involvement within the classroom or in a mathematics laboratory, it requires that the teacher be familiar with commonly available, free, and inexpensive material and supplies. It also requires knowledge of commercial aids and teacher resource material.

Included in each module introduction is a section titled "Resources for Active Learning." This list correlates appropriate "Activities" from teacher sourcebooks to the objectives of that module. The "Manipulative Devices" and "Commerical Games" can be used to support the mathematical objectives. They also provide for a means of reinforcement, extension, application, and fun.

At the time of this writing, the resources included were those which directly complement the active-learning approach and which are of good quality and present useful ideas.

Familiarize yourself with these materials and others which have been marketed subsequently. Check with your principal, supervisor, or mathematics coordinator for more recent materials. Only a few suppliers are listed for each device or game. Often they can be obtained from your local supplier. A complete bibliography and list of Suppliers of Resources for Active Learning is included immediately following this list.

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Childcraft Education Corp., Chicago.

Creative Publications, Palo Alto, California.

Cuisenaire Company of America, Inc., New Rochelle, New York.

Edmund Scientific Co., Barrington, New Jersey.

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LaPine Scientific Co., Chicago.

The Macmillan Co., New York (Available from Collier-Macmillan Canada Ltd., Don Mills, Ontario.

Math Media Inc., Danbury, Conn.

Metric Aids Ltd., Toronto.

Milton Bradley, Springfield, Mass.

Moyer-Vico Ltd., Weston, Ontario.

Responsive Environments Corp., Englewood Cliffs, New Jersey.

Sargent-Welch Scientific Co. of Canada Ltd., Weston, Ontario.

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Sigma Division, Scott Scientific Inc., Fort Collins, Colorado.

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School Mathematics 1

1974 Edition

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UNIT A: THE ARITHMETIC OF WHOLE NUMBERS

Module 1: Numbers and Numerals

General Objectives

To review the ideas of sets, union of sets, and intersection of sets.
To review the decimal numeration system, strengthen place value concepts, and work with larger numbers.
To provide experience with powers of ten and scientific notation.
To review rounding whole numbers.
To work with inequalities.
To provide experiences working with a numeration system other than base ten, in particular base eight.

	Performance Objectives	Pupil Text	Reteach-Reinforce	Related Activities	
1	**RED** Given a numeral of twelve digits or less, the student can give the place value of any given digit.	A-6, 7	**WB** 2		
2	**RED** The student can demonstrate the ability to use exponents in writing powers of ten.	A-8, 9	**WB** 3 **DM** 2		
3	**RED** Given a numeral of eight digits or less, the student can round the numeral to a specified place.	A-12, 13	**WB** 5 **DM** 4		
4	**RED** Given two whole numbers a and b, the student can tell whether $a < b$, $a = b$, or $a > b$.	A-14, 15	**WB** 6 **DM** 5	**SP** a-1	**SWM 1** 52, 53
5	**YELLOW** Given a pair of sets, the student can determine the elements of the union or intersection of the sets.	A-2, 3 A-4, 5	**WB** 1 **DM** 1	**SP** b-1, 2	**SWM 1** 70 **SWM 2** 172, 173
6	**YELLOW** Given a large number, the student can express it in scientific notation, and vice versa.	A-10, 11	**WB** 4 **DM** 3	**ASC** AP-3, 4 **SP** a-3	
7	***GREEN*** *Given a base-eight numeral of four digits or less, the student can give the equivalent base-ten numeral.*	A-18,19	**WB** 7		**SWM 2** 22, 23, 246
8	***GREEN*** *Given addition, subtraction, multiplication, or division problems using the base-eight numeration system, the student can find the solutions.*	A-20, 21 A-22, 23	**WB** 8, 9 **DM** 7, 8		
9	***GREEN*** *Given a Roman numeral, the student can write an equivalent decimal numeral.*	A-24, 25			**SWM 1** 69
	Reviewing the Ideas	A-26	**WB** 10		

MATHEMATICS

The concept of a set is considered by many mathematicians to be one of the great unifying ideas of modern mathematics, for it permeates much of the mathematics from kindergarten through the most advanced branches of mathematics.

In a formal study of set theory, the word *set* is usually taken as an undefined term. Therefore, only intuitive descriptions are given in such treatments. A set is sometimes described as any group or collection of objects which is considered as a single entity. The objects in a set are called the *elements* or *members* of the set. Other synonyms for the word set are "class" or "aggregate."

Two ways of specifying set membership are often given:

(1) A verbal description of the set
(2) A listing of the elements of the set

Both methods have their advantages and uses. Thus the verbal description

"The set of all whole numbers greater than 5 but less than 9"

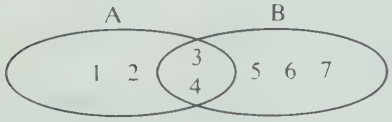
can be greatly simplified by listing the elements: $\{6, 7, 8\}$. However, to attempt to list the elements of a set such as "the set of whole num-

bers less than 1000" would be both tedious and undesirable. In this case, the verbal description of the set is much simpler.

Sets X and Y are *equal* if the elements of the two sets are identical. For example, if $A = \{0, 1, 2, 3, 4\}$ and B is the set of whole numbers less than 5, then $A = B$ because the elements of each set are identical.

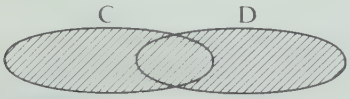
Union and *intersection* can be thought of as ways of combining two sets to give another set. Thus, union and intersection are operations on pairs of sets. An element is in the union of two sets if it is in *at least one* of the two sets. An element is in the intersection of two sets if it is in *both* sets.

Venn diagrams are useful teaching devices to help students distinguish between the union of two sets and the intersection of two sets. The figures below illustrate how Venn diagrams may be used.

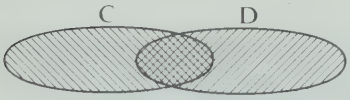


$A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$
$A \cap B = \{3, 4\}$

Venn diagrams may also be shaded to illustrate union and intersection.



C ∪ D is shaded ▨.



C ∩ D is shaded ▨.

The operation of addition on the whole numbers is closely related to operation of union of two sets. Thus, if A and B are two sets which have no common element ($A \cap B = \phi$) and if $n(A) = x$ (the number of elements in set A is x) and $n(B) = y$, then $n(A \cup B) = x + y$.

TEACHING THE MODULE

Materials

Geoboards, dot paper or graph paper; index cards; pencils or sets of sticks (for numeration work); clock or stopwatch; newspapers and magazines.

Vocabulary

abacus
base
decimal system
digit
element
empty set
equal sets
equivalent sets
expanded notation
exponent
greater than
intersection
less than
number
numeral
numeration system
one-to-one correspondence
period
place value
power
Roman numerals
Rounding
scientific notation
set
subset
union

Module 1 provides a vital foundation for the overall development of the text, since a clear understanding of place-value concepts is essential both for algorithmic work in computing with whole numbers and for development of decimal notation in work with rational numbers. Module 1 also provides introductory work with such basic concepts as inequalities and exponents—topics that will appear throughout the text. The introduction of other systems of numeration is not necessary as background material for later work in the text; however, the introduction may provide insight into the decimal system and capture the interest of many students.

Lesson Schedule

The background of your students and the amount of material covered will be the main determinants of how much time you should allow for this module, but generally, 8 to 10 days should provide adequate coverage.

Evaluation

In evaluating student progress, keep in mind that mastery in computation with base-eight numerals is *not* a basic goal of the module. The module review problems should help you decide whether any reteaching is needed. The self-evaluation test *(Test Yourself)* will aid the students as well as you in assessing their understanding of the main concepts of the module. In addition, you may wish to administer the Module Achievement Test, which is available from the publisher as part of the Evaluation Program designed for this series.

RESOURCES FOR ACTIVE LEARNING

Activities

Chip Trading Activities, Sets IV & V, Scott Resources
Computers, Cuisenaire Co.
ESS: Peas and Particles, Webster, McGraw-Hill
Experiments in Mathematics, Stage 2, pp. 22–23, 48–51, Houghton Mifflin
Let's Play Games in Mathematics, Vol. 7, "Expand a Number," pp. 88–89, National Textbook Co.
Mathematics Around the Clock, pp. 18–37, Rand McNally
Nuffield Project: *Computers; Problems*—Red Set, No. 1, Wiley
Sourcebook for Substitutes (Nim-type games), pp. 87–89, Addison-Wesley

Manipulative Devices

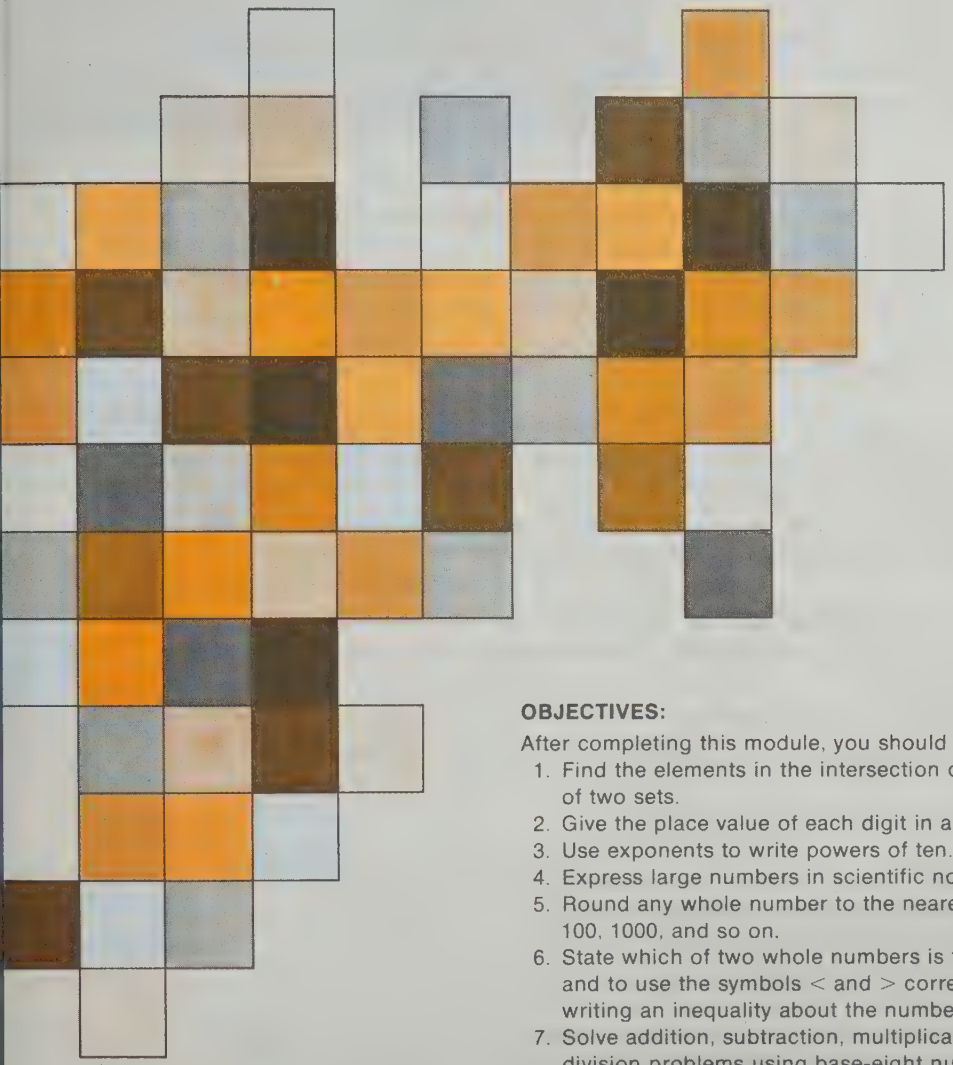
Abacuses—a variety (Creative Publications; Selective Educational Equipment)
Base Ten Blocks (Addison-Wesley)
Binary Counter (Creative Publications; Sargent-Welch)
Chip Trading materials (Creative Publications; Scott Resources)
Geoboards (Addison-Wesley)
Multi-base and binary materials (Educational Teaching Aids; Selective Educational Equipment)
Multi-base Blocks (Educational Teaching Aids)
Multi-base Converter (Educational Teaching Aids)

Games and Puzzles

Inter-union Logic Game (Sigma)
On-Sets (Wff'N Proof)
Operational Systems Games (Webster, McGraw-Hill)

UNIT A: *The Arithmetic of Whole Numbers*

MODULE 1: *Numbers and Numerals*



OBJECTIVES:

After completing this module, you should be able to:

1. Find the elements in the intersection or union of two sets.
2. Give the place value of each digit in a numeral.
3. Use exponents to write powers of ten.
4. Express large numbers in scientific notation.
5. Round any whole number to the nearest 10, 100, 1000, and so on.
6. State which of two whole numbers is the larger and to use the symbols $<$ and $>$ correctly when writing an inequality about the numbers.
7. Solve addition, subtraction, multiplication, and division problems using base-eight numerals.
8. Write a decimal numeral for a Roman numeral and write a Roman numeral for a decimal numeral.

Objective

Given a set, the student can list subsets of the set, identify the empty set, and show a one-to-one correspondence between pairs of equivalent sets.

PREPARATION

No special preparation is needed before the students begin the lesson. If you wish, however, you might review some ideas and terminology of sets, to determine how much experience your students have had with the topic.

INVESTIGATION

The investigation provides a challenging context in which to introduce the idea of a *subset* of a given set. Few of your students are likely to find *all* of the subsets of the set of 5 problems. When sufficient time has been devoted to the Investigation, have your students list on the chalkboard the different sets of problems that could have been completed. If the students' results are combined, it is likely that all 32 subsets will be found.

Mathematically, if a set S contains n elements, then there are 2^n subsets of S. Since the set in the Investigation has 5 elements, it has 2^5, or 32 subsets. During the Investigation, allow the students to express their subsets in any convenient manner. The standard notation used for sets can be introduced during the Discussion.

DISCUSSION

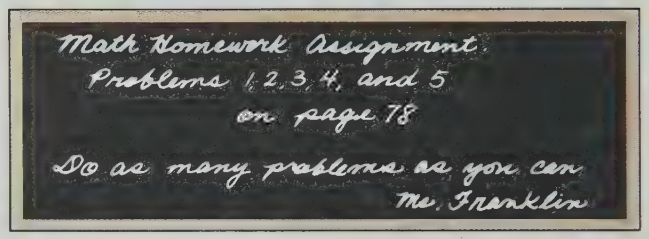
Exercise 1 introduces the standard notation of braces to denote sets. Many students will be familiar with this notation from their previous work in mathematics. Exercise 2 introduces the empty set. The use of the definite article *the* in the term *the empty set* indicates that there is only one empty set, the set containing no elements. *The empty set is a subset of every set.*

In response to Exercise 4, encourage students to suggest such one-to-one matchings as the fingers of one hand to those of the other, the set of occupied desks to the set of seated students, or the set of textbooks to the set of students, and so forth.

Sets

Investigating the Ideas

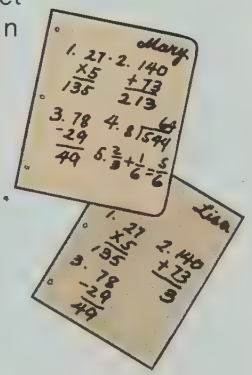
When Ms Franklin's class turned in their homework papers, she found that **no two students had completed exactly the same set** of homework problems.



Can you determine the largest possible number of students that could be in Ms Franklin's class?	List the different sets of problems that the students might have completed.

Discussing the Ideas

1. Mary completed all five of the homework problems. The set of numbers designating these problems could be written in **set notation** as $\{1, 2, 3, 4, 5\}$.
   Lisa completed only problems 1 and 3. Use set notation to show which problems she completed.



2. Bill did not complete any of the problems. The set of problems that he completed is the **empty set**. This set is denoted by $\{\ \}$ or ϕ. The set of men who have walked on the sun is the empty set.
   - A Is the set of men who have walked on the moon the empty set?
   - B Give an example of the empty set.

3. The sets $\{1, 2\}$, $\{2, 3, 4\}$, $\{5\}$, $\{\ \}$, $\{1, 3, 4, 5\}$, and many others are **subsets** of the set $\{1, 2, 3, 4, 5\}$. However, the set $\{1, 3, 6\}$ is not a subset of $\{1, 2, 3, 4, 5\}$. How can you decide when one set is a subset of another set?

4. The double-headed arrows show a **one-to-one correspondence** between set R and set S. Each element of either set is matched with exactly one element of the other set. R and S are **equivalent sets.** That is, each one contains the same number of elements as the other.
   Give another example of equivalent sets.

$$R = \{a, b, c\}$$
$$\updownarrow\ \updownarrow\ \updownarrow$$
$$S = \{1, 2, 3\}$$

Using the Ideas

1. List all the subsets of the set of letters {d, e, f}.
   Do not forget to list the set itself and the empty set.

2. Use set notation to show the elements of each set described below.
   - **A** The set of whole numbers greater than zero and less than eight.
   - **B** The set of whole numbers between 4998 and 5004.
   - **C** The set of the last seven letters of the alphabet.
   - **D** The set of months whose names begin with the letter *M*.
   - **E** The set of whole numbers that are divisors of 12.

3. For each set described below, indicate whether or not it is the empty set. If it is not empty, list the elements in the set.
   - **A** The set of students in your mathematics class who are less than six years old.
   - **B** The set of months of the year which have less than 30 days.
   - **C** The set of days of the week that begin with the letter *C*.
   - **D** The set of odd numbers that are exactly divisible by 2.

4. Suppose set *X* contains 23 elements and there is a one-to-one correspondence between set *X* and another set *Y*.
   - **A** How many elements must set *Y* contain?
   - **B** Is set *X* equivalent to set *Y*?

5. Set $R = \{a, b, c\}$ contains exactly the same elements as set $T = \{b, c, a\}$ except that the order in which they are listed is different. We say that set *R* **equals** set *T* and we write $R = T$.
   Suppose set *A* consists of the letters in the word triangle and set *B* consists of the letters in the word altering
   - **A** Is there a one-to-one correspondence between sets *A* and *B*?
   - **B** Is $A = B$?

think

How many different words can you find in the word ARITHMETIC if the letters in the words occur in the same order as in ARITHMETIC?
EXAMPLE: ART is in ARITHMETIC but TAR is not.

UTILIZATION

Before assigning the exercises, you may wish to discuss further the idea of *equivalent sets* and emphasize the distinction between *equality of sets* and *equivalence of sets* (see Exercises 4 and 5). For example set *A* of whole numbers which are greater than zero and less than 4 is *equal* to set *B* where *B* is the set of numbers consisting of 1, 2, and 3, and we denote this by $A = B$. Equivalent sets, on the other hand, have the same number of elements and can be placed in one-to-one correspondence, but the elements in a pair of equivalent sets need not be identical.

EXTENSION

Enrichment: Have your students determine the number of different one-to-one correspondences that can be established between two sets with the same number of elements. For example, if $X = \{a, b, c\}$ and $Y = \{1, 2, 3\}$, there are six one-to-one correspondences that can be established between *X* and *Y*:

$$\begin{array}{ccc} \{a,b,c\}, & \{a,b,c\}, & \{a,b,c\} \\ \updownarrow\ \updownarrow\ \updownarrow & \updownarrow\ \updownarrow\ \updownarrow & \updownarrow\ \updownarrow\ \updownarrow \\ \{1,2,3\} & \{1,3,2\} & \{2,1,3\} \end{array}$$

$$\begin{array}{ccc} \{a,b,c\}, & \{a,b,c\}, & \{a,b,c\} \\ \updownarrow\ \updownarrow\ \updownarrow & \updownarrow\ \updownarrow\ \updownarrow & \updownarrow\ \updownarrow\ \updownarrow \\ \{2,3,1\} & \{3,1,2\} & \{3,2,1\} \end{array}$$

In general, if two sets each have *n* elements, there are

$$1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n, \text{ or } n!$$

one-to-one correspondences that can be made between them.

Investigation Answers

{1, 2, 3, 4, 5}	{1, 2, 5}	{1, 2}	{3, 5}
{1, 2, 3, 4}	{1, 3, 4}	{1, 3}	{4, 5}
{1, 2, 3, 5}	{1, 3, 5}	{1, 4}	{1}
{1, 2, 4, 5}	{1, 4, 5}	{1, 5}	{2}
{1, 3, 4, 5}	{2, 3, 4}	{2, 3}	{3}
{2, 3, 4, 5}	{2, 3, 5}	{2, 4}	{4}
{1, 2, 3}	{2, 4, 5}	{2, 5}	{5}
{1, 2, 4}	{3, 4, 5}	{3, 4}	ϕ

Assignments

Minimum 1–3. Average 1–4. Maximum 1–5.

Objective

Given the elements in two sets, the student can find the intersection and the union of the sets.

PREPARATION

Although union and intersection of sets are introduced in the Investigation, you may prefer to introduce the two concepts as a pre-book activity. If you do so, also include an example in which the intersection of two sets is the empty set.

INVESTIGATION

Have the students read the Investigation carefully. For the sets shown, the sum of the numbers in set A is 29, the sum of the numbers in set B is 29, and the sum of the numbers in $A \cap B$ is 13. Students will be able to find several different solutions for the Investigation problem but one is sufficient. Here are four possibilities in which the sum in each set is 30 and the sum in the intersection is 15.

1. $A = \{1, 3, 4, 5, 8, 9\}$, $B = \{1, 2, 5, 6, 7, 9\}$, $A \cap B = \{1, 5, 9\}$
2. $A = \{1, 3, 5, 6, 7, 8\}$, $B = \{2, 3, 4, 5, 7, 9\}$, $A \cap B = \{3, 5, 7\}$
3. $A = \{3, 5, 6, 7, 9\}$, $B = \{1, 2, 3, 4, 5, 7, 8\}$, $A \cap B = \{3, 5, 7\}$
4. $A = \{1, 3, 5, 6, 7, 8\}$, $B = \{1, 2, 3, 4, 5, 6, 9\}$, $A \cap B = \{1, 3, 5, 6\}$

DISCUSSION

A common error made by students in listing the elements in the union of two sets is to list more than once an element which is in the intersection of the two sets. Therefore, when discussing Exercise 3, it would be helpful to have the students represent the sets C and D by means of a Venn diagram, as in the Investigation and Exercise 2. Exercise 5 illustrates that since set N is a subset of M, $M \cap N = N$ and $M \cup N = M$.

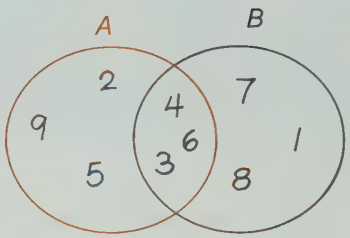
Union and Intersection of Sets

Investigating the Ideas

Think of the colored ring as containing the elements of set A and the black ring as containing the elements of set B.

The union of sets A and B is the set containing the elements in A or in B, or in both A and B.
We write: $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

The intersection of sets A and B is the set containing the elements in both A and B.
We write: $A \cap B = \{3, 4, 6\}$.

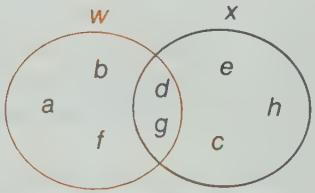


Can you place the numbers 1 through 9 into two sets like A and B so that the sum of the numbers in one set is the same as the sum of the numbers in the other set and the sum of the numbers in the intersection is 15?

Discussing the Ideas

1. Refer to sets A and B above to answer the following questions:
   - A What number in $A \cap B$ is an odd number?
   - B What is the largest number in set B?
   - C What is the number in set A that is not in set B and is less than 4?

2. Refer to the sets at the right.
   - A What elements are in $W \cup X$?
   - B What elements are in $W \cap X$?



3. If $C = \{1, 2, 3, 4\}$ and $D = \{3, 4, 5, 6, 7\}$, what are the elements in the following sets?
   - A $C \cup D$
   - B $C \cap D$

4. If $R = \{m, n, o\}$ and $S = \{x, y, z\}$, you can see that sets R and S have no common elements and $R \cap S = \phi$. What elements are in $R \cup S$?

5. Let $M = \{1, 2, 3, 4\}$ and $N = \{2, 3, 4\}$.
   - A What is $M \cup N$? What do you notice?
   - B What is $M \cap N$? What do you notice?

Using the Ideas

1. Let $A = \{a, c, d, e\}$, $B = \{b, c, d, e\}$, $C = \{e, f\}$, and $D = \{b, f, g\}$.
List the elements in each of the following sets.

A $A \cup B$ **C** $A \cup D$ **E** $D \cap B$ **G** $A \cup C$ **I** $C \cup D$
B $B \cup C$ **D** $A \cap B$ **F** $C \cap D$ **H** $A \cap C$ **J** $B \cap C$

2. Let $R = \{0, 1, 2, 3, 4, 5\}$, $S = \{0, 2, 4\}$, and $T = \{1, 3, 5\}$.
Which statements are true and which are false?

A $R \cup S = R$ **C** $S \cup T = R$ **E** $R \cup T = T$ **G** $R \cap T = R$
B $R \cap S = R$ **D** $T \cap S = \phi$ **F** $S \cap T = R$ **H** $S \cap T = T$

3. **A** If A and B are any sets, is $A \cup B$ the same as $B \cup A$?
B Is $A \cap B$ the same as $B \cap A$?

4. If $M = \{1, 2, 3, 4, 5\}$, what set is
A $M \cup M$? **B** $M \cap M$? **C** $M \cup \phi$? **D** $M \cap \phi$?

5. Let $A = \{1, 2, 3\}$, $B = \{2, 3, 4, 5\}$, and $C = \{3, 4, 5, 6\}$.
Find the elements in each of the following sets.
Consider the sets in the parentheses first.

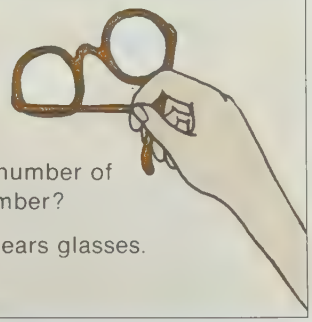
A $(A \cup B) \cup C$ **D** $(A \cap B) \cup C$ **G** $(A \cap B) \cup (A \cap C)$
B $(A \cap B) \cap C$ **E** $A \cup (B \cap C)$ **H** $A \cap (B \cap C)$
C $A \cap (B \cup C)$ **F** $A \cup (B \cup C)$ **I** $(A \cup B) \cap (A \cup C)$

★ **6.** Which parts of Exercise 5 are sets which have the same elements?

★ **7.** Suppose set A contains 5 elements and set B contains 4 elements.
A What is the greatest possible number of elements in $A \cup B$?
B What is the fewest possible number of elements in $A \cup B$?

think

In a set of 24 students, 12 of them wear glasses.



1. If you know that none of the boys wear glasses, what is the greatest number of boys possible in the set?
2. If there are 9 boys in the set, what is the fewest possible number of girls who wear glasses? What is the greatest possible number?
3. Now suppose there are 2 boys for every girl and no girl wears glasses. How many boys do not wear glasses?

More practice, page S-2, Set 1

UTILIZATION

After completing the Discussion, you may want to give examples of combining three or more sets like those given in Exercise 5, and discuss the use of parentheses when combining three or more sets. Then assign the exercises.

You may want to point out to your students that the operations of both union and intersection are associative and commutative. (It would be better, though, for the students themselves to discover these ideas.)

Exercise 5 illustrates two varieties of a distributive principle. Thus, if X, Y, and Z are sets,

$$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$$

and

$$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z).$$

However, the purpose of this lesson is not to teach theoretical set theory but simply to provide students with a working knowledge of the ideas of union and intersection. It is better to give the lesson a light touch than to overteach it.

EXTENSION

To provide further practice, make selective assignments from Supplementary Exercise Set 1 on page S-2, *Workbook* page 1, and *Duplicator Masters* page 1.

Think Solution

1. Since 12 students wear glasses, 12 must not wear glasses. Thus, if none of the boys wear glasses, there can be at most 12 boys in the set.
2. If there are 9 boys, then there are 15 girls. If we assume that all 9 boys wear glasses, then 3 girls must wear glasses. As fewer boys wear glasses, more girls must wear them, up to the maximum of 12 girls.
3. If there are 2 boys for every 1 girl, then there are 16 boys and 8 girls. Since none of the girls wear glasses, 12 of the boys must, thus leaving 4 boys who do not wear them.

Assignments

Minimum 1–2. Average 1–4. Maximum 1–7.

Objective

Given a numeral in the decimal system, the student can give the place-value meaning of each digit in the numeral and represent the number using expanded notation.

PREPARATION

Materials: Clock or stopwatch.

The day before this lesson is to be presented, ask the students to bring in newspaper or magazine clippings of statistical articles which use large numbers (for example, any article about the space program which discusses great distances). Have the students present these large numbers and explain to the class how they are used in conjunction with Discussion Exercise 4.

INVESTIGATION

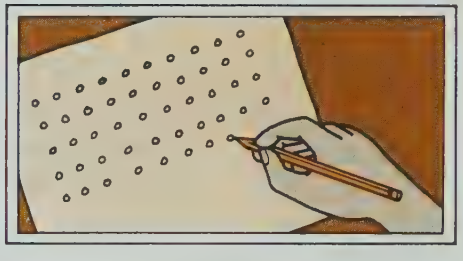
The purpose of the Investigation is to provide a concrete experience with large numbers. Pairs of students can work together and time each other in making 100 dots. This activity may take about 1 minute for students who work rapidly. Working at such a rate, it would take them about 10 minutes to make 1000 dots. To make 1 000 000 dots they would have to work about 1000 times as long as for 1000 dots, or about 10 000 minutes. This is nearly 167 hours. Thus, one person would need to work 8 hours a day for nearly 22 days in order to mark one million dots.

DISCUSSION

The amount of time spent on the Discussion should be adjusted to the students' understanding of place-value concepts. Since place value plays a vital role in decimal concepts, this material is also important as preparation for the later development involving decimal notation. Exercise 2 reviews the meaning of place value, and Exercise 3 introduces expanded notation. Exercise 4 reviews the period names through the billions. Students will need to understand these ideas before undertaking the exercises on the following page.

If some students have difficulty reading larger numerals, devote more time to this activity, using examples from the newspaper and magazine articles as suggested in the Preparation.

The Decimal Numeration System



Investigating the Ideas

Mark one hundred dots on a piece of paper so that they are easy to count. Time yourself to see how long it takes you to mark all of the dots.

How long do you think it would take you to mark a thousand dots if you worked at the same rate? Could you ever mark a million dots?

Discussing the Ideas

1. How many students would be needed to combine their papers in order to show a thousand dots?

2. Our numeration system is called a **decimal system** because we group by tens. The decimal system uses **place value** because each digit in a numeral, depending upon its place tells how many ones, tens, hundreds, and so on.

   6543 → The 3 tells how many ones. ——→ (3×1)

   6543 → The 4 tells how many tens. ——→ (4×10)

   6543 → The 5 tells how many hundreds. ——→ (5×100)

   6543 → The 6 tells how many thousands. ——→ (6×1000)

   In the numeral **7839**, give the place value of each digit.

3. The numeral 6543 can be written as $(6 \times 1000) + (5 \times 100) + (4 \times 10) + 3$ using **expanded notation**. How would you write 23 746 using expanded notation?

4. To write numerals with several digits, we separate the digits into groups of three. Each group is called a **period**. The diagram shows the periods and place-value names through the billions.

PERIOD NAMES	BILLIONS			MILLIONS			THOUSANDS			UNITS		
PLACE-VALUE NAMES	hundred billions	ten billions	one billions	hundred millions	ten millions	one millions	hundred thousands	ten thousands	one thousands	hundreds	tens	ones
NUMERAL	5	4	7	2	0	8	7	9	3	1	6	5

   A What digit is in the hundred millions' place? ten thousands' place?

   B How would you read the numeral above?

Using the Ideas

1. Each sentence is about a digit in the numeral **321 896 574**.
Copy the sentences. Give the missing words and numbers.

   A 7 in the tens place means 7×10
   B 9 in the ? place means $9 \times$ ▨.
   C 3 in the ? place means $3 \times$ ▨.
   D 5 in the ? place means $5 \times$ ▨.
   E 6 in the ? place means $6 \times$ ▨.
   F 2 in the ? place means $2 \times$ ▨.

2. In the example, the number is written using **expanded notation.**
Use expanded notation to represent each number.

   EXAMPLE: $7423 = (7 \cdot 1000) + (4 \cdot 100) + (2 \cdot 10) + 3$

   A 358 **C** 53 729 **E** 458 297 **G** 24 364 200
   B 4562 **D** 70 364 **F** 6 200 876 **H** 647 870 000

3. Each number is written using expanded notation. Give the usual numeral for each number.

   A $(9 \times 100) + (7 \times 10) + 4$
   B $(8 \times 1000) + (6 \times 100) + (7 \times 10) + 5$
   C $(5 \times 10\,000) + (3 \times 1000) + (4 \times 100) + (2 \times 10) + 7$
   D $(3 \times 100\,000) + (5 \times 1000) + (8 \times 100) + (9 \times 10) + 4$

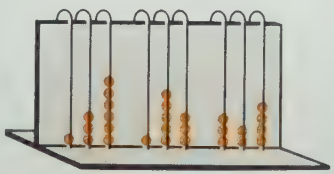
4. The diagram shows the names of the periods through the sextillions. Use it to help you write the numeral mentioned in each of the statements below.

SEXTILLIONS	QUINTILLIONS	QUADRILLIONS	TRILLIONS	BILLIONS	MILLIONS	THOUSANDS	UNITS

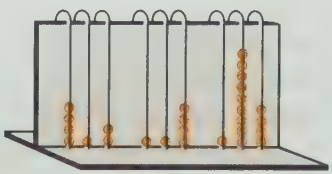
   A The speed of light is about three hundred million metres per second.
   B The Earth's mass is about seven sextillion tonnes.
   C The estimated age of the sun is about one hundred forty quadrillion seconds.

5. An **abacus** is a simple computing device that uses place value.
Write the numeral shown by each abacus.

A



B



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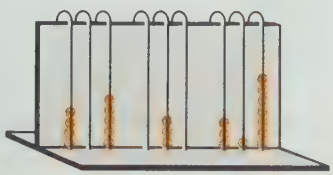


UTILIZATION

Exercises 1, 2, and 3 give practice in place value concepts and expanded notation. Exercise 4 introduces the names of the periods through sextillions. Students should not be expected to memorize these names; rather, you will want them to understand the general scheme of writing numerals for large numbers. For those students interested in the names of the larger periods, suggest Research Project A on page A-27.

Exercise 5 provides work with the *abacus*. Refer students to Research Project B on page A-27, and provide a classroom demonstration of how an abacus is used, if one is available.

Answers, Exercise 2

A. $358 = (3 \times 100) + (5 \times 10) + 8$
B. $4562 = (4 \times 1000) + (5 \times 100) + (6 \times 10) + 2$
C. $53\,729 = (5 \times 10\,000) + (3 \times 1000) + (7 \times 100) + (2 \times 10) + 9$
D. $70\,364 = (7 \times 10\,000) + (3 \times 100) + (6 \times 10) + 4$
E. $458\,297 = (4 \times 100\,000) + (5 \times 10\,000) + (8 \times 1000) + (2 \times 100) + (9 \times 10) + 7$
F. $6\,200\,876 = (6 \times 1\,000\,000) + (2 \times 100\,000) + (8 \times 100) + (7 \times 10) + 6$
G. $24\,364\,200 = (2 \times 10\,000\,000) + (4 \times 1\,000\,000) + (3 \times 100\,000) + (6 \times 10\,000) + (4 \times 1000) + (2 \times 100)$
H. $647\,870\,000 = (6 \times 100\,000\,000) + (4 \times 10\,000\,000) + (7 \times 1\,000\,000) + (8 \times 100\,000) + (7 \times 10\,000)$

EXTENSION

Page 2 of the *Workbook* may be assigned for students who would benefit from additional practice with the concepts of this lesson.

Enrichment: Some students might be interested in preparing a display of 1 million dots after their experience with the Investigation. You might suggest that they type or mark as many dots as they can on a duplicator master. Then the master can be reproduced enough times to make one million dots.

Assignments

Minimum 1–3. Average 1–4. Maximum 1–5.

OBJECTIVES

Given a power of ten, the student can write the usual numeral.

Given a number which is a power of ten, the student can represent the number as a power of ten.

The student can find products and quotients involving powers of ten.

PREPARATION

You may wish to review some place value ideas before beginning the Investigation. Ask students questions such as: "How many tens make 100?" "How many hundreds make 1000?" "How many thousands make 10 000?" and so on.

INVESTIGATION

Many students will be able to determine quickly that 100 tens must be multiplied to give a googol as a product. Those students who are unable to discover this probably do not know the usual shortcut for multiplying by tens or powers of ten. Point out to the students how unwieldy it is to write numerals representing the magnitude of a googol and thus introduce the exponential notation for such numbers. Some students may wonder what is the place-value name for a googol. Of course, it is not really practical to use a place-value name for such a large number, but since the name of the 22nd period is vigintillion you could name a googol as *ten undecillion vigintillions*.

DISCUSSION

In Exercise 1, it is important for students to understand that the number of factors of ten for numbers such as 100, 1000, 10 000, and so on can be determined by simply counting the number of times the digit 0 occurs in the numeral.

Exercise 2 introduces exponential notation and the use of a centered dot for multiplication. You will want to give additional examples and practice in reading powers of ten.

Powers of Ten

Investigating the Ideas

You know that $10 \times 10 = 100$,
$10 \times 10 \times 10 = 1000$,
and $10 \times 10 \times 10 \times 10 = 10\,000$.
Look at the numeral at the right.

This number is called a **googol**.

10 000 000 000 000 000 000 000 000 000
000 000 000 000 000 000 000 000 000
000 000 000 000 000 000 000 000 000
000 000 000 000 000 000

Can you find how many tens would have to be multiplied together in order to make a googol?

Discussing the Ideas

1. How many times does the digit 0 appear in the numeral for a googol? How can you use this fact to help you answer the Investigation question?

2. When the same number is used as factor several times, you can use **exponents** to simplify the writing of a numeral.
(NOTE: The middle dot " · " indicates multiplication.)
The first power of ten, 10^1, is usually written as 10.
10^2 is read as "10 squared" or "ten to the second power."

$10^3 = 10 \cdot 10 \cdot 10$
10^3 is the third **power of ten.**
10^3 is read "10 cubed" or "ten to the third power."
10 is called the **base.**
3 is called the **exponent.**

Give the missing words and numbers.

   A 10^4 means 10 used as a factor ? times $\rightarrow 10^4 = 10 \cdot 10 \cdot 10 \cdot 10 =$ ▮▮▮▮.
   B 10^5 means 10 used as a factor ? times $\rightarrow 10^5 = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 =$ ▮▮▮▮▮.
   C 10^6 means 10 used as a factor ? times $\rightarrow 10^6 = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 =$ ▮▮▮▮▮▮.

3. Read and complete each statement.
   A One hundred is ten to the ? power.
   B One thousand is ten to the ? power.
   C Ten thousand is ten to the ? power.
   D A googol is ten to the ? power.

Using the Ideas

1. Write each of the following by using 10 with an exponent.
   - **A** $10 \cdot 10 \cdot 10$
   - **B** $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$
   - **C** $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$

2. Give the correct number of zeros for each blank.
   - **A** To write 10^3, we can write a 1 followed by ? zeros.
   - **B** To write 10^6, we can write a 1 followed by ? zeros.
   - **C** To write 10^{19}, we can write a 1 followed by ? zeros.
   - **D** To write 10^{100}, we can write a 1 followed by ? zeros.

3. Write each of the following using 10 with an exponent.
   - **A** 100 **B** 1000 **C** 10 000 **D** 100 000 **E** 1 000 000
   - **F** Light can travel about 10 000 000 000 000 kilometres per year.
   - **G** Some rocks from the moon are more than 100 000 000 000 000 000 seconds old.
   - **H** The approximate mass of Uranus is 100 000 000 000 000 000 000 000 tonnes.

4. Give the number for each ▒.
   - **A** $10^2 =$ ▒ **C** $10^4 =$ ▒ **E** $10^6 =$ ▒ **G** $10^8 =$ ▒ **I** $10^{10} =$ ▒
   - **B** $10^3 =$ ▒ **D** $10^5 =$ ▒ **F** $10^7 =$ ▒ **H** $10^9 =$ ▒ **J** $10^{11} =$ ▒

5. For each part, give the product **P**. Then give the missing power of ten for each ▒.
   - **A** $10 \cdot 100 = \mathbf{P}$; $10^1 \cdot 10^2 =$ ▒
   - **B** $100 \cdot 100 = \mathbf{P}$; $10^2 \cdot 10^2 =$ ▒
   - **C** $1000 \cdot 100 = \mathbf{P}$; $10^3 \cdot 10^2 =$ ▒

6. Find the products.
   - **A** $10^3 \cdot 10^1$ **C** $10^2 \cdot 10^4$ **E** $9 \cdot 10^2$ **G** $87 \cdot 10^4$ **I** $3 \cdot 10^2 \cdot 10^3$
   - **B** $10^3 \cdot 10^3$ **D** $10^3 \cdot 10^5$ **F** $42 \cdot 10^3$ **H** $68 \cdot 10^5$ **J** $47 \cdot 10^3 \cdot 10^4$

7. For each part, give the quotient **Q**. Then give the missing power of ten for each ▒.
   - **A** $100 \div 10 = \mathbf{Q}$; $10^2 \div 10^1 =$ ▒
   - **B** $10\,000 \div 10 = \mathbf{Q}$; $10^4 \div 10^1 =$ ▒
   - **C** $100\,000 \div 100 = \mathbf{Q}$; $10^5 \div 10^2 =$ ▒

8. Observe that $10^1 \cdot 10^1 = 10^{1+1} = 10^2$ and $10^2 \cdot 10^3 = 10^{2+3} = 10^5$. Therefore, $10^2 \cdot 10^0 = 10^{2+0} = 10^2$. According to the example, does the number 10^0 represent **1**, **10**, or **0**?

More practice, page S-2, Set 2

UTILIZATION

You may find it desirable to divide the exercise set into two parts. Exercises 1, 2, 3, 4 are exercises which are designed to strengthen the students' understanding and skill in use of powers of ten. Exercises 5, 6, and 7 introduce multiplication and division with powers of ten. Some students may discover the index laws for multiplication and division in these exercises.

$$10^a \cdot 10^b = 10^{a+b} \text{ and}$$
$$10^a \div 10^b = 10^{a-b},\ a \geq b$$

Others may need additional practice with powers of ten before discovering these laws.

Exercise 8 deserves special emphasis because it introduces 10^0. If the index laws for multiplication and division are to be preserved, it is necessary to define 10^0 as equal to 1. In general, for any base $b \neq 0$, $b^0 = 1$. If you wish to give additional arguments that $10^0 = 1$, you might use a division example such as this:

$$\frac{10^4}{10^4} = 10^{4-4} = 10^0$$

But $\frac{10^4}{10^4} = 1$, therefore $10^0 = 1$.

EXTENSION

Additional practice is provided by Supplementary Exercise Set 2, page S-2; page 3 of the *Workbook;* and page 2 of the *Duplicator Masters.*

Enrichment: Ask students to build a table of powers of 3.

3^0	3^1	3^2	3^3	3^4	3^5	3^6	3^7	3^8
1	3	9	27	81	243	729	2187	6561

Write multiplication and division equations using the numbers in the table, and then have students write the corresponding equation using powers of 3. Examples:

1. $27 \cdot 81 = 2187$; $3^3 \cdot 3^4 = 3^7$
2. $6561 \div 243 = 27$; $3^8 \div 3^5 = 3^3$

Assignments

Minimum 1, 2, 4, 8. Average 1–5, 8. Maximum 1–8.

Objective

The student can represent appropriate numbers in scientific notation.

PREPARATION

Before students begin the Investigation, briefly review powers of ten, especially the fact that $10^0 = 1$. You might also have a student prepare a chalkboard display showing the standard numerals for powers of 10 through 10^9.

INVESTIGATION

Students should be able to complete this Investigation quite rapidly. Observe that when the students write each number as a product of two numbers, one from set *A* and one from set *B*, they are expressing the number in scientific notation. (However, the term *scientific notation* is not introduced until the Discussion stage.) You may wish to give other numbers like those in the Investigation before proceeding to the Discussion.

DISCUSSION

Some students may have found it difficult to represent 8 as the product of a whole number and a power of 10, so a further review of 10^0 may be necessary when discussing Exercise 2. The term *scientific notation* is defined in Exercise 3. While scientific notation may not be particularly easy for slower students, they should at least be exposed to the ideas involved. Later, the work with scientific notation will be expanded to include the use of negative exponents to represent very small numbers. Therefore, you will want to take extra care at this time to see that most of your students understand the use of scientific notation to represent large numbers.

Scientific Notation

Investigating the Ideas

Study sets *A* and *B*.

$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$B = \{10^0, 10^1, 10^2, 10^3, 10^4, 10^5, 10^6, 10^7, 10^8, 10^9, 10^{10}\}$

Can you express each of the numbers below as a product of two numbers such that one number is from set *A* and the other is from set *B*?

1. 6000
2. 9 000 000
3. 200 000
4. 70 000
5. 30 000 000 000
6. 8

Discussing the Ideas

1. Larry and Carol did not agree on the way to represent 6000. Who do you think is correct? How can you tell?

Larry: $6 \cdot 10^4 = 6000$

Carol: $6 \cdot 10^3 = 6000$

2. Jerry did not think that it was possible to represent 8 using one number from set *A* and the other from set *B*. So Max tried to help him by writing $8 \cdot \blacksquare = 8$. What power of ten from set *B* is correct for the $\blacksquare$?

3. A number is represented in scientific notation when it is written as **a number from 1 through 9 times a power of ten.**

EXAMPLES: $300 = 3 \cdot 100 = 3 \cdot 10^2$

$700\,000 = 7 \cdot 100\,000 = 7 \cdot 10^5$

$1000 = 1 \cdot 1000 = 1 \cdot 10^3$

How would you express these numbers in scientific notation?

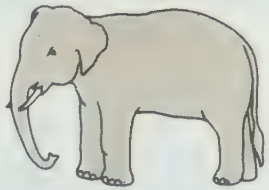
A 9000 **B** 80 **C** 2 000 000 **D** 300 000

4. What number does each of these represent?

A $6 \cdot 10^4$ **B** $9 \cdot 10^3$ **C** $7 \cdot 10^6$ **D** $4 \cdot 10^0$

Using the Ideas

1. Express the number in each statement in scientific notation.
   - **A** A jet airliner can travel 900 kilometres (km) per hour (km/h).
   - **B** A large telescope lens has a 500-centimetre (cm) diameter.
   - **C** A large elephant weighs about 4000 kilograms (kg).
   - **D** An artificial satellite travels 32 000 km/h.
   - **E** A whale weighs about 90 000 kilograms.
   - **F** The diameter of the Earth is over 10 000 000 metres (m).
   - **G** The radius of the sun is about 700 000 000 metres.
   - **H** Light travels at about 30 000 000 000 cm/s.
   - **I** The sun's mass is about 8 000 000 000 000 000 000 000 000 000 000 kg.



2. Each number below is represented in scientific notation. Write a usual numeral for each of them.

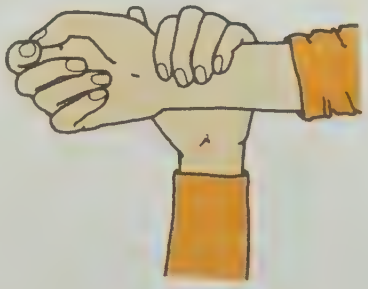
A $6 \cdot 10^4$	**D** $7 \cdot 10^5$	**G** $2 \cdot 10^6$	**J** $8 \cdot 10^9$
B $3 \cdot 10^7$	**E** $1 \cdot 10^8$	**H** $9 \cdot 10^2$	**K** $6 \cdot 10^{15}$
C $4 \cdot 10^{10}$	**F** $6 \cdot 10^0$	**I** $5 \cdot 10^3$	**L** $3 \cdot 10^{23}$

3. Use scientific notation to represent each number name.

A three million	**D** five quadrillion	**G** nine quintillion
B six trillion	**E** two sextillion	**H** six quadrillion
C fifty	**F** four billion	**I** two hundred million

think

1. Check your pulse rate to find how many times your heart beats in one minute.
2. **Estimate** the answer to each question below. Then calculate the answer and compare it with your estimate.
   - **A** If your heartbeat rate remains the same, how many days (to the nearest day) will it take for your heart to beat one million times?
   - **B** How many years (to the nearest year) will it take for your heart to beat one billion times?



More practice, page S-3, Set 3

UTILIZATION

The exercises give the student practice in expressing numbers in scientific notation and in finding the usual numeral for a number given in scientific notation.

Answers, Exercise 2

A. 60 000	G. 2 000 000
B. 30 000 000	H. 900
C. 40 000 000 000	I. 5000
D. 700 000	J. 8 000 000 000
E. 100 000 000	K. 6 000 000 000 000 000
F. 6	L. 300 000 000 000 000 000 000 000

EXTENSION

For more practice, suggest Supplementary Exercise Set 3, page S-3; page 4 of the *Workbook*; and page 3 of the *Duplicator Masters*.

Think Solution

1. The average heartbeat rate is about 72 beats per minute.
   - A. In one day, there are $60 \cdot 24$ or 1440 minutes. The average heart would beat $1440 \cdot 72$ or 103 680 times in 1 day. Thus, 1 000 000 beats $\div$ 103 680 beats/day $\approx$ 10 days.
   - B. If the average heart beats 103 680 times in 1 day, then it beats $103\,680 \cdot 365$ days or 37 843 200 times in 1 year. Thus, 1 000 000 000 beats $\div$ 37 843 200 beats/year $\approx$ 26 years.

Assignments

Minimum 1A–D, 2A–F, Average 1–2.
Maximum 1–3.

Objective

The student can round a number to a specified place.

PREPARATION

The Investigation could be carried out without introduction, but if you prefer, you might ask students to use an almanac or other book of facts and find examples of commonly used large numbers that are rounded to various degrees of accuracy. Such examples might include the current provincal and national budgets; the Gross National Product; the size of various provincal nations, and continents; and the like.

INVESTIGATION

The Investigation gives the student an opportunity to investigate and review the basic idea of rounding numbers, in this case, to the nearest thousand. You can extend the Investigation and increase student interest by showing a list of nearby towns or cities and having the students choose the nearest multiple of 1000 or other large multiples of 10 for the population.

DISCUSSION

The Discussion exercises should be used to help the student review the scheme used to round numbers. You may wish to give additional examples on the chalkboard. Although Exercise 3 suggests that we agree to round *up* when a number is halfway between two multiples, students should understand that this is an arbitrary choice. For convenience, however, we do need to agree always to do it the same way. You might point out to the students that we could even invent a rule whereby we round up if the digit before the five is even and round down if the digit before the five is odd.

Rounding Numbers

Investigating the Ideas

The 1970 population of four small towns is shown in the table.

TOWN	POPULATION
Brookings	2720
Burns	3293
Canby	3813
Central Point	4004

Can you match each town with the multiple of 1000 that is nearest to the population of that town?

A 2000 B 3000 C 4000 D 5000

Discussing the Ideas

1. When you chose the multiple of 1000 that was nearest to the population of a town, you were **rounding the number to the nearest thousand.** Explain how you made your choice for each of the numbers in the Investigation.

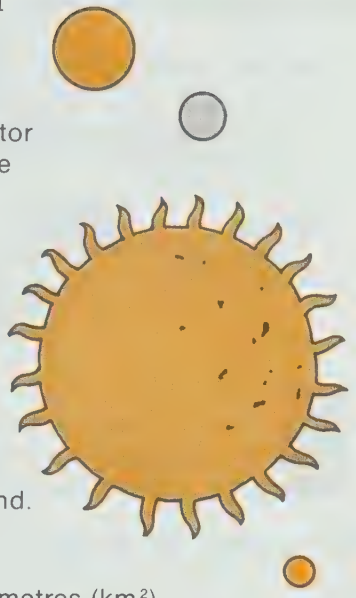
2. To round 274 793 **to the nearest hundred thousand,** you must decide whether the number is nearer to 200 000 or 300 000. What is 274 793 rounded to the nearest hundred thousand?

3. When a number is halfway between two multiples, we agree to **round up.** What is 4500 rounded to the nearest thousand?

4. The population of Centerville is between 5000 and 6000. How can you decide which number is closer to the actual population even though two digits of the numeral are hidden?

EARTH FACTS

Using the Ideas



1. Earth's average diameter (distance through) is about 12 755 kilometres (km). Round this number to the
   - **A** nearest hundred.
   - **B** nearest thousand.

2. Earth's circumference (distance around) at the equator is about 40 074 kilometres. Round this number to the
   - **A** nearest thousand.
   - **B** nearest ten thousand.

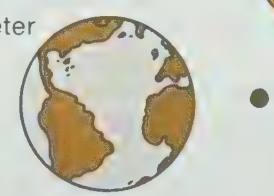
3. The average distance from Earth to the moon is 384 393 kilometres. Round this number to the
   - **A** nearest hundred.
   - **B** nearest thousand.
   - **C** nearest hundred thousand.

4. The average distance from Earth to the sun is 149 593 690 kilometres. Round this number to the
   - **A** nearest hundred.
   - **B** nearest hundred thousand.
   - **C** nearest million.

5. The surface area of Earth is 410 171 500 square kilometres (km^2). Round this area to the nearest hundred million.

6. Mount Everest, the highest mountain on Earth, is 8848 metres tall. Round this height to the nearest thousand metres.

7. The Marianas Trench in the Pacific Ocean is 11 033 metres deep.
   - **A** Round this depth to the nearest thousand metres.
   - **B** Round this depth to the nearest ten thousand metres.



★8. The diameter of the sun is about 109 times the diameter of Earth. (See Exercise 1.)
   - **A** Find the diameter of the sun and round it to the nearest hundred thousand kilometres.
   - **B** Express this number in scientific notation.

★9. The volume of a ball can be measured approximately by finding the third power of the diameter (d^3 or $d \cdot d \cdot d$) and then dividing this number by 2.
   - **A** Using your answer to Exercise 1**b**, find the approximate volume of the earth.
   - **B** Using your answer to Exercise 8**a**, find the approximate volume of the sun.
   - **C** About how many "earths" would it take to fill as much space as the sun fills?

UTILIZATION

The exercises on this page involve practice in rounding numbers related to interesting statistical facts about the Earth.

Exercises 8 and 9 offer a special challenge in that they involve some computation as well as rounding.

EXTENSION

Additional practice is provided by the exercises on *Workbook* page 5 and *Duplicator Masters* page 4.

Remedial: Prepare some exercises such as the ones below to improve rounding concepts.

Directions: Decide whether the middle number of each three is nearer to the larger or smaller of the other two numbers. If the middle number is halfway, choose the larger number.

A. 60 **64** 70
B. 80 **86** 90
C. 300 **350** 500
D. 700 **778** 800
E. 3000 **3623** 4000
F. 8000 **8499** 9000
G. 50 000 **52 965** 60 000
H. 10 000 **14 999** 20 000
I. 400 000 **451 600** 500 000
J. 200 000 **248 771** 300 000
K. 6 000 000 **6 399 999** 7 000 000
L. 9 000 000 **9 764 862** 10 000 000

Enrichment: On the chalkboard, write one statistical fact that involves a large number; for example. "The World population in 1973 was approximately 3 631 797 000." Ask the students to round the number to the nearest 10, 100, 1000, and so on until the rounding process reaches the billions' place.

Assignments

Minimum 1–4. Average 1–7. Maximum 1–9.

Objective

Given two whole numbers, the student can use inequality symbols (<, >) to state which of two numbers is the larger or smaller of the pair.

PREPARATION

Materials: Index cards.

A helpful preparation for this lesson might be to have students describe various sets of objects and tell, without actually identifying the number of the set, which set has more. An example of such pairs of sets would be the seats in the room and the students in the room. These two sets can be easily compared by having each student attempt to take a seat. This makes the pattern of comparing sets by one-to-one matching obvious. You may also find it useful to lead a brief review of place value and expanded notation prior to the Investigation.

INVESTIGATION

Distribute ten index cards to each student or to each group of students, depending upon whether they are to do the Inv stigation individually or in small groups. Students may have some difficulty arranging the cards properly on the basis of the number names shown, so you might encourage them to write the usual numeral on the reverse side of each card before they attempt to put the cards in order. This will help the students appreciate how place value can aid them in comparing the sizes of numbers. (Of course, you might simply have the students write the numerals on a sheet of paper; however, the Investigation will be more effective if the numerals are on cards that students can manipulate to achieve the correct order.)

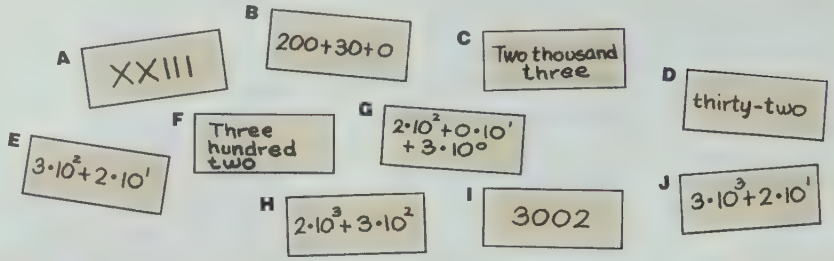
DISCUSSION

The Discussion Exercises emphasize the inequality symbols for "is less than" and "is greater than" and the use of place value in deciding which of two numbers is the greater. Since this will be a review for most students you can probably move through these exercises quite rapidly.

Inequalities

Investigating the Ideas

A symbol for a number is printed on each card.

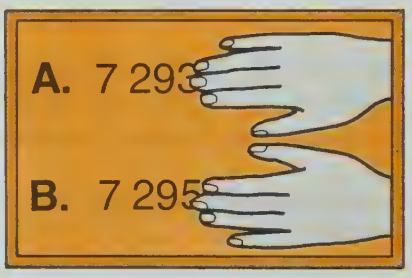


Can you arrange the numbers shown by the cards in order from the smallest to the largest?

Discussing the Ideas

1. In order to show that 23 is less than 32, we write $23 < 32$. We also can say that 32 is greater than 23 and write $32 > 23$. Can you think of some ways to help you remember which symbol, $<$ or $>$, means "is less than" or means "is greater than"?

2. The same number of digits are covered in both A and B. How can you use your understanding of place value to decide which is the larger number?



3. What is the largest number that can be represented by a 3-digit numeral?

4. What is the smallest number that can be represented by a 3-digit numeral?

5. Which symbol ($<$ or $>$) would you put in the ▨ in order to make a true statement.

   A 76 ▨ 84 C 3248 ▨ 3748 E 72 865 ▨ 92 765

   B 219 ▨ 209 D 7036 ▨ 7063 F 3 030 300 ▨ 3 003 003

Using the Ideas

1. Give the correct symbol (<, >, or =) for each ◯.

A 68 ◯ 64	D 6758 ◯ 6578	G 49 753 ◯ 94 753
B 89 ◯ 98	E 5999 ◯ 6001	H 748 932 ◯ 784 923
C 349 ◯ 359	F 3989 ◯ 3899	I 6 004 400 ◯ 6 040 040

2. Copy each pair of numerals placing the correct inequality symbol between. EXAMPLE: For 29, 92, write $29 < 92$.

A 73, 75	D 3574, 3534	G 4672, 4592
B 39, 93	E 8878, 8888	H 1000, 999
C 452, 442	F 6491, 6492	I 0, 364

3. Give the correct symbol (<, >, or =) for each ◯.
Try to do this without finding the sums.

A $24 + 37$ ◯ $25 + 37$	D $343 + 68$ ◯ $86 + 343$
B $62 + 93$ ◯ $72 + 83$	E $487 + 352$ ◯ $497 + 342$
C $249 + 356$ ◯ $239 + 346$	F $6941 + 7213$ ◯ $6841 + 7513$

4. Write the numeral for each number indicated.

A 100 less than one million	G 52 million more than 846
B 1000 less than one million	H 8 billion more than 59
C 100 000 less than one million	I A half million more than 6 587 346
D 10 000 less than one million	J A half million less than 456 972 863
E 101 less than one billion	K The amount that must be added to 456 972 863 to give 457 000 000
F Half as much as one billion	

★ **5.** How old, to the nearest year, would a person be who had lived one million hours? Is he older or younger than a person who has lived one billion seconds?

think

1. Suppose you had **one billion** dollars to give away. If you give away a $100 bill each minute, how long will it take you to give it all away? Guess which answer is closest to the correct answer.

A A day	C A month	E 10 years	G A century
B A week	D A year	F 20 years	H 2000 years

2. Compute the correct answer to the nearest year.

More practice, page S-3, Set 4

UTILIZATION

Exercises 1 and 2 involve placing the correct inequality symbol between pairs of given numbers. Encourage students to use mental computation in doing exercise 4. You can stimulate some interesting discussion by having students describe the shortcuts they may have used to arrive at their answers.

EXTENSION

To provide further practice, make selective assignments from Supplementary Exercise Set 4 on page S-3, *Workbook* page 6, and *Duplicator Masters* page 5.

Enrichment: Students will enjoy the place value game of larger, Larger, LARGEST. It is a game for 2 players. Write different digits on 5 slips of papers. Mix the slips, and then place them in a row to form a 5-digit number.

3	9	2	6	7

The two players take turns exchanging the positions of any two digits so that *after each exchange a larger number is formed.* The player whose exchange results in the largest number wins. A sample exchange for 39 267 is shown below.

A: 7 9 2 6 3
B: 7 9 6 2 3
A: 7 9 6 3 2
B: 9 7 6 3 2 Player B wins!

Students can extend the game to larger numbers by using additional cards.

Think Solution

One billion dollars would be 10 000 000 one-hundred-dollar bills, so it would take 10 million minutes to give them away. There are $60 \cdot 24 \cdot 365$ minutes in 1 year (525 600 minutes); so $10\,000\,000 \div 525\,600$ gives the number of years needed to give all the money away.

Assignments

Minimum 1, 2. Average 1, 2, 3A–C, 4A–F. Maximum 1–5.

Objective

The student can write a numeral in a specified base other than ten, to show the number of objects in a set.

PREPARATION

Materials: Geoboards, dot paper or graph paper (*Duplicator Masters*, page 82 or 83).

Review the idea of place value and the fact that our numeration system is based on grouping by tens.

INVESTIGATION

The Investigation may help you determine how much students may recall about other numeration systems. Have the students study the geoboards shown in the Investigation and discuss the symbols used for base ten, base eight, and base four. Then have the students show the other groupings on their geoboard and write the numerals that show their groupings. If geoboards are not available, dot paper or graph paper can be effective substitutes. An advantage of the geoboard, though, is that it does not preserve student errors with the same permanency as does pencil and paper. Thus students are not inhibited by earlier mistakes.

DISCUSSION

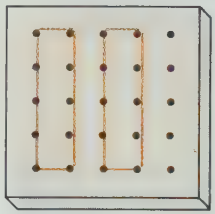
The amount of time that should be devoted to the Discussion will depend upon students' previous experience with other number bases. Encourage students to approach these ideas and concepts as though they are playing an adventurous, creative game. An atmosphere of enjoyment and lively interest can more effectively nourish improved understanding of the role of place value in grouping than can a rigorous presentation of the concepts. When the base of a numeration system is less than ten, the base can be denoted by a simple digit. Thus $12_{(5)}$ indicates a numeral with base five, and $11_{(2)}$ indicates a numeral with base two. However, a numeral such as $101_{(10)}$ is somewhat ambiguous because the symbol for the base $_{(10)}$ could be interpreted as 1 ten and zero or 1 nine and zero and so on. To make certain that there will be no misinterpretation, we shall use "$_{(ten)}$" to denote the base instead of "$_{(10)}$."

Other Number Bases

Investigating the Ideas

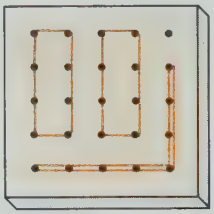
How many nails are on each geoboard? The number is the same for each of them, but the numeral we write depends upon the **base** or **grouping** that we use. In our decimal system, the base is ten. However, we can use other bases to write numerals much the same as we do in base ten.

BASE TEN



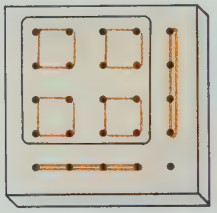
2 tens and 5
We write: $25_{(ten)}$

BASE EIGHT



3 eights and 1
We write: $31_{(8)}$

BASE FOUR



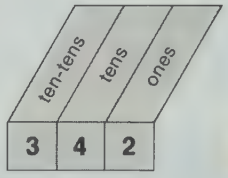
1 four-fours, 2 fours, and 1
We write: $121_{(4)}$

Can you show the groupings and write the numerals for the number of nails on a geoboard using base-seven, base-six, and base-five numeals?

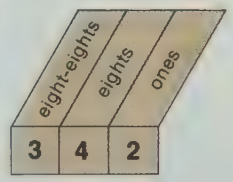
Discussing the Ideas

1. In the base-ten numeration system, the ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 are used with place value to write any numeral. In base eight, only the eight digits 0, 1, 2, 3, 4, 5, 6, and 7 are used. What digits do you think are used in base seven?

2. The names of the first three places are shown for a base-ten and a base-eight numeral. What are the names of the first three places for a base-six numeral?



Base ten



Base eight

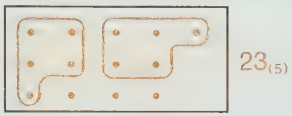
3. Complete the statement.
2 groups of eight and 3 ones in base eight is the same as ? groups of ten and ? ones in base ten.

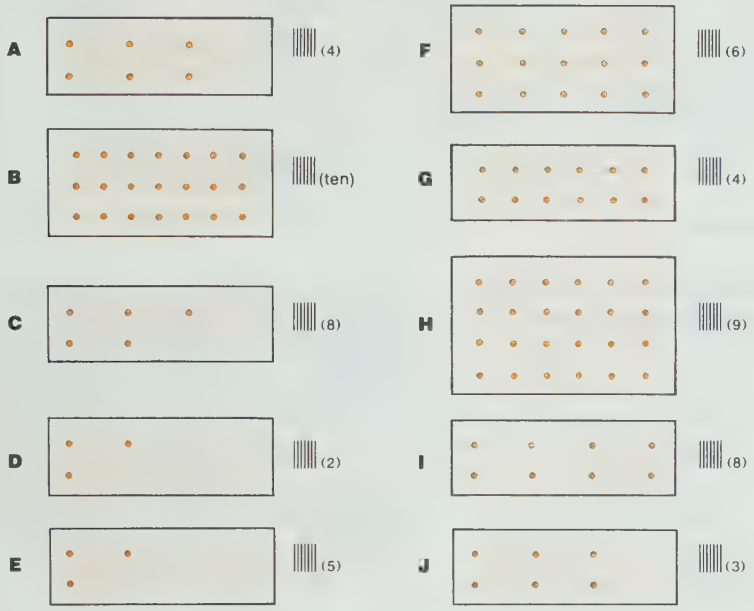
4. Use your understanding of place value to explain what each numeral shows. EXAMPLE: $23_{(5)}$ means 2 **fives** and 3
   A $14_{(6)}$ B $27_{(8)}$ C $30_{(5)}$ D $101_{(ten)}$ E $121_{(3)}$

Using the Ideas

1. Think about grouping the following sets of dots as indicated by the given base. Write a numeral which indicates the number of dots in each box.

EXAMPLE:





2. Give the missing base-eight numerals for the number of dots shown in each block.

3. For each part, write a numeral using the indicated base.

- **A** 3 eights and 2
- **B** 1 six-sixes, 3 sixes, and 5
- **C** 5 eight-eights, 7 eights and 1
- **D** 2 four-fours, 3 fours, and 3
- **E** 3 five-fives, 0 fives, and 2
- **F** 1 two-twos, 1 two, and 0

★**4.** "$312_{(4)}$" is a base four numeral that represents a certain whole number. Draw $312_{(4)}$ dots. (**Hint:** Think of the place-value scheme that we use in base ten.)

UTILIZATION

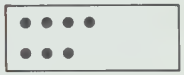
As you present this section, keep in mind that the material's primary purpose is to stimulate interest and create excitement in mathematics. The students are not expected to attain a high degree of mastery in work with numbers written in other bases. The students should not feel obliged to memorize rules or procedures for arriving at numerals in other bases.

When the students have finished the exercises, give them an opportunity to discuss the methods they used to find the other base numerals for each of the sets.

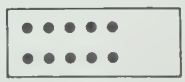
EXTENSION

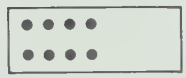
To provide further practice, assign page 6 of the *Duplicator Masters*.

Remedial: Prepare some additional problems like the ones below. Ask students to think about grouping the dots by the given base and then give the appropriate symbol for each |.

A. 
1 group of four and 3. We write $13_{(4)}$.

B.
1 group of eight and 6. We write $16_{(8)}$.

C. 
2 groups of five and 0. We write $20_{(5)}$.

D. 
2 groups of three and 2. We write $22_{(3)}$.

Enrichment: Call attention to Research Project D on page A-27. Several of your students may be interested in making a calendar like the one suggested.

Assignments

Minimum 1, 2. Average 1–3. Maximum 1–4.

Objectives

The student can write a base-eight numeral to name the number of objects in a set.

The student can write a base-eight numeral for a base-ten numeral and vice versa.

PREPARATION

Materials: Geoboards or dot paper (*Duplicator Masters*, page 82 or 83).

This lesson focuses on a single numeration system, base eight. Review grouping in various bases before introducing the Investigation.

INVESTIGATION

Students can place four geoboards together and use rubber bands to group by eights. (Dot paper may be used as a substitute for geoboards.) As students work on the Investigation you might ask questions such as: "Are there as many as 8 eights?" "How many eights are left over?" Questions such as these will help the student to write the correct numeral, $144_{(8)}$, for the number of nails.

DISCUSSION

The table in Exercise 1 is designed to help the student compare place-value names of base-eight and base-ten numerals. The students must understand that in base eight the digit in the third place represents 8 *eights*, 8^2, or 64 just as in base ten the digit in the third place represents 10 *tens*, 10^2, or 100.

After discussing Exercise 2, you might discuss the division process for converting from base ten to base eight. Using $165_{(ten)}$ as an example, we have:

$64\overline{)165}$
$\underline{128}$ ② sixty-fours
$8\overline{)37}$
$\underline{32}$ ④ eights
5 ⑤ ones Hence, $165_{(ten)} = 245_{(8)}$

Exercise 3 introduces expanded notation for base eight. By knowing place-value names, students can convert base-eight numerals to base-ten numerals. Thus, $325_{(8)}$ means

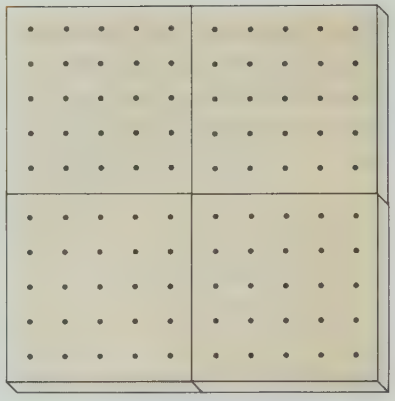
3 sixty-fours + 2 eights + 5 ones

or $3 \cdot 8^2 + 2 \cdot 8^1 + 5 \cdot 8^0 = 213_{(ten)}$.

A Base-Eight Numeration System

Investigating the Ideas

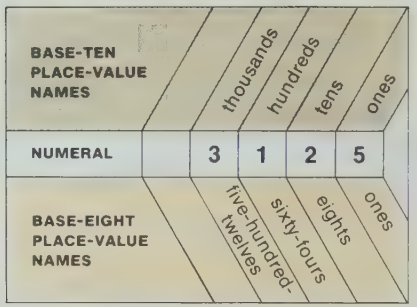
When four geoboards are placed together, the base-ten numeral for the number of nails is $100_{(ten)}$.



What is the base-eight numeral for the number of nails on the geoboards?

Discussing the Ideas

1. The diagram at the right compares the place-value names for base ten and for base eight.
   - A In a base-ten numeral, what is the name of the next place to the left of the thousands place?
   - B In a base-eight numeral, what would be the name of the next place to the left of the five-hundred-twelves place?



2. By answering each question below, you can see how to rewrite the base-ten numeral $89_{(ten)}$ as a base-eight numeral.
   - A How many sixty-fours in $89_{(ten)}$?
   - B How many eights in $25_{(ten)}$?
   - C How many ones left over?
   - D What is the base-eight numeral for $89_{(ten)}$?

$$\begin{array}{rl} 89 & \\ -64 & \leftarrow \text{① sixty-fours} \\ \hline 25 & \\ -24 & \leftarrow \text{③ eights} \\ \hline 1 & \leftarrow \text{① ones} \end{array}$$

3. You can express base-eight numerals in expanded notation in much the same way as in the base-ten system. However, you use powers of eight rather than powers of ten. How would you express each of these base-eight numerals in expanded notation?

   A $245_{(ten)}$ means $(2 \cdot 10^2) + (4 \cdot 10^1) + 5$.

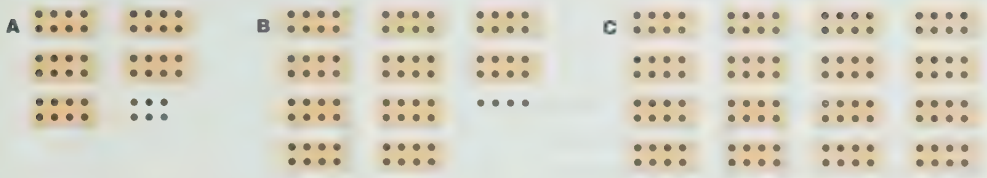
   B $245_{(8)}$ means $(2 \cdot 8^2) + (4 \cdot 8^1) + 5$, or $165_{(ten)}$.

   Note that just as 10^2 means $10 \cdot 10$, 8^2 means $8 \cdot 8$.

   A $37_{(8)}$ B $137_{(8)}$ C $416_{(8)}$ D $506_{(8)}$

Using the Ideas

1. Give the base-eight numeral for the number of dots shown in each part.



2. Complete each sentence with the correct place-value name.
   - A $276_{(8)}$: The 6 tells how many ? .
   - B $276_{(8)}$: The 7 tells how many ? .
   - C $276_{(8)}$: The 2 tells how many ? .

3. Solve the equations.
   - A $19_{(ten)} = (2 \cdot 8) + n$
   - B $30_{(ten)} = (n \cdot 8) + 6$
   - C $58_{(ten)} = (n \cdot 8) + 2$
   - D $62_{(ten)} = (7 \cdot 8) + n$
   - E $45_{(ten)} = (n \cdot 8) + 5$
   - F $48_{(ten)} = (6 \cdot 8) + n$

4. Write a 2-digit base-eight numeral for each base-ten numeral in Exercise 3.

5. Write a 3-digit base-eight numeral for each base-ten numeral.
   - A $77_{(ten)}$
   - B $102_{(ten)}$
   - C $99_{(ten)}$
   - D $149_{(ten)}$
   - E $215_{(ten)}$
   - F $511_{(ten)}$

6. Write each numeral in base-eight expanded notation.
   EXAMPLE: $137_{(8)}$ $1 \cdot 8^2 + 3 \cdot 8^1 + 7$
   - A $35_{(8)}$
   - B $125_{(8)}$
   - C $740_{(8)}$
   - D $346_{(8)}$
   - E $407_{(8)}$
   - F $623_{(8)}$

7. Write a base-ten numeral for each base-eight numeral in Exercise 6. EXAMPLE: $137_{(8)} = 1 \cdot 8^2 + 3 \cdot 8^1 + 7$
   $= 64 + 24 + 7 = 95_{(ten)}$

★8. What number comes immediately after each of these numbers? Use base-eight numerals.
   - A $57_{(8)}$
   - B $127_{(8)}$
   - C $277_{(8)}$
   - D $7077_{(8)}$
   - E $100777_{(8)}$

★9. In base-eight notation, $3412_{(8)}$ means $(3 \cdot 8^3) + (4 \cdot 8^2) + (1 \cdot 8^1) + 2$
Copy each exercise and give the base-ten numeral for each ▨.
   - A $3412_{(8)} =$ ▨ $_{(ten)}$
   - B $5632_{(8)} =$ ▨ $_{(ten)}$
   - C $4273_{(8)} =$ ▨ $_{(ten)}$
   - D $2541_{(8)} =$ ▨ $_{(ten)}$
   - E $7675_{(8)} =$ ▨ $_{(ten)}$
   - F $7777_{(8)} =$ ▨ $_{(ten)}$

UTILIZATION

The exercises provide practice in writing base-eight numerals and converting from base ten to base eight, and vice versa.

Exercises 3 and 4 involve the use of expanded notation to convert a base-ten numeral to base eight. In Exercise 3A, $19_{(ten)} = (2 \cdot 8) + 3$, hence the base-eight numeral for $19_{(ten)}$ is $23_{(8)}$.

Exercises 6, 7, and 9 involve the use of expanded notation with powers of eight to convert to base ten. In Exercise 7, $125_{(8)} = 1 \cdot 8^2 + 2 \cdot 8 + 5 = (64 + 16 + 5)_{(ten)} = 85_{(ten)}$.

EXTENSION

More practice is provided on page 7 of the *Workbook*.

Remedial: Provide additional practice problems in writing base-eight numerals. Use sets of dots grouped in eights as in Exercise 1 of Using the Ideas.

Enrichment: Provide practice problems involving base-eight numerals of 4, 5, and 6 digits. (See Exercise 9 of the Using the Ideas section.)

Assignments

Minimum 1, 2. Average 1–7. Maximum 1–9.

Objective

The student can find sums and differences using one- and two-digit base-eight numerals.

PREPARATION

Materials: Sets of 14 small objects such as counters or buttons.

You might want to prepare duplicated copies of a base-eight table like that shown in the Investigation, to conserve class time.

INVESTIGATION

Have the students study the addition example in the Investigation. You will want to provide additional examples of adding with one-digit base-eight numerals before having the students fill out the addition table. Some students may find it helpful to use concrete materials such as counters to demonstrate addition. For example, to arrive at the sum $6 + 7$, they might use a set of 6 counters and a set of 7 counters, push them together, and then group them by eights to get 1 group of eight and 5. Hence, $6 + 7$ in base eight yields a result of $15_{(8)}$.

Again, we stress that although you are now dealing with computation in a base other than ten, your primary purpose remains to stimulate student interest. We do not expect students to compute in another base with proficiency. The idea is to present computing in another base as a means of broadening the students' general understanding of numeration systems and how these ideas operate in problems involving computation.

DISCUSSION

Extend Discussion Exercise 1 by encouraging students to compare and discuss their Investigation findings. Exercise 2 presents the addition algorithm for base-eight numerals. Contrast it with addition using base-ten numerals.

The subtraction algorithm is presented in Exercise 4. Further examples should be provided as necessary.

Encourage the students to "think in base eight," as shown in Exercises 2 and 4; otherwise, some may convert each numeral to base-ten, find the base-ten (or difference) and then convert it to a base-eight numeral.

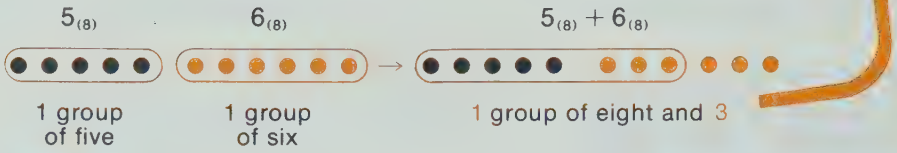
Adding and Subtracting with Base-Eight Numerals

Investigating the Ideas

Addition and subtraction with base-eight numerals may be performed in much the same way as with base-ten numerals. The example shows how you can use sets to think about addition in base eight.

Base-eight addition table

+	0	1	2	3	4	5	6	7
0								
1								
2								
3								
4								
5								13
6								
7								



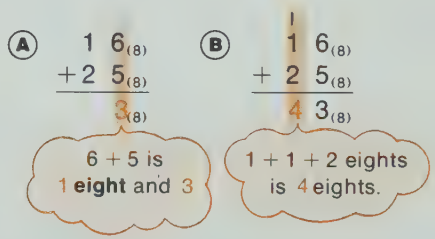
Can you fill in the addition table like the one above using base-eight numerals?

Discussing the Ideas

1. Explain how to find $7_{(8)} + 6_{(8)}$: Use sets of dots, if necessary.

2. Study the steps for finding a sum using base-eight numerals. Can you explain how to find the sums below?

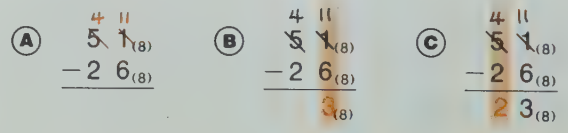
   A $27_{(8)} + 46_{(8)}$ B $53_{(8)} + 35_{(8)}$



3. How can the base-eight addition table be used to find these differences?

   A $13_{(8)} - 5_{(8)}$ B $16_{(8)} - 7_{(8)}$ C $10_{(8)} - 7_{(8)}$ D $11_{(8)} - 3_{(8)}$

4. Explain the steps in this subtraction problem.



Using the Ideas

1. Find the sums. Use the addition table completed in the Investigation, if necessary.

A $15_{(8)} + 2_{(8)}$ **B** $15_{(8)} + 3_{(8)}$ **C** $16_{(8)} + 4_{(8)}$ **D** $26_{(8)} + 1_{(8)}$ **E** $26_{(8)} + 2_{(8)}$ **F** $26_{(8)} + 3_{(8)}$

G $24_{(8)} + 33_{(8)}$ **H** $24_{(8)} + 34_{(8)}$ **I** $24_{(8)} + 37_{(8)}$ **J** $54_{(8)} + 23_{(8)}$ **K** $44_{(8)} + 36_{(8)}$ **L** $54_{(8)} + 24_{(8)}$

M $67_{(8)} + 21_{(8)}$ **N** $135_{(8)} + 216_{(8)}$ **O** $344_{(8)} + 125_{(8)}$ ★ **P** $626_{(8)} + 547_{(8)}$ ★ **Q** $316_{(8} + 712_{(8)}$ ★ **R** $1735_{(8)} + 2276_{(8)}$

2. Find the sums. Use base-eight numerals.

A $2_{(8)} + 3_{(8)} + 4_{(8)}$ **B** $6_{(8)} + 5_{(8)} + 4_{(8)}$ **C** $3_{(8)} + 7_{(8)} + 6_{(8)}$ **D** $6_{(8)} + 6_{(8)} + 6_{(8)} + 6_{(8)}$ **E** $7_{(8)} + 1_{(8)} + 4_{(8)} + 4_{(8)}$ **F** $5_{(8)} + 7_{(8)} + 6_{(8)} + 4_{(8)}$

3. Find the differences. Use the addition table, if necessary.

A $13_{(8)} - 4_{(8)}$ **B** $7_{(8)} - 2_{(8)}$ **C** $10_{(8)} - 5_{(8)}$ **D** $12_{(8)} - 6_{(8)}$

E $16_{(8)} - 6_{(8)}$ **F** $15_{(8)} - 4_{(8)}$ **G** $15_{(8)} - 6_{(8)}$ **H** $14_{(8)} - 5_{(8)}$

I $13_{(8)} - 5_{(8)}$ **J** $15_{(8)} - 7_{(8)}$ **K** $10_{(8)} - 7_{(8)}$ **L** $16_{(8)} - 7_{(8)}$

4. Find the differences. Use base-eight numerals.

A $26_{(8)} - 13_{(8)}$ **B** $60_{(8)} - 27_{(8)}$

C $41_{(8)} - 5_{(8)}$ **D** $74_{(8)} - 47_{(8)}$

E $55_{(8)} - 16_{(8)}$ **F** $100_{(8)} - 77_{(8)}$

★ **G** $205_{(8)} - 173_{(8)}$ ★ **H** $431_{(8)} - 163_{(8)}$

think

Place base-eight numerals in the empty spaces of the square to form a **magic square**. The sum in each row, column, and both diagonals must be the same.

$15_{(8)}$		$13_{(8)}$
	$12_{(8)}$	
		$7_{(8)}$

UTILIZATION

In assigning the exercises, keep in mind that you should not expect students to be proficient in computation with base-eight numerals. They should not be expected to memorize any base-eight facts and should be permitted to refer to their addition tables when necessary. The main purpose of this work is for students to gain a deeper insight into the nature of a numeration system. Some computers utilize base-eight numeration, but aside from this practical significance, the use of base-eight rather than some other base in these lessons is arbitrary.

EXTENSION

See Supplementary Exercise Set 5 on page S-4, *Workbook* page 8, and *Duplicator Masters* page 7 for additional practice with the concepts presented in this lesson.

Think Solution

Using the three given numerals in the diagonal of the square, the sum is found to be $36_{(8)}$. With this information, the numerals for the remaining squares can be found by addition and subtraction.

Assignments

Minimum 1A–F, 2A–C, 3A–D.
Average 1A–I, 2, 3A–H, 4A–D.
Maximum 1–4.

Objective

The student can find products and quotients using base-eight numerals.

PREPARATION

Materials: 50 small objects (counters, buttons, etc.).

It would be helpful to make duplicated copies of the base-eight multiplication table for distribution to the students when they begin on the Investigation.

INVESTIGATION

Have the students study the multiplication example in the Investigation. The students may need to try some additional examples, preferably with concrete materials.

To demonstrate the product $7 \cdot 3$ using physical objects, the student would exhibit 3 sets of 7 counters each, push the three sets together, and then group by eights to get 2 sets of eight and 5. Thus $3 \cdot 7$ would be $25_{(8)}$. Give the students considerable latitude in working out these tables for themselves. At this stage they should not be discouraged from converting to base ten, computing in this familiar base, and then converting back to base eight. Distribute the duplicated multiplication tables and have the students complete them.

DISCUSSION

Give careful attention to the Discussion Exercises, especially to Exercise 3 which illustrates the algorithm for multiplying a two-digit base-eight numeral by a one-digit numeral. Give additional examples as needed.

Exercises 4 and 5 introduce division with base-eight numerals. Allow the students to use their tables to find additional quotients for division problems that you or the students themselves suggest.

Multiplying and Dividing with Base-Eight Numerals

Investigating the Ideas

You can multiply and divide using base-eight numerals in much the same way as you do using base-ten numerals. The example below shows how you can use sets to think about multiplication in base eight.

Base-eight multiplication table

×	0	1	2	3	4	5	6	7
0								
1								
2								
3								22
4								
5								
6								
7								

Can you fill in a multiplication table like the one above using base-eight numerals?

Discussing the Ideas

1. **A** What multiplication fact does the array of dots suggest?
   B Write the fact using base-eight numerals.

2. Explain how to find $4_{(8)} \cdot 7_{(8)}$. Use an array of dots, if necessary.

3. Explain the steps in the multiplication problem.

A
$$\begin{array}{r} {\scriptstyle 2}\ 3\ 4_{(8)} \\ \times\ 5_{(8)} \\ \hline 4_{(8)} \end{array}$$

B
$$\begin{array}{r} {\scriptstyle 2}\ 3\ 4_{(8)} \\ \times\ 5_{(8)} \\ \hline 2\ 1\ 4_{(8)} \end{array}$$

4. **A** Since $3_{(8)} \cdot 6_{(8)} = 22_{(8)}$, what is $22_{(8)} \div 6_{(8)}$?
   B What is $22_{(8)} \div 3_{(8)}$?

5. Explain how the base-eight multiplication table can be used to find these quotients.
   A $17_{(8)} \div 3_{(8)}$ **B** $34_{(8)} \div 7_{(8)}$ **C** $10_{(8)} \div 2_{(8)}$ **D** $52_{(8)} \div 7_{(8)}$

Using the ideas

1. Find the products. Use the base-eight multiplication table in Investigation.

A $12_{(8)} \times 3_{(8)}$ **B** $12_{(8)} \times 4_{(8)}$ **C** $12_{(8)} \times 5_{(8)}$ **D** $23_{(8)} \times 3_{(8)}$ **E** $34_{(8)} \times 2_{(8)}$ **F** $17_{(8)} \times 6_{(8)}$

G $15_{(8)} \times 7_{(8)}$ **H** $42_{(8)} \times 5_{(8)}$ **I** $36_{(8)} \times 2_{(8)}$ **J** $33_{(8)} \times 3_{(8)}$ **K** $57_{(8)} \times 6_{(8)}$ **L** $66_{(8)} \times 5_{(8)}$

2. Find the quotients. Use the multiplication table, if necessary.

A $6_{(8)} \div 2_{(8)}$ **B** $10_{(8)} \div 2_{(8)}$ **C** $17_{(8)} \div 3_{(8)}$ **D** $16_{(8)} \div 2_{(8)}$

E $30_{(8)} \div 6_{(8)}$ **F** $44_{(8)} \div 6_{(8)}$ **G** $43_{(8)} \div 7_{(8)}$ **H** $36_{(8)} \div 5_{(8)}$

3. Use base-eight numerals to continue the doubling process until you reach $10000_{(8)}$.

$1_{(8)}$, $2_{(8)}$, $4_{(8)}$, $10_{(8)}$, $20_{(8)}$, ▮, ▮, ▮, . . .

4. Is $10000_{(8)}$ greater than or less than $5000_{(ten)}$?

5. Find the products. Use base-eight numerals.

A $23_{(8)} \times 15_{(8)}$ **B** $34_{(8)} \times 26_{(8)}$

C $304_{(8)} \times 12_{(8)}$ **D** $537_{(8)} \times 46_{(8)}$

6. Find the quotients. Use base-eight numerals.

A $6_{(8)}\overline{)556_{(8)}}$

B $5_{(8)}\overline{)1572_{(8)}}$

C $23_{(8)}\overline{)503_{(8)}}$

D $34_{(8)}\overline{)2444_{(8)}}$

Think

Numbo was in grade 22 and was 110 years old. He slept 101 hours each night, awoke at 21 o'clock, and ran 1212 metres in 22 seconds. About all the history he knew was that Columbus discovered North America in the year 2001021.

1. The numbers in this story are represented using a number base other than ten. Find what number base is used and recopy the story using the correct base ten numerals.
2. Write a story using a number base other than base ten. Give the story to a classmate to figure out.

UTILIZATION

Again, in multiplication, you should not expect great computational skill using base-eight numerals. If students are allowed to use their multiplication tables freely in working the exercises, they will undertake the assignment with greater confidence.

Exercises 5 and 6 provide more complex multiplication and division exercises, and should be assigned selectively depending on students' abilities.

EXTENSION

Further practice with the concepts of this lesson is provided in Supplementary Exercise Set 6 on page S-4, in the *Workbook* on page 9, and in the *Duplicator Masters* on page 8.

Think Solution

Since 0, 1, and 2 are the only digits used, it can be determined that the base is three. A check that Columbus discovered America in 2001021 (or 1492) assures us that base three is correct.

Here are some selected conversions from base three to base ten in the story:

$$22_{(3)} = (2 \cdot 3^1) + (2 \cdot 3^0) = 8$$

$$110_{(3)} = (1 \cdot 3^2) + (1 \cdot 3^1) = 12$$

$$1212_{(3)} = (1 \cdot 3^3) + (2 \cdot 3^2) + (1 \cdot 3^1) + (2 \cdot 3^0)$$
$$= 27 + 18 + 3 + 2 = 50$$

$$2001021_{(3)} = (2 \cdot 3^6) + (1 \cdot 3^3) + (2 \cdot 3^1) + (1 \cdot 3^0)$$
$$= 1458 + 27 + 6 + 1$$
$$= 1492$$

Assignments

Minimum 1A–F, 2A–D. Average 1–4. Maximum 1–6.

Objective

The student can write a Roman numeral for a given decimal numeral, and conversely.

UTILIZATION

The material on this page and the following page represents a departure from the usual teaching strategy of the previous material. Both pages can be regarded as optional material and can be assigned as enrichment or extension work.

The pedagogical significance of this section may lie chiefly in the study of the development of Roman numerals, which are still used in a variety of familiar situations. While the Roman numeration employs both an *additive* and a *subtractive* scheme, originally the system was strictly additive. Thus, 9 originally was written VIIII rather than IX. The advantage of the additive system is apparent if one attempts to compute using Roman numerals. The abacus was used in ancient Rome for computing and the numerals were used for number recording purposes.

The subtractive scheme introduces some ambiguities unless certain agreements are made. For example 49 might be written XLIX or IL. For the answers given, we have made these agreements:

I is subtracted when preceding V or X.
X is subtracted when preceding L or C.
C is subtracted when preceding D or M.

Answers, Exercise 5

A. I, II, III, IV, V, VI, VII, VIII, IX, X
B. XV, XVI, XVII, XVIII, XIX, XX
C. XL, XLI, XLII, XLIII, XLIV, XLV, XLVI, XLVII, XLVIII, XLIX, L
D. CXX, CXXI, CXXII, CXXIII, CXXIV, CXXV, CXXVI, CXXVII, CXXVIII, CXXIX, CXXX
E. CCCXCV, CCCXCVI, CCCXCVII, CCCXCVIII, CCCXCIX, CD
F. CMXC, CMXCI, CMXCII, CMXCIII, CMXCIV, CMXCV, CMXCVI, CMXCVII, CMXCVIII, CMXCIX, M

Assignments

Minimum 1–3. Average 1–5. Maximum 1–8.

ROMAN NUMERALS

The **Roman numeration system** does not use place value. When two Roman symbols are placed side by side, we add or subtract to determine the number represented. We **add** when the smaller number is written on the right.

XI means 10 + 1

We **subtract** when the smaller number is written on the left.

IX means 10 − 1

The seven basic Roman numerals	
I = 1	C = 100
V = 5	D = 500
X = 10	M = 1000
L = 50	

1. Write a decimal numeral for each Roman numeral. Each numeral is an "addition numeral."

A	VI	C	XV	E	LX	G	CX	I	DC	K	MC
B	LI	D	XVI	F	CL	H	CV	J	DXI	L	MD

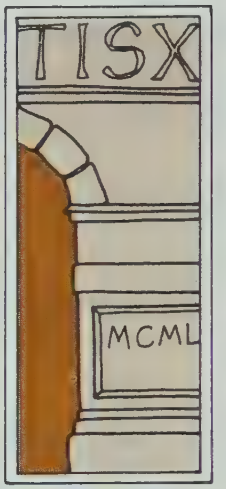
2. Write a decimal numeral for each of the six basic "subtraction numerals" shown below.

A	IV	B	IX	C	XL	D	XC	E	CD	F	CM

3. Write a Roman numeral for each decimal numeral.

EXAMPLES: 7 = VII 24 = XXIV

A	3	C	11	E	38	G	49	I	529	K	1372
B	6	D	34	F	19	H	126	J	340	L	2968



4. Write a decimal numeral for each Roman numeral.

A	CCLXII	C	CDXLIV	E	CMLIX	G	MCDXCII
B	DXLV	D	CDXXXVII	F	MMLI	H	MCMLXXVI

5. Write the Roman numerals for the indicated sets of numbers.

A	1 through 10	C	40 through 50	E	395 through 400
B	15 through 20	D	120 through 130	F	990 through 1000

6. Add each of the following without changing to decimal numerals.

A	CXI + XVI	B	CM + D	C	LXVII + CLXVI

7. Subtract each of the following without changing to decimal numerals.

A	XXXIII − XII	B	MDCCXXXVIII − MCCXXI	C	MDXI − MCX

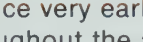
8. Find the product of CLVI and XV without changing to decimal numerals.

OTHER ANCIENT NUMERALS

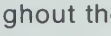
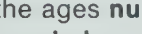
Since very early times, people have needed numbers. Throughout the ages **numbers have not changed, but many different numerals have been used to represent them.**

The **Egyptian numeration system** did not use place value. To show larger numbers, they wrote symbols over again and **added** the value of each symbol.

Egyptian numerals 1700 B.C.		Our numerals
	=	1
	=	10
	=	100
	=	1000
	=	10 000
	=	100 000
	=	1 000 000

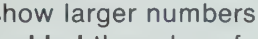
For example,  is a symbol for 132

1. Write a decimal numeral for each Egyptian numeral.

A  **B**  **C**

D **E**

F **G**

H 

2. Write an Egyptian numeral for each decimal numeral.

A 18	**C** 426	**E** 1 000 010	**G** 1968	**I** 2 432 324
B 42	**D** 100 100	**F** 1 111 111	**H** 10 001	**J** 3 005 418

The **Babylonian numeration system** used the idea of place value. It was a **base sixty** numeration system! The Babylonians used a piece of wood called a **stylus** to write in clay. They wrote numerals for numbers up to 59, using combinations of the two symbols shown in the Babylonian numeral table.

Babylonian numerals 1700 B.C.		Our numerals
	=	1
	=	10

For example, is a symbol for 6 and is a symbol for 44

3. Write a decimal numeral for each Babylonian numeral.

A **B** **C** **D**

4. Write a Babylonian numeral for each decimal numeral.

A 7 **B** 16 **C** 36 **D** 40 **E** 55 **F** 11

The ancient **Mayas** of Yucatan used a numeration system with a base of 20. For example, the Mayan numeral is a symbol for 12, and would represent $1 \cdot 20 + 12$ or 32

Mayan numerals		Our numerals
	=	0
	=	1
	=	5
	=	7
	=	27

5. Write a decimal numeral for each Mayan numeral.

A **B** **C** **D** **E**

Objective

The student can represent numerals by Egyptian, Babylonian, or Mayan numerals.

UTILIZATION

This lesson is also a departure from the usual teaching strategy. You may wish to cover only one of the systems at a time. Exercises 1 and 2 deal with the Egyptian system, which is a primitive additive system employing no place value.

Exercises 3 and 4 involve the Babylonian system, a base-sixty system.

Exercise 5 introduces the Mayan numeration system, which had a base of 20.

The emphasis you give to these systems should be determined by the interest of your students.

Answers, Exercises 2 and 4

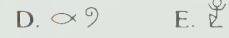
2A. B.

C.

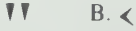
D. E. F.

G.

H. I.

J. 

4A. B. C.

D. E.  F.

EXTENSION

Call the students' attention to Research Projects C, D, and E at the end of this module. These projects can serve as valuable motivational devices since they permit students to depart from the usual classroom work and place no limit on individual creativity.

Assignments

Minimum 1, 2. Average 1–4. Maximum 1–5.

Objective

The student will demonstrate the ability to work with the concepts presented in this module.

UTILIZATION

You may assign these exercises either for independent study or use them as a basis for small or large group discussion. They are not intended to be used for evaluation purposes, although they can form the basis for the construction of a teacher-made test.

If you have made varied assignments or have elected to omit some topics in the module, keep this in mind in assigning the review problems. For example, if you have not emphasized computation in bases other than ten, you will want to omit Exercises 16 and 17.

EXTENSION

More review of the ideas contained in this module is provided on page 10 of the *Workbook*.

REVIEWING THE IDEAS

1. Let $R = \{1, 2, 3, 4, 5\}$ and $S = \{0, 2, 4, 6, 8\}$.

A List the elements in $R \cup S$.

B List the elements in $R \cap S$.

2. Write 7369 using expanded notation.

3. In the numeral **563 412 989**, what digit is in each of these places?

A ones' D ten millions'

B thousands' E hundreds'

C millions' F hundred thousands'

4. Write the usual numeral for $9 \cdot 10\,000 + 8 \cdot 1000 + 5 \cdot 100 + 3 \cdot 10 + 7$.

5. Which expression is not a numeral for ten?

A $10 \cdot 1$ C $10 + 0$ E 10^0

B 10^1 D $10 \div 1$ F $10^3 \div 10^2$

6. A 10^4 means 10 used as a factor ? times.

B $10^4 =$ ▨

7. Write 100 000 using 10 with an exponent.

8. Give the product **P** and the missing power of ten.

$$\begin{array}{ccccc} 100 & \cdot & 1000 & = & \mathbf{P} \\ \downarrow & & \downarrow & & \downarrow \\ 10^2 & \cdot & 10^{\square} & = & \blacksquare \end{array}$$

9. Write the usual numeral for each.

A $4 \cdot 10^3$ B $7 \cdot 10^9$ C $5 \cdot 10^0$

10. Express 400 000 in scientific notation.

11. Round 7 856 467 to the nearest

A hundred. C hundred thousand.

B thousand. D million.

12. Give the correct symbol ($<$, $>$, or $=$) for each ▨.

A 347 ▨ 374

B $637 + 146$ ▨ $627 + 156$

C 34 843 ▨ 34 749

13. What is one less than one million?

14. Write a numeral telling how many dots there are using

A base-ten.

B base-eight.

C base-five.

15. A How many eights in 43?

B How many ones left over?

C Write the base-eight numeral for $43_{(ten)}$.

16. Find the sums and differences.

A $5_{(8)} + 5_{(8)} =$ ▨

B $11_{(8)} - 5_{(8)} =$ ▨

C $\begin{array}{r} 44_{(8)} \\ +\,16_{(8)} \\ \hline \end{array}$ D $\begin{array}{r} 21_{(8)} \\ -\,14_{(8)} \\ \hline \end{array}$

17. Find the products and quotients.

A $4_{(8)} \cdot 3_{(8)} =$ ▨

B $24_{(8)} \div 5_{(8)} =$ ▨

C $\begin{array}{r} 23_{(8)} \\ \times\,5_{(8)} \\ \hline \end{array}$ D $7_{(8)}\overline{)430_{(8)}}$

18. Write a decimal numeral for each Roman numeral.

A CLXVII B MDCCIX

19. Write a Roman numeral for each decimal numeral.

A 629 B 1984

TEST YOURSELF

1. Let $X = \{0, 1, 3, 4, 6\}$ and $Y = \{2, 3, 4, 5, 7\}$.
   - A List the elements in $X \cup Y$.
   - B List the elements in $X \cap Y$.

2. In the numeral **3 459 726**, which digit is in the
   - A thousands' place?
   - B hundred thousands' place?

3. What is the exponent of 10 in the equation $1\,000\,000 = 10^{\square}$?

4. Express 700 000 in scientific notation.

5. Round 3 456 726 to the nearest
   - A thousand.
   - B hundred thousand.

6. Give the correct symbol ($<$ or $>$) for each ◍.
   - A 649 ◍ 694
   - B 21 749 ◍ 21 726
   - C 23 388 635 ◍ 23 386 835

7. Write the base-eight numeral for the number of dots shown below.

8. Find the sum and difference.
   - A $5_{(8)} + 7_{(8)}$
   - B $13_{(8)} - 6_{(8)}$

9. Find the product and quotient.
   - A $5_{(8)} \cdot 7_{(8)}$
   - B $30_{(8)} \div 4_{(8)}$

10. Write a decimal numeral for MDLXIV.

RESEARCH PROJECTS

A Do you know how large a decillion is? A quintillion? Find the names of the periods of very large numbers and make a chart to show your findings. (See *Lore of Large Numbers* by Philip Davis, New Mathematics Library; Syracuse, New York: Singer Company, 1961 available from Random House of Canada Ltd.)

B Read and report about the **abacus**. There are several different kinds. What people have used the abacus? What people still use the abacus today (See *I Can Learn About Calculators and Computers* by Raymond Kenyon; Evanston, Illinois: Harper and Row, 1961 available from Fitzhenry and Whiteside Ltd.)

C Make a model of a Babylonian clay tablet using modeling clay. Use a triangular stylus to mark Babylonian numerals on the tablet. (See *History of Mathematics* by David E. Smith; New York: Dover Publications, 1953 available from General Publishing, Ltd.)

D Make a calender with numerals for the days of the month in a base other than base ten.

E Find out about the ancient Greek numeration system. Make a poster showing how their numerals compare to ours. (See *A History of Mathematics* by Carl B. Boyer; New York: John Wiley & Sons, Canada, Ltd. 1968)

TEST YOURSELF

This criterion-referenced test is intended to provide the students with a means of evaluating their mastery of the major concepts and skills treated in this module. The numerals annotated beside each test question show the correspondence between the test items and the objectives listed on the module-opening page.

Answers to the test problems are provided for the students on the following page so that they can check their own work and gain immediate reinforcement.

The Module Achievement Tests, which are part of the test package that is available from the publisher, are also criterion referenced and may be used for further student evaluation. Students who perform well on the self-evaluation test can be expected to perform well on the achievement test also, since the latter is designed to test performance for the same objectives but with alternative questions.

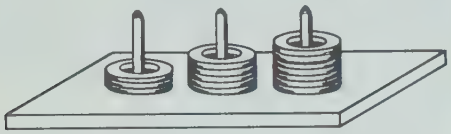
RESEARCH PROJECTS

The research projects are designed to encourage active student involvement in other mathematical topics. Many of the projects could be the basis for math fair projects. Generally the projects require that the student seek out additional information on a particular topic. Some references are given as aids in finding information about the project. For some students, work on such projects will be extremely motivating and will result in increased interest in the subject of mathematics.

Encourage students who undertake Research Project B to find out how the Japanese abacus, called the *soroban*, can be used in computing. Some students may have a model at home that they could bring to class.

MATHEMATICAL RECREATION

The Game of Nim is an ancient game which provides good practice in building strategies and using logical reasoning. The game is available commercially in the form of a 3-place abacus. A homemade version could easily be made with a 3-peg board and washers.



There are various strategies that students can discover. For example if you can play and leave a 1, 2, 3 combination (1 in one row, 2 in another row, and 3 in the third row), you can win. Your opponent has six moves at his disposal and all lose.

If he leaves:	Then you leave:
0, 2, 3	0, 2, 2
1, 1, 3	1, 1, 1
1, 0, 3	1, 0, 0
1, 2, 2	0, 2, 2
1, 2, 1	1, 1, 1
1, 2, 0	1, 0, 0

If you leave a 1, 4, 5 combination, your opponent then has 10 different plays at his disposal, but all lose. You next move can force him into (a) a 1, 2, 3 combination; or (b) two rows with four in each; or (c) one counter in each of three rows; or (d) a single counter.

The first player can win at Nim by taking one counter from any row and then playing to force (a) a 2, 4, 6 combination; or (b) a 1, 4, 5 combination; or (c) a 1, 2, 3 combination; or (d) two rows with the same number in each row.

MATHEMATICAL RECREATIONS

THE GAME OF NIM

The game Nim is played by two persons using three rows of counters, 3 in the first row, 5 in the second, and 7 in the third. Each player in turn takes one or more counters from just one row. The player who is forced to remove the last of the 15 counters loses the game.

For example, the first player might take all 7 counters in the third row, and the second might remove 2 from the second row. This leaves two rows of 3.

The second player can now win the game regardless of how many counters the first player removes if he thinks carefully.

Play the game with a classmate. Try to develop some strategies that will help you win the game.

MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. A $\{0, 1, 2, 3, 4, 5, 6, 7\}$; B $\{3, 4\}$ **2.** A 9; B 4 **3.** 6
4. $7 \cdot 10^5$ **5.** A 3 457 000; B 3 500 000 **6.** A $<$; B $>$; C $>$
7. $63_{(8)}$ **8.** A $14_{(8)}$; B $5_{(8)}$ **9.** A $43_{(8)}$; B $6_{(8)}$ **10.** 1564

UNIT A: THE ARITHMETIC OF WHOLE NUMBERS

Module 2: Whole Numbers—Operations and Computation

General Objectives

To review the whole number operations of addition, subtraction, multiplication, and division.

To increase the students' computational skills using these operations.

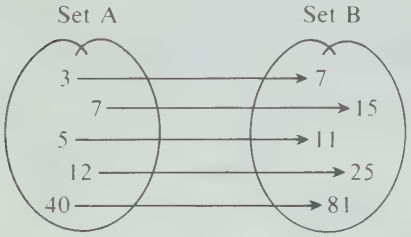
	Performance Objectives	Pupil Text	Reteach-Reinforce	Related Activities	
10	**RED** Given flow charts or function tables involving various operations, the student can find the output number.	A-30, 31 A-34, 35	**DM** 9		**SWM 1** 151
11	**RED** Given appropriate equations, the student can show a knowledge of the inverse relationship between addition and subtraction, and multiplication and division.	A-32, 33	**WB** 11	**PD** f-1	**SWM 1** 197
12	**RED** Given appropriate expressions, the student can demonstrate knowledge of the order of operations by evaluating the expressions correctly.	A-36, 37	**WB** 12 **DM** 10		**SWM 2** 247 **SWM 3** 58, 59
13	**RED** Given appropriate equations, the student can solve the equations.	A-36, 37	**WB** 11		
14	**RED** Given an addition problem, the student can find the sum.	A-42, 43 A-44, 45	**WB** 14 **DM** 11	**ASC** W-1, 2 **SP** a-1 **PD** a-1	**SWM 1** 66 **SWM 2** 194
15	**RED** Given a subtraction problem, the student can find the difference.	A-42, 43 A-44, 45	**WB** 14 **DM** 11	**ASC** W-3, 4 **SP** a-2 **PD** a-1	**SWM 1** 66
16	**RED** Given a multiplication problem, the student can find the product.	A-46 through 51	**WB** 15-17 **DM** 12-14	**ASC** W-5, 6 **SP** a-3 **PD** a-2, 3	**SWM 1** 195 **SWM 2** 36 **SWM 3** 28
17	**RED** Given a division problem, the student can find the quotient.	A-52, 53 A-54, 55	**WB** 18, 19 **DM** 15, 16	**ASC** W-7, 8, 9, 10 **SP** a-4 **PD** a-4	
18	**YELLOW** Given appropriate examples, the students can identify the basic principles for addition and multiplication of whole numbers.	A-38, 39 A-40, 41	**WB** 13		
19	**YELLOW** The student can solve equations by using the basic principles.	A-38, 39 A-40, 41	**WB** 13		
	Reviewing the Ideas	A-56	**WB** 20		

MATHEMATICS

An important concept introduced in this module and used throughout the text is the concept of *function*. Students who have studied from books of the *Investigating School Mathematics* series will be familiar with the "function machine," which is used to give the student semiconcrete experiences with functions.

The idea of a function involves two sets, A and B, and a rule of correspondence between the two sets, as shown by the example below.

Function Rule: Multiply by 2 and then add 1.



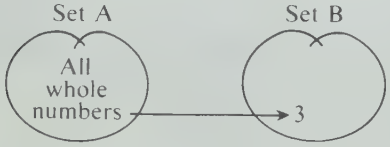
If we take a number from set A, say 7, and apply the rule, we get exactly one number in Set B. Hence, we get this set of pairs:

(3,7), (7,15), (5,11), (12,25), (40,81).

One vital feature of these pairs is that for each first number there is only one second number.

If x denotes any element of set A, then we often denote the corresponding number in B by the symbol $f(x)$. The number x, the number $f(x)$, and the rule of correspondence can then be related by the equation $f(x) = 2x + 1$.

Here is another example. Rule: Multiply by 0 and then add 3; that is $0x + 3$.



Some of the pairs of numbers are:

(17,3), (2,3), (1,3), (0,3), (285,3), . . .

Notice that, although the second number of each ordered pair is 3, it remains true that, given any first number, we get only one second number (in this case, 3).

In summary, we have a set and a rule for each function. When we apply the rule to an element of the set, we get just one answer. Thus, for each function, we have a set of ordered pairs, no two of which have the same first number.

TEACHING THE MODULE

Materials

Centimetre rulers, graph paper.

Vocabulary

addend	function
addition	identity
arithmetic mean	input
associative	inverse
average	multiplication
commutative	operation
difference	output
distributive	parentheses
division	product
equation	quotient
estimate	remainder
factor	subtraction
flow chart	sum

Students who have studied from the earlier books of *Investigating School Mathematics* will already be familiar with flow charts. The flow charts in this module are sufficiently simple that they should not be difficult for students.

The most important teaching suggestion for this module is to cover the elements that are essentially review at a rapid pace but to place special emphasis on constructing a firm bridge between the students' skill in use of the basic algorithms and their ability to bring these skills to bear in problem-solving situations.

Lesson Schedule

Although each lesson is intended to constitute about one day's work, you may find that some lessons will require slightly longer while others may be covered at the rate of two per day. Depending upon the abilities of your students you should allow from 10 to 12 days for adequate coverage of the module.

Evaluation

It is quite simple to evaluate the students' ability to compute sums, differences, products, and quotients. Beyond this, however, the individual teacher must decide the degree of emphasis that is to be placed on speed and accuracy in computation.

The review problems and the self-test at the end of the module will help you and your students assess their progress in attaining the module objectives.

RESOURCES FOR ACTIVE LEARNING

Activities

ESS: Mapping, Webster, McGraw-Hill

Experiments in Mathematics, Stage 3, pp. 54–57, Houghton Mifflin

From Fingers to Computers, Rand McNally

Madison Project: *Explorations*, Parts 4 & 5, Addison-Wesley

Making and Using Graphs and Nomographs (whole numbers), pp. 5–24, Rand McNally

Nuffield Project: *Number Patterns 1; Problems*-Red Set, Nos. 9, 9A, Wiley

Number Lines, Cuisenaire Co.

Sourcebook for Substitutes, pp. 73–79, 82–83, Addison-Wesley

Manipulative Devices

Calculators (Cuisenaire Co.; Selective Educational Equipment)

Madison Project: Independent Exploration Kits (Mind/Matter)

Napier's Bones (Rods) (Creative Publications; Midwest Publications)

Slide rules (Creative Publications; Geyer; Pickett)

Games and Puzzles

Base Check (Ideal; Sigma)

Equations (SRA; Wff'N Proof)

Foo (Cuisenaire Co.)

In-order Game Kit (Midwest Publications)

Krypto (Creative Publications)

Math Match (Creative Publications)

Numo (Gamco; Midwest Publications)

Orbiting the Earth (Scott Foresman)

Quinto (Sigma)

Radix (Lang)

Ranko (Midwest Publications)

Real Numbers Game (Wff'N Proof)

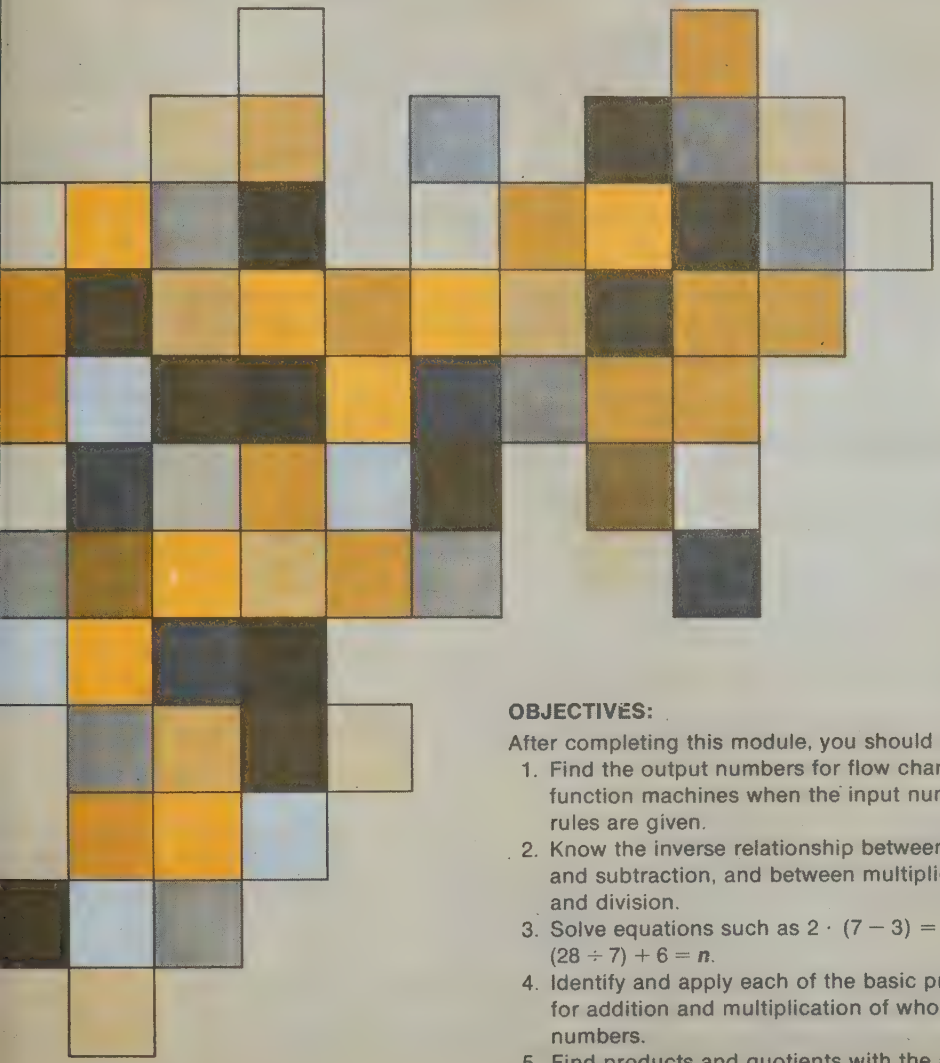
Tower Puzzle (Creative Publications)

TUF (Cuisenaire Co.; TUF)

Winning Touch (Ideal; Sigma)

UNIT A: *The Arithmetic of Whole Numbers*

MODULE 2: *Whole Numbers—Operations and Computation*



OBJECTIVES:

After completing this module, you should be able to:

1. Find the output numbers for flow charts and function machines when the input numbers and rules are given.
2. Know the inverse relationship between addition and subtraction, and between multiplication and division.
3. Solve equations such as $2 \cdot (7 - 3) = n$ or $(28 \div 7) + 6 = n$.
4. Identify and apply each of the basic principles for addition and multiplication of whole numbers.
5. Find products and quotients with the aid of multiples.
6. Add, subtract, multiply, and divide whole numbers.

Objective

Given a flow chart with an input number and various instructions, the student can find the output number.

PREPARATION

Materials: Overhead transparencies or chalkboard drawings of the Investigation flow charts and blank flow charts.

To introduce this lesson, ask students to do some simple mental computing such as: "Start with 3, add 4, multiply by 2, subtract 1. What is the result?" This will aid students in thinking about the step-by-step instructions in the flow charts in this lesson and the next.

INVESTIGATION

Flow charts are devices which list steps to be performed in a prescribed sequence. Though the most notable application of flow charts is in the writing of computer programs, it is not the purpose of this or subsequent lessons to make computer programmers of the students. Rather, the purpose of the flow charts in this Investigation is simply to give students practice in following instructions in flow chart form.

As students begin the Investigation activity, suggest that they use the whole numbers from 1 to 10 as input numbers and make a table showing each input number and the corresponding output. The students thus should find that when the input number is 9 both flow charts will have the same output—50. Be certain that students understand that, though the input numbers can vary, the instructions remain the same.

DISCUSSION

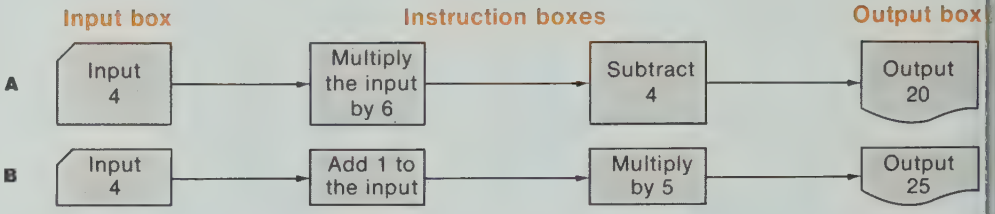
The main purpose of using flow charts is to emphasize to the students the need to organize their thinking in problem solving into a sequence of logical steps.

The student is introduced to only four different shapes in the diagrams—information or input boxes, instruction boxes, question boxes, and output or answer boxes. The shapes used in the text are like those frequently used in many standard works on computer programming. Arrows connecting the boxes indicate the "flow" of the instructions.

Flow Charts

Investigating the Ideas

A **flow chart** is a diagram that gives instructions arranged in a logical order. Study flow charts A and B below.



Choose other input numbers for each flow chart and then find corresponding outputs.

Can you find an input number for both flow charts that will give the same number as an output?

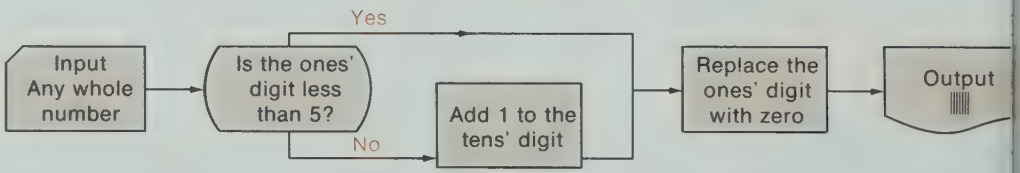
Discussing the Ideas

1. The flow chart below is a simplified version of flow chart A.

   A What does the instruction "·6" tell you to do?
   B What does the instruction "−4" mean?

2. How would you simplify flow chart B?

3. Some flow charts have **question boxes**. Flow chart questions must be answered with either "yes" or "no."



   A Explain how the flow chart works if the input number is 243. What is the output number?
   B Explain how the flow chart works if the input number is 517. What is the output number?
   C Describe in your own words what the flow chart does to each input number.

Using the Ideas

1. Find the output number ***n*** in each flow chart.
Part A is worked as an example.

A 6 → · 9 → + 21 → ***n*** ANSWER: $n = 75$

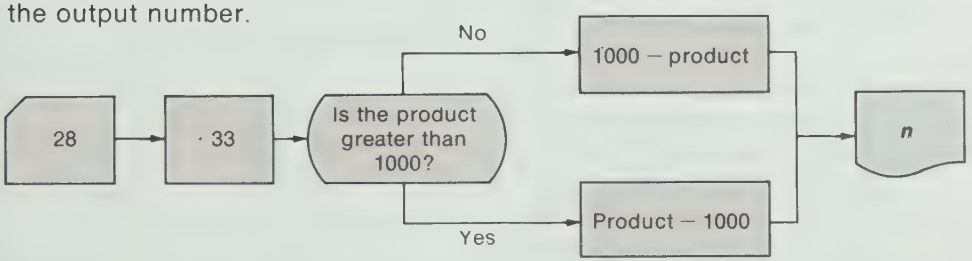
B 32 → ÷ 8 → · 16 → − 5 → ***n***

C 10 → · 10 → · 10 → · 10 → ***n***

D 47 → · 7 → − 1 → ÷ 8 → ***n***

E 101 → − 1 → ÷ 10 → ÷ 2 → ***n***

2. Find the output number.

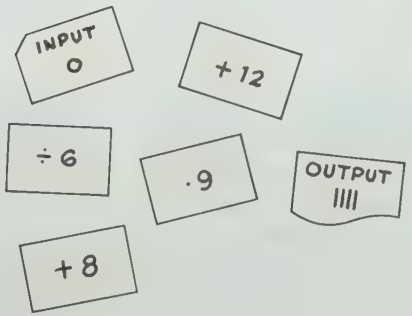


★**3.** Arrange these cards into flow charts so that they will have the following outputs.

A 14 **D** 10

B 26 **E** 24

C 30 **F** 180



UTILIZATION

Exercise 1 provides practice in computing output numbers given the input numbers and two or more instructions. Exercise 2 involves the use of a question box in a flow chart.

Exercise 3 is intended as a special challenge for the students.

Solutions, Exercise 3

A. 0 → + 8 → · 9 → + 12 → ÷ 6 → 14

B. 0 → + 12 → ÷ 6 → · 9 → + 8 → 26

C. 0 → + 12 → + 8 → · 9 → ÷ 6 → 30

D. 0 → · 9 → + 12 → ÷ 6 → + 8 → 10

E. 0 → + 8 → · 9 → ÷ 6 → + 12 → 24

F. 0 → ÷ 6 → + 12 → + 8 → · 9 → 180

EXTENSION

Remedial: Supply additional flow charts like those in Exercise 1 for additional practice.

Enrichment: Challenge the students to find as many different outcome numbers as they can by using different arrangements of the instructions given in Exercise 3.

Suggest that students design their own flow charts. Do not place any restrictions on the complexity, but require that each student be able to use and explain his own flow chart.

Assignments

Minimum 1, 2. Average 1, 2. Maximum 1–3.

Objective

The student can solve simple equations involving one operation.

PREPARATION

Review the use of flow charts. Have students mentally compute output numbers for some simple flow charts you might write on the chalkboard or overhead projector.

INVESTIGATION

The Investigation will require only a brief amount of time. The purpose of the Investigation is to introduce the idea of inverse operations to the student. The inverse relationship between addition and subtraction and between multiplication and division is basic to the understanding of the program of this series.

DISCUSSION

The main emphasis of the discussion should be on Exercises 2 and 3. Subtraction is introduced in terms of addition. A formal definition of subtraction is as follows:

If a, b, and c are whole numbers and $a + b = c$, then $b = c - a$.

Thus b, the difference of two whole numbers, is an addend in the addition statement. If it is emphasized now that every subtraction problem involves finding a missing addend, the student will be better prepared for later work with subtraction involving integers.

Similarly, emphasis should be placed on the inverse relationship between multiplication and division. Formally:

If a, b, and c are whole numbers and if $a \cdot b = c$ and $a \neq 0$, then $b = c \div a$.

The quotient in a division problem is a factor in a multiplication statement. This relationship is a concept vital to further work in the text.

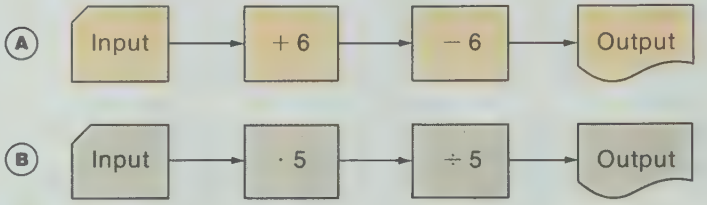
In Exercise 2, point out the meaning of the letters **A, A,** and **S,** which represent Addend, Addend, and Sum.

In Exercise 3, **F, F,** and **P** represent Factor, Factor, and Product.

The Basic Operations

Investigating the Ideas

Study flow charts A and B below. Try various input numbers for each flow chart.



What do you notice about the output numbers in the two flow charts?

Discussing the Ideas

1. Adding and subtracting the same number are opposite in effect and are called **inverse operations**. Are the operations multiplying and dividing by the same number inverse operations?

2. The table shows how you can think of subtraction as finding a missing addend. In each equation, tell whether the missing number represents an addend or a sum. Then give the missing number.

   A $7 + 3 = \square$ **C** $7 - 3 = \square$

   B $6 + \square = 10$ **D** $8 - \square = 3$

When you find this addend,	A A S $n + 6 = 9$
you have found this difference.	S A A $9 - 6 = n$

3. Study the table at the right Tell whether the missing number in each equation is a factor or a product. Then give the missing number.

   A $9 \cdot \square = 36$ **C** $24 \div \square = 4$

   B $5 \cdot 7 = \square$ **D** $\square \div 8 = 7$

When you find this factor,	F F P $n \cdot 4 = 28$
you have found this quotient.	P F F $28 \div 4 = n$

Using the Ideas

1. Solve the equations. EXAMPLE: $6 + 5 = a$, $a = 11$

A $4 + 5 = a$	D $8 + 4 = m$	G $y + 0 = 8$	J $6 + 6 = n$
B $9 + n = 10$	E $d + 9 = 17$	H $6 + 9 = q$	K $9 + d = 12$
C $b + 6 = 15$	F $0 + r = 13$	I $5 + f = 5$	L $r + 3 = 11$

2. Think about finding the missing addend. Then solve each subtraction equation.

A $? + 6 = 13$ $13 - 6 = r$	B $? + 5 = 11$ $11 - 5 = a$	C $? + 7 = 16$ $16 - 7 = n$
D $12 - 4 = x$	E $8 - 0 = c$	F $14 - 7 = t$
G $10 - 9 = q$	H $7 - 1 = m$	I $16 - 9 = s$

3. Solve the equations.

A $2 \cdot 6 = a$	D $c \cdot 4 = 28$	G $7 \cdot y = 21$	J $t \cdot 4 = 36$
B $8 \cdot n = 40$	E $6 \cdot d = 36$	H $3 \cdot 5 = r$	K $8 \cdot 1 = p$
C $9 \cdot 3 = b$	F $x \cdot 8 = 56$	I $8 \cdot s = 16$	L $7 \cdot m = 49$

4. Think about finding the missing factor. Then solve each division equation.

A $? \cdot 4 = 24$ $24 \div 4 = n$	B $? \cdot 7 = 28$ $28 \div 7 = a$	C $? \cdot 5 = 30$ $30 \div 5 = r$
D $3 \div 3 = t$	E $42 \div 6 = q$	F $40 \div 8 = s$
G $18 \div 2 = h$	H $0 \div 9 = b$	I $7 \div 7 = d$

5. Find the output number for the flow chart if the missing operation in the instruction box is

A $+$ B $-$
C $\cdot$ D $\div$

UTILIZATION

The exercises involve only one operation in each equation and students need only write the solution to the equations. Since students may complete Exercises 1–4 rather quickly, plan to devote some time to discussing the exercises.

For example, in Exercise 1, ask students to decide whether the variable in each equation stands for an addend or a sum. In Exercise 3, ask whether the variable represents a factor or a product. Note the use of the "think cloud" in Exercises 2 and 4 to help students think about missing addends and missing factors.

EXTENSION

Additional practice exercises are provided on *Workbook* page 11.

Remedial: Provide additional equations like those in Exercises 1–4 for students to solve.

Enrichment: Have students complete the sum–difference pairs and product–quotient pairs in the tables shown below.

Sums — Differences

A.	9	a	7	—	2	b	5
B.	12	a	9	—	3	b	6
C.	14	a	8	—	6	b	2
D.	14	a	7	—	7	b	0
E.	17	a	12	—	5	b	7

Products — Quotients

A.	18	a	6	$\div$	3	b	2
B.	12	a	6	$\div$	2	b	3
C.	27	a	9	$\div$	3	b	3
D.	32	a	8	$\div$	4	b	2
E.	20	a	10	$\div$	2	b	5

Assignments

Minimum 1A–F, 2A–F, 3A–F, 4A–F.
Average 1–4. Maximum 1–5.

Objectives

Given an input number and a function rule, the student can give the output number.

Given a set of input-output numbers related by a simple unstated function rule such as "Add 4" or "Subtract 6," the student can state the function rule.

PREPARATION

Review some simple flow charts in which the instruction consists of one operation.

DISCUSSION

This section begins by introducing the basic idea of how the function machine is used with respect to input and output numbers. Functional notation is also introduced. The work with functions at this time plays an important role as background preparation for later work with graphing. The lesson also provides practice with the basic facts.

It is vital for students to understand how the function machine operates; that is, they should realize that each input number and function rule determine a unique output number. Provide several examples of simple function rules to see whether students can correctly obtain the output numbers.

Try to make sure that students are not daunted by the $f(n)$ notation. If they simply think of $f(n)$ as a symbol for the output number, it should not cause any real problem. This is standard function notation that the student will encounter again and again in later work with mathematics.

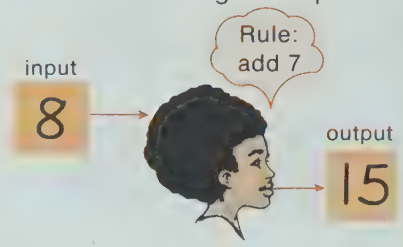
A great deal of enthusiasm can be generated by playing the guess-the-function-rule game. For example, when student A says 3, student B says 7; when A says 5, B says 11; when A says 4, B says 9; when A says 7, B says 15; when A says 2, B says 5; when A says 0, B says 1. Other members of the class are to guess the function rule—in this case, $2 \cdot n + 1$.

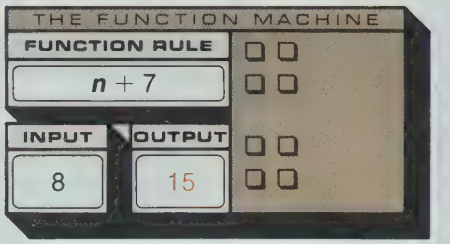
Playing this game before working the exercises will provide background for successful experience with this lesson.

The Function Machine and Operations

Discussing the Ideas

The picture below can help you understand the **function machine.** The function rule on the function machine tells how to find the output number for a given input number.





For the example above, you can describe the function rule as

output number = input number + 7.

If n denotes the input number and $f(n)$ denotes the output number, we can write $f(n) = n + 7$.

For the input number 8, we write $f(8) = 15$ and we read "f of 8 equals 15."

1. Using the function rule above, tell what number each of these outputs represents.

   A $f(4)$ B $f(9)$ C $f(6)$ D $f(0)$ E $f(10)$

2. The input and output numbers for a function rule can be arranged in a **function table.** Give the missing numbers for the table.

Function Rule: $n + 7$

	Input n	Output $f(n)$
A	4	
B	3	
C	7	
D	0	
E	5	

3. Suppose the function rule was $7 - n$. How would you complete the function table for Exercise 2?

4. Explain why the function tables for rules A and B would be the same.

   A $f(n) = 3 \cdot n$ B $f(n) = n + n + n$

5. In Exercise 4 above, you can see how repeatedly adding the same number is an operation related to multiplication. Explain how subtraction and division are related in a similar way.

Using the Ideas

1. Using the function rule $f(n) = n \cdot 9$, give the output number for each of the following input numbers.

A 5 **B** 3 **C** 7 **D** 1 **E** 10 **F** 2 **G** 0 **H** 100

In Exercises 2 through 4, give the missing numbers for each table.

2. Function Rule $n \cdot 4$

	Input n	Output $f(n)$
	2	8
	3	12
A	4	
B	1	
C	6	

3. Function Rule $n + 4$

	Input n	Output $f(n)$
	5	9
A	3	
B	0	
C	6	
D	9	

4. Function Rule $n - 3$

	Input n	Output $f(n)$
A	24	
B	6	
C		3
D		20
E	15	

For Exercises 5 through 8, study each table and invent a suitable function rule.

5.

input n	6	1	4	0	10
output $f(n)$	10	5	8	4	14

6.

input n	3	5	0	8	10
output $f(n)$	6	10	0	16	20

7.

input n	28	70	21	42	7
output $f(n)$	4	10	3	6	1

★8.

input n	2	5	1	7	4
output $f(n)$	6	30	2	56	20

think

Copy the triangular array of numbers and try to give the missing numbers. Find the sum of the numbers in each row. Do you see a pattern? Now find the next row.

Sums

1 → 1
1 1 → 2
1 2 1 → 4
1 3 3 1 → ▨
1 4 6 4 1 → ▨
1 5 10 ▨ ▨ 1 → ▨
1 6 ▨ ▨ ▨ ▨ 1 → ▨

UTILIZATION

Exercises 1, 2, and 3 provide practice in computing output numbers, $f(n)$, for given input numbers and function rules. Exercise 4 is slightly harder because in parts C and D the students must think about inverse operations in order to compute the input number when the output number and function rule are given.

Exercises 5 through 8 extend the work with functions and require that the student find the function rule when both the input and the output numbers are given. The guess-the-function-rule game suggested in the Discussion on the previous page will help students be successful with this type of problem. Exercise 8 should be assigned as a challenge problem.

EXTENSION

Page 9 of the *Duplicator Masters* offers additional appropriate practice exercises.

Remedial: Have students compute output numbers for the following function rules, using input numbers from the set {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}.

1. $f(n) = n + 8$
2. $f(n) = 20 - n$
3. $f(n) = 7 \cdot n$
4. $f(n) = n + n$
5. $f(n) = n$
6. $f(n) = 8 \cdot n$

Enrichment: Have the students construct some function tables like those in Exercises 5 through 8, and ask other students to discover the function rule for each table.

Think Solution

The array shown is often referred to as "Pascal's triangle." Each number is obtained by adding the two numbers to the right and to the left in the preceding line. Thus, in rows 6 and 7, we see this pattern:

5 → 15 ← 10 → 20 ← 10

The patterns and significance of the triangle are many and varied. For example, the sum of each row is found to be a power of 2. The power of 2 for the n^{th} row would be 2^{n-1}.

Assignments

Minimum 1–4. Average 1–7. Maximum 1–8.

Objective

The student will be able to perform operations with numbers grouped with parentheses.

PREPARATION

It would be reasonable to begin work immediately with the Investigation. However, you may prefer to introduce the lesson with some expressions like

$12 - 6 - 5$; $7 \cdot 5 - 3$; and $16 + 4 \div 2$.

Ask the students what numbers each expression can represent. Since several responses may be given for each expression, encourage students to try to find a way to resolve the ambiguity. Then place parentheses in two ways for each expression:

$12 - (6 - 5)$ and $(12 - 6) - 5$

$7 \cdot (5 - 3)$ and $(7 \cdot 5) - 3$

$(16 + 4) \div 2$ and $16 + (4 \div 2)$

Explain that the parentheses signify that the enclosed computation must be performed first. Since different numbers result from different groupings, students will gain a deeper appreciation of the importance of the parentheses and their placement.

INVESTIGATION

Allow enough time for the Investigation to enable almost all of the students to complete it. Have students display their groupings.

DISCUSSION

The main objective of this lesson is to have students gain familiarity with parentheses in expressions and equations. The concept of associativity is not a point of the lesson and will be treated separately in the following lesson, although Exercises 2 and 3 indicate that, for the single operation of either addition or multiplication, placement of parentheses is of no significance.

Exercise 4 introduces an agreement concerning this basic idea: We will often not use parentheses in expressions and equations where the combination of multiplication and either addition or subtraction are involved. Therefore, we agree to multiply adjoining pairs of numbers first. This allows us to write equations like $2 \cdot n + 3 = 11$ or $7 \cdot x - 3 = 4$ without parentheses, yet free from ambiguity.

Using Parentheses with Operations

Investigating the Ideas

When more than one operation is used in the same problem, you can use **parentheses** to show which operations to perform first.

EXAMPLES: $(8 + 3) \cdot 2 = 22$

$8 + (3 \cdot 2) = 14$

Can you place parentheses in the equations so that each equation will be a true statement?

1. $16 - 7 + 8 = 1$
2. $16 - 7 + 8 = 17$
3. $12 \div 2 + 4 = 2$
4. $12 \div 2 + 4 = 10$
5. $3 \cdot 5 - 4 = 11$
6. $3 \cdot 5 - 4 = 3$
7. $18 \div 6 \cdot 3 = 1$
8. $18 \div 6 \cdot 3 = 9$

Discussing the Ideas

1. Show how the expression $9 - 3 + 2$ can be made to represent two different numbers depending upon how parentheses are placed in the expression.

2. Can you place parentheses in the expression $8 + 5 + 7$ so that it will represent two different numbers?

3. Can you place parentheses in the expression $2 \cdot 3 \cdot 4$ so that it will represent two different numbers?

4. When a combination of **multiplication and addition** or of **multiplication and subtraction** is used in an expression without parentheses, we agree to **multiply adjoining numbers first** and then do the adding or subtracting.

   For example, $3 \cdot 7 + 4 = 25$ and $6 + 5 \cdot 2 = 16$.

   Tell what number each of these expressions represents.

   A $9 \cdot 2 + 3$ C $18 - 3 \cdot 5$ E $3 \cdot 5 + 2 \cdot 6$

   B $7 \cdot 3 - 10$ D $12 + 4 \cdot 3$ F $4 \cdot 2 \cdot 3 - 5$

Using the Ideas

1. Tell what number each expression represents. Perform the operation in the parentheses first.

A $12 + (3 \cdot 4)$	C $(16 \div 4) + 2$	E $20 - (8 \div 4)$
B $8 - (5 + 2)$	D $(6 \cdot 5) - 15$	F $(20 - 8) \div 4$

2. The examples illustrate how to use parentheses in equations. Solve the equations.

Ⓐ $(9 - 3) + 2 = n$

$6 + 2 = n,\ n = 8$

Ⓑ $16 \div (6 + 2) = m$

$16 \div 8 = m,\ m = 2$

A $(14 - 9) + 7 = x$	G $(3 \cdot 6) \div 2 = n$	M $6 \cdot (6 \div 6) = u$
B $9 + (10 - 2) = n$	H $5 + (8 - 3) = a$	N $6 \div (6 \div 6) = v$
C $(12 + 4) - 9 = a$	I $4 \cdot (7 + 2) = c$	O $(6 \cdot 6) \div 6 = w$
D $16 - (8 + 1) = d$	J $(21 \div 3) + 6 = f$	P $6 + (6 \div 6) = x$
E $10 + (6 - 2) = q$	K $18 - 6 \cdot 3 = b$	Q $(6 + 6) \div 6 = y$
F $(17 - 8) + 4 = h$	L $(24 \div 6) \cdot 8 = m$	R $6 - (6 \div 6) = z$

3. Solve the equations.

A $7 \cdot 4 + 5 = r$	C $8 \cdot 3 - 7 \cdot 2 = x$	E $9 \cdot 8 - 9 \cdot 7 + 9 \cdot 6 = n$
B $12 - 2 \cdot 4 = t$	D $2 \cdot 3 \cdot 5 + 1 = y$	F $(10 \cdot 9 - 14) \div 2 = s$

4. Give the missing numbers in each function table.

	Function Rule $n \cdot 3 - 2$	
	n	$f(n)$
A	2	
B	5	
C	8	
D	10	
E	7	

	Function Rule $20 - 2 \cdot n$	
	n	$f(n)$
F	1	
G	3	
H	6	
I	7	
J	10	

	Function Rule $(n \div 3) - 1$	
	n	$f(n)$
K	15	
L	9	
M	21	
N	30	
O	18	

★ **5.** Solve the equations.

A $2 \cdot x + 3 = 13$	C $7 + 2 \cdot n = 27$	E $2 \cdot b + 2 \cdot b = 20$
B $3 \cdot n + 4 = 10$	D $10 - 5 \cdot a = 0$	F $3 \cdot d + 2 \cdot d = 35$

More practice, page S-5, Set 7

UTILIZATION

Exercises 1 and 2 stress work with parentheses. In Exercise 3 the student must use the agreement stated in Discussion Exercise 4.

Exercise 5, which is designed primarily for the more able students, presents a variety of equations to be solved. You are urged not to attempt to teach any specific techniques for solving these equations. The equations are included primarily to help students think about the operations in order to discover the solutions. Encourage students to think about missing addends and factors. Thus, for Equation 5A, the student can reason:

"Since $2 \cdot x$ adds to 3 to give 13, I know $2 \cdot x$ must be 10. Then x must equal 5."

EXTENSION

Supplementary Exercise Set 7 on page S-5, *Workbook* page 12, and *Duplicator Masters* page 10 provide exercises appropriate for students who would benefit from further practice with the concepts of this lesson.

Remedial: Provide additional equations for more practice.

1. $(2 \cdot 7) - 3 = x$
2. $(36 - 21) - 5 = n$
3. $15 - (8 + 3) = y$
4. $(6 \cdot 8) \div 4 = s$
5. $21 + (12 - 9) = t$
6. $2 \cdot 3 + 4 = k$
7. $3 \cdot 9 - 16 = l$
8. $(9 \div 3) \div 3 = z$
9. $9 \div (3 \div 3) = p$
10. $2 \cdot 7 + 2 \cdot 3 = m$

Enrichment: Have the students create equations like the ones in the Investigation. Then ask them to exchange equations and insert parentheses in the correct positions.

Assignments

Minimum 1, 2A–F, 3A–C. Average 1–4. Maximum 1–5.

Objectives

The student can identify and use the associative and commutative principles for addition and multiplication, and can identify and use the zero and one principles.

PREPARATION

No preparation is needed before beginning the Discussion.

DISCUSSION

Although the associative and commutative principles are important in their own right, the most important idea to be brought out is the fact that the commutative and associative principles together allow us to rearrange addends (or factors) in any way that is convenient. In actual practice this rearranging of factors or addends is the most common use for these two principles. (See Discussion Exercise 3.)

Although the zero and one principles are quite familiar to most students, they should be emphasized at this time. Of particular importance in this lesson is the fact that there is no such operation as division by zero. Thus in Exercise 6A if n is any whole number, then $0 \div n = 0$ because $0 \cdot n = 0$ for every whole number n. However, in Exercise 6B, we cannot divide any number by zero. For example, consider the equation $7 \div 0 = x$. Then $x \cdot 0 = 7$. But there is no number x which multiplied by 0 gives 7.

Also consider the trivial case of $0 \div 0 = x$. Then $x \cdot 0 = 0$ *for every whole number* x. Thus, for a nonzero dividend, we can never find a quotient when dividing by zero. For the zero dividend, every number could be a quotient if a zero divisor were permitted. For these reasons, we agree never to divide by zero.

This lesson plays an important mathematical role in that the commutative principle, the associative principle, and the properties of zero and one are seen to be a part of the basic structure of the whole number system.

The material should prove quite easy for most students and should be covered rapidly.

Basic Principles for Addition and Multiplication

Discussing the Ideas

1. Study the statements concerning the commutative principles.
   - A Can you tell, in your own words, what the commutative principle means?
   - B Illustrate the commutative principle for addition when $a = 12$ and $b = 34$.

Commutative ( ) principles

If a and b are any whole numbers, then

$a + b = b + a$ and $a \cdot b = b \cdot a$

   - C Illustrate the commutative principle for multiplication when $a = 9$ and $b = 12$.

2. Study the associative principles.
   - A Is $289 + (746 + 868)$ the same sum as $(289 + 746) + 868$?
   - B Illustrate the associative principle for multiplication using $a = 5$, $b = 4$, and $c = 2$.

Associative ( ) principles

If a, b and c are any whole numbers, then

$a + (b + c) = (a + b) + c$ and

$a \cdot (b \cdot c) = (a \cdot b) \cdot c$

3. Together, the commutative and associative principles assure us that **we can arrange three or more addends or factors in any convenient way and still get the same sum or product.**

   EXAMPLES: $(7 + 9) + 3 = (7 + 3) + 9$ $\quad 4 \cdot (25 \cdot 9) = (25 \cdot 4) \cdot 9$

   - A Explain how the commutative and associative principles for multiplication can help you quickly find the product $5 \cdot 39 \cdot 20$.

4. Study the zero and one principles. Use the inverse relation between addition and subtraction and the **zero principle** to explain why

   $a - 0 = a$ and $a - a = 0$

The zero principle

Zero is the **identity element** for addition. This means that for each whole number a,

$a + 0 = a$

5. Use the inverse relation between multiplication and division and the **one principle** to explain why

   $a \div 1 = a$ and $a \div a = 1$

The one principle

The number 1 is the **identity element** for multiplication. This means that for each whole number a, $a \cdot 1 = a$

6. Use the inverse relation between multiplication and division to explain the following statements:
   - (A) We can divide zero by every whole number except zero.
   - (B) We never divide any number by zero.

Using the Ideas

1. Tell whether each equation illustrates the commutative principle, the associative principle, or both principles.

A $8 + 7 = 7 + 8$

B $5 + (6 + 2) = (6 + 2) + 5$

C $(4 + 3) + 1 = 4 + (3 + 1)$

D $8 \cdot 7 = 7 \cdot 8$

E $5 \cdot (6 \cdot 2) = (6 \cdot 2) \cdot 5$

F $(3 \cdot 4) + 6 = 6 + (3 \cdot 4)$

2. Solve the equations. Think of the basic principles to help you solve them quickly.

A $37 + 26 = n + 37$

B $51 \cdot 24 = 24 \cdot a$

C $(56 + 2) + 13 = 56 + (d + 13)$

D $8 \cdot (7 \cdot 6) = (b \cdot 7) \cdot 6$

E $(4 \cdot 3264) \cdot 7 = 4 \cdot (s \cdot 7)$

F $6 + (156 + 38) = (n + 156) + 38$

3. Use the commutative and associative principles to find each sum or product.

A $99 + (47 + 1)$

B $96 + (79 + 4)$

C $999 + (356 + 1)$

D $(200 + 653) + 800$

E $5 \cdot (13 \cdot 2)$

F $(25 \cdot 41) \cdot 4$

G $25 \cdot (37 \cdot 4)$

H $(50 \cdot 97) \cdot 2$

I $(125 \cdot 93) \cdot 8$

4. Solve the equations.

A $67 + x = 67$

B $92 - x = 0$

C $74 - x = 74$

D $67 \cdot a = 67$

E $92 \div a = 1$

F $74 \div a = 74$

G $87 + (x - 12) = 87$

H $87 \cdot (a \div 12) = 87$

I $96 - (x + 96) = 0$

5. Both addition and multiplication are associative. Do you think that subtraction is associative? That is, is it true that if the subtractions can be performed for all numbers a, b, and c, then $a - (b - c) = (a - b) - c$?

Answer the same question for division. Is it true that for all numbers a, b, and c, then $a \div (b \div c) = (a \div b) \div c$?

6. Which of the expressions do not represent numbers?

A $0 \div 6$ **B** $6 \div 0$ **C** $0 \div 0$

★7. Suppose two students both choose a whole number. What can you say about the numbers if

A their product is zero?

B their product is 1?

think

In numbering the pages of a book 495 digits were used.

How many pages were there?
How many 9's were used?

UTILIZATION

Exercises 1 through 4 provide practice in identification and application of the basic principles.

Solution, Exercise 5

In order to disprove a general principle, only one counterexample is necessary. Thus, for the question regarding subtraction, pick any three numbers for a, b, and c (say, 6, 3, and 2) and show

$$6 - (3 - 2) \stackrel{?}{=} (6 - 3) - 2$$
$$6 - 1 \stackrel{?}{=} 3 - 2$$
$$5 \neq 1$$

To answer the question for division, consider the numbers 24, 6, and 2.

$$24 \div (6 \div 2) \stackrel{?}{=} (24 \div 6) \div 2$$
$$24 \div 3 \stackrel{?}{=} 4 \div 2$$
$$8 \neq 2$$

Solution, Exercise 7

If P = 0, at least one number is 0.
If P = 1, both numbers are 1.

Think Solution

There are 9 digits for the pages 1 through 9. There are 180 digits for the two-digit numeral pages from 10 through 99. For the three-digit numerals from pages 100 through 199, there are 300 digits. For pages 1 through 199, therefore, we have $9 + 180 + 300 = 489$ digits. Thus, we need 6 more digits, and these are obtained from pages 200 and 201.

The number of nines used in the digits for 201 pages is as follows: pages 1–9, 1 nine; pages 10–89, 8 nines; pages 90–99, 11 nines; pages 100–189, 9 nines; pages 190–199, 11 nines; pages 200–201, 0 nines; or a total of 40 nines.

Assignments

Minimum 1–3, 4A–F. Average 1–6.
Maximum 1–7.

Objective

The student can demonstrate an understanding of the distributive principle of multiplication with respect to addition in evaluating expressions and solving equations.

PREPARATION

Plan a brief review of flow charts or play the guess-the-function-rule game, previously described in the commentary for page A-34 of this module.

INVESTIGATION

The Investigation involves the use of the distributive principle; however, the principle itself is not specifically mentioned in the Investigation stage. Students are expected to discover the fact that regardless of the three input numbers selected for the two flow charts, the output numbers will be the same.

There is no need to emphasize early verbalization of the distributive principle. Many students who can use it correctly may have some difficulty in precisely describing it.

DISCUSSION

Exercise 1 formalizes the results of the Investigation and states the distributive principle algebraically.

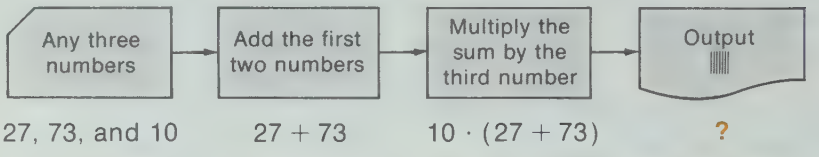
Exercise 2 uses sets to present a more concrete example of the distributive principle. Note that the principle is presented as "breaking apart" one of the two factors in a multiplication problem. It might be helpful for students to use graph paper to demonstrate this "breaking apart" idea.

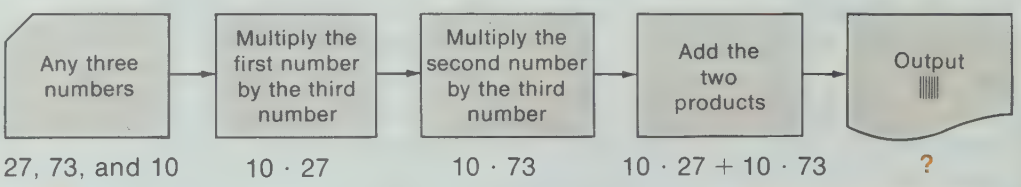
Exercise 3 should be covered carefully in order to prepare students for the problems on the following page.

The Distributive Principle

Investigating the Ideas

Study the flow charts and then find the output for each one.




Any three numbers
Multiply the first number by the third number
Multiply the second number by the third number
Add the two products
Output
27, 73, and 10
10 · 27
10 · 73
10 · 27 + 10 · 73
?

Choose three more input numbers. Then find the output.

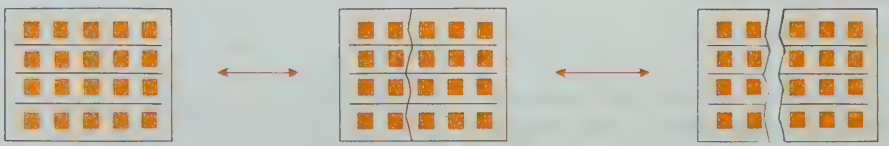
What can you say about the two output numbers in the flow charts above for any three numbers?

Discussing the Ideas

1. The flow charts illustrate the **distributive principle for multiplication with respect to addition**.
The flow charts show that
$10 \cdot (27 + 73) = (10 \cdot 27) + (10 \cdot 73)$.
Illustrate the distributive principle using $a = 20$, $b = 34$, and $c = 9$.

Distributive principle
If a, b, and c are any whole numbers, then
$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$

2. The sets below may help you to better understand the distributive principle. Describe and explain each display and then give the number for n.



3. Explain how to find the number for n in each equation.
   A $5 \cdot 6 + 5 \cdot 3 = 5(6 + 3) = 5 \cdot n$
   B $3 \cdot 7 + 3 \cdot 6 = 3(n + 6) = 3 \cdot 13$

Using the Ideas

1. Complete each equation in the left column by choosing the correct expression from the right column.

A	$8 \cdot (7 + 5) =$ ▒▒	A	$7 \cdot (8 + 5)$
B	$(6 \cdot 2) + (9 \cdot 2) =$ ▒▒	B	$(9 \cdot 2) + (9 \cdot 6)$
C	$(7 \cdot 8) + (7 \cdot 5) =$ ▒▒	C	$(8 \cdot 7) + (8 \cdot 5)$
D	$5 \cdot (7 + 8) =$ ▒▒	D	$(6 + 9) \cdot 2$
E	$9 \cdot (2 + 6) =$ ▒▒	E	$(9 + 2) \cdot 6$
F	$(9 \cdot 6) + (2 \cdot 6) =$ ▒▒	F	$(5 \cdot 7) + (5 \cdot 8)$

2. Solve the equations.

A $5 \cdot 8 = 5 \cdot (6 + a) \rightarrow 5 \cdot 8 = (5 \cdot 6) + (5 \cdot a)$

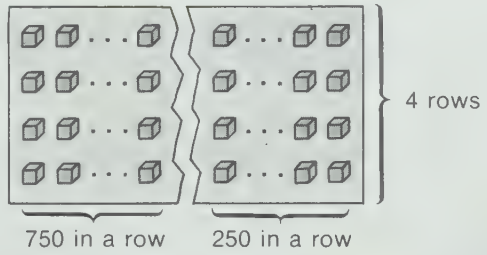
B $6 \cdot 4 = 6 \cdot (n + 1) \rightarrow 6 \cdot 4 = (6 \cdot n) + (6 \cdot 1)$

C $4 \cdot 9 = 4 \cdot (3 + b) \rightarrow 4 \cdot 9 = (4 \cdot 3) + (4 \cdot b)$

3. Solve the equations.

A	$(6 \cdot 5) + (6 \cdot 1) = 6 \cdot a$	F	$(4 \cdot 70) + (4 \cdot 8) = 4 \cdot r$
B	$(5 \cdot 3) + (5 \cdot 4) = 5 \cdot m$	G	$(7 \cdot 100) + (7 \cdot 50) = 7 \cdot a$
C	$(8 \cdot 10) + (8 \cdot 3) = 8 \cdot n$	H	$(6 \cdot 600) + (6 \cdot 10) + (6 \cdot 9) = 6 \cdot c$
D	$(5 \cdot 10) + (5 \cdot 7) = 5 \cdot s$	I	$(8 \cdot 400) + (8 \cdot 30) + (8 \cdot 8) = 8 \cdot m$
E	$(6 \cdot 30) + (6 \cdot 5) = 6 \cdot y$	J	$(5 \cdot 900) + (5 \cdot 90) + (5 \cdot 9) = 5 \cdot q$

4. The small black dots are to indicate that many cubes have been left out. Give the total number of cubes, without pencil-and-paper computation.



★**5.** Copy and complete the equations.

A	$a \cdot (b + c) = (a \cdot b) +$ ▒▒	E	$r \cdot (a +$ ▒▒$) = (r \cdot a) + (r \cdot y)$
B	$s \cdot (r + t) =$ ▒▒ $+ (s \cdot t)$	F	$f \cdot (g + h) =$ ▒▒ $+$ ▒▒
C	▒▒ $\cdot (c + d) = (n \cdot c) + (n \cdot d)$	G	▒▒ $= (a \cdot n) + (a \cdot m)$
D	$b \cdot ($▒▒$ + n) = (b \cdot m) + (b \cdot n)$	H	▒▒ $= (b \cdot s) + (b \cdot s)$

UTILIZATION

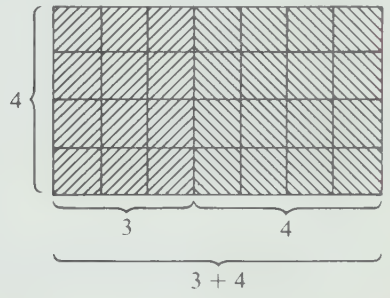
When students have finished the exercises, be sure to allow adequate time for discussing the application of the distributive principle. As preparation for work with the multiplication algorithm, students should clearly understand the role of the distributive principle in problems such as those in Exercise 3.

EXTENSION

For additional practice exercises, see page 13 of the *Workbook*.

Remedial: Ask students to cut out rectangular shapes from graph paper and shade them so as to illustrate the distributive principle. The example below shows that

$4 \cdot 7 = 4 \cdot (3 + 4) = 4 \cdot 3 + 4 \cdot 4$



Assignments

Minimum 1–3F. Average 1–3. Maximum 1–5.

Objective

The student can find sums and differences by using the usual addition and subtraction algorithms.

PREPARATION

Most of your students will have some skill at finding sums and differences. This section provides your students with an opportunity to review the addition and subtraction algorithms and to improve their computational skills. To gain some idea of the proper emphasis and the amount of reteaching necessary for this lesson, you may wish to provide a short diagnostic test prior to this lesson.

INVESTIGATION

The missing-digit exercises in the Investigation will offer much insight into the students' understanding of the regrouping process in adding and subtraction. When students have completed the Investigation, ask some of them to display their completed problems and explain how they found the missing digits.

DISCUSSION

Exercises 1 through 4 provide additional discussion opportunities concerning regrouping in addition and subtraction.

Answers, Discussion Exercises 1 and 2

1. If you added the two largest digits possible in the ones' place ($9 + 9$), you would carry 1 ten to the tens' column, which would give the digit 7 in the tens' place in the sum. It is possible that there would be no tens to carry, which gives the digit 6 in the tens' place in sum.
2. From the sum of the digits in the ones' place, the greatest number of tens that could be carried would be two, since $9 + 9 + 9 = 27$. The greatest number of hundreds that could be carried from the sum of the tens' digits would be two, since $(90 + 90 + 90) + 20 = 290$. The possibilities for the hundreds' digit in the sum are

$(8 + 5 + 8) + 0 = 21$ (one)
$(8 + 5 + 8) + 1 = 22$ (two)
$(8 + 5 + 8) + 2 = 23$ (three)

Adding and Subtracting Review

Investigating the Ideas

Some of the digits are missing in the addition and subtraction problems shown below.

1.
$$\begin{array}{r} \square 937 \\ 8\square \\ 109 \\ +4\square 85 \\ \hline 85\square 7 \end{array}$$

2.
$$\begin{array}{r} 5373\square \\ \square 1059 \\ \square 006 \\ +\quad 69 \\ \hline \square 31\square\square 1 \end{array}$$

3.
$$\begin{array}{r} 7\square 25 \\ -384\square \\ \hline \square 7\square 3 \end{array}$$

4.
$$\begin{array}{r} 6\square 02 \\ -43\square 5 \\ \hline 162\square \end{array}$$

Can you find the missing digits in each of the problems?

Discussing the Ideas

In Examples A, B, C, and D, some of the digits are covered by screens.

1. Explain why the tens' digit in the sum for Example A can be either 6 or 7.
2. Explain why the hundreds' digit in the sum for Example B can be 1, 2, or 3.
3. Explain why the number of tens in the difference for Example C is either 3 or 2.
4. What are the possible digits for the ▒ in Example D?

A
$$\begin{array}{r} \square 7\square \\ +\square 9\square \\ \hline \square\square\square \end{array}$$

B
$$\begin{array}{r} \square 8\square \\ \square 5\square \\ +\square 8\square\square \\ \hline \square\square\square \end{array}$$

C
$$\begin{array}{r} \square 7\square \\ -\square 4\square \\ \hline \square\square\square \end{array}$$

D
$$\begin{array}{r} \square 5\square \\ -\square 8\square \\ \hline \square\square\square \end{array}$$

5. What are some methods you can use to help you find sums rapidly and accurately?
6. A How can you check your work in an addition problem?
   B How can you check your work in a subtraction problem?

Using the Ideas

1. Find the sums.

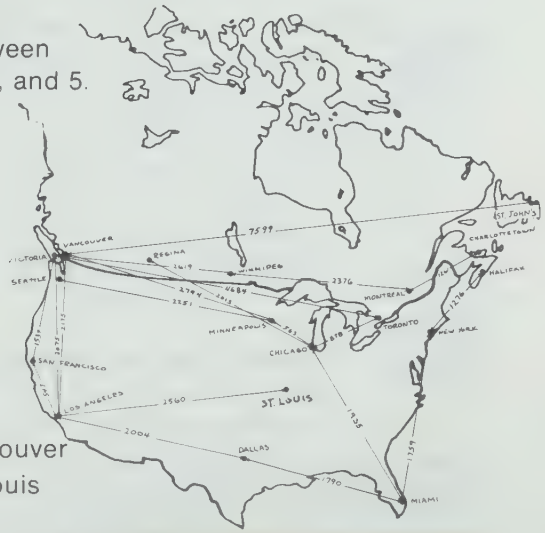
A 37 +52	**B** 86 +37	**C** 534 +256	**D** 744 +257	**E** 2898 +7265
F 217 314 +253	**G** 328 426 153 +867	**H** 2347 981 49 + 6	**I** 6243 1527 4639 +7254	**J** 28 705 44 192 89 763 +17 541

K 57 + 41 + 96 **L** 123 + 174 + 6209 **M** 2338 + 5620 + 23 746

2. Find the differences.

A 87 −35	**B** 92 −68	**C** 131 − 49	**D** 862 −777	**E** 1207 − 849
F 5200 −2637	**G** 7891 −4987	**H** 9451 −5789	**I** 23 742 −19 888	**J** 987 654 −321 098

The map shows airline distances (km) between some cities. Use the map for Exercises 3, 4, and 5.



3. Give the total distance for each trip.

- **A** St. John's–Vancouver–Los Angeles
- **B** Regina–Chicago–Miami
- **C** Seattle–Minneapolis–Chicago
- **D** Miami–New York–Halifax

4. Which of the two trips is the shorter? How much shorter is it?

- **A** Toronto–Chicago–Minneapolis–Vancouver
- **B** Toronto–Victoria–Los Angeles–St. Louis

5. What is the total airline distance covered on a trip from Charlottetown to Montreal to Winnipeg to Victoria to San Francisco to Los Angeles to Dallas to Miami to New York?

think

The sum of two numbers is 56.
Their difference is 18.
What are these two numbers?

$a + b = 56$
$a - b = 18$

More practice, pages S-5 and 6, Sets 8 and 9

UTILIZATION

The first two exercises provide straightforward practice in adding and subtracting. Exercises 3, 4, and 5 provide experiences in solving story problems that involve addition and subtraction.

EXTENSION

Appropriate additional practice exercises are available on pages S-5 and S-6, Supplementary Exercise Sets 8 and 9; page 14 of the *Workbook;* and page 11 of the *Duplicator Masters. Arithmetic Skill Cards* W-1 through W-4 might also be used with this lesson.

Enrichment: Students might be interested in planning a trip from their home town to one of the cities on the map on this page. This kind of activity, appealing to students' individual travel interests, may prove particularly motivating.

Think Solution

Although algebraic techniques may be implicit in their methods, most students will probably solve this problem by trial and error. For example:

(1) Try $30 + 26 + 56$,
but $30 - 26 < 18$.

The difference between the numbers must be greater.

(2) Try $35 + 21 = 56$,
but $35 - 21 < 18$.

The difference must again be made greater.

(3) Try $38 + 18 = 56$,
but $38 - 18 > 18$.

Just a bit too great a difference!

(4) Try $37 + 19 = 56$,
and $37 - 19 = 18$.

Algebraically: $\begin{cases} a + b = 56 \\ a - b = 18 \end{cases}$

$2 \cdot a = 74$

$a = 37,\ b = 19$

Assignments

Minimum 1A–H, 2A–H. Average 1–4. Maximum 1–5.

Objective

The student can solve simple word problems that involve numbers in the thousands and the millions by using addition and subtraction.

UTILIZATION

These word problems involve facts about the areas and populations of the countries of North America. This problem set offers an excellent opportunity to relate mathematics to social studies.

Assignments

Minimum 1–3. Average 1–4. Maximum 1–7.

NORTH AMERICA

REGION	AREA (km²)	POPULATION (1970 est.)
British Honduras	22 965	130 000
Canada	9 974 890	21 681 000
Canal Zone	1432	45 000
Costa Rica	50 699	1 800 000
Cuba	109 345	8 553 000
El Salvador	21 393	3 534 000
Guatemala	108 889	5 189 000
Honduras	112 087	2 582 000
Mexico	1 963 891	48 313 000
Nicaragua	139 699	1 984 000
Panama	75 649	1 464 000
United States	9 346 460	204 766 000

1. How much larger in area is Canada than the United States?

2. What is the difference in the areas of Nicaragua and Cuba?

3. How many more people live in Mexico than in Canada?

4. A Find the total population of Guatemala and Honduras.
   B Find the difference between the total in A and the population of Cuba.

5. Find the total area of North America.

6. Find the total population of North America.

7. How much more is the population of the United States than all the remaining North American regions together?

Money Problems

1. What is the total cost of these groceries?

2. Janet had $10.00. She bought a blouse for $5.69. How much change did she get back?

3. A sports car has a list price of $3096. With "optional equipment," the price is $3435. How much extra does the "optional equipment" cost?

4. Mr. Johnson bought a house for $17 960. He made a down payment of $3592. How much more does he owe on the house?

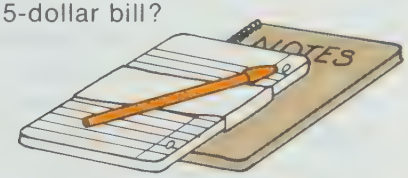
5. The Watson family had these expenses for a certain month:

$265.00	Housing
210.85	Food
36.95	Clothing
46.72	Utilities
30.90	Recreation
79.53	Transportation

What is the total amount of the expenses for the month?

6. Paul saw a 10-speed bicycle advertised for $83.60. He has already saved $25.95 for a bicycle. How much more money must Paul save before he can purchase a bicycle?

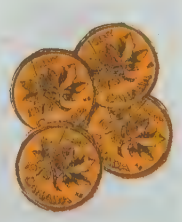
7. Lisa bought some school supplies. She paid 69¢ for paper, 98¢ for a ball point pen, and 79¢ for a notebook. How much change should she get back if she paid for the supplies with a 5-dollar bill?



8. Martin earns $10.00 per week doing part time work. One week he spent $3.75 for recreation and $2.60 for snacks. He put the rest in his savings account. How much did Martin save that week?

think

Rick has 44 pennies. Can he put a different number of pennies in each of his pockets if he has a total of ten pockets?



Objective

The student can solve word problems involving money by using addition and subtraction.

UTILIZATION

Most students can successfully solve the word problems involving money with little instruction. It is probably unnecessary to discuss the role of the decimal point in money notation other than to remark that "It separates the dollars from the cents." A few students may need to be reminded to be sure to keep the decimal points aligned in addition and subtraction problems.

Think Solution

The sum of the first ten nonzero numbers is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55$. Therefore the ten pockets cannot be filled with pennies numbering from 1 through 10. Suppose 0 pennies are placed in one of the pockets. Then $0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$, which is one more than the number of pennies available. If the number of pennies is less than in any pocket above, two pockets will have the same number of pennies. Therefore the 44 pennies cannot be placed in the ten pockets so that there is a different number in each pocket.

Objectives

The student can find products of numbers that are multiples of 10, 100, or 1000.

The student can find quotients for pairs of numbers that are multiples of 10, 100, or 1000.

PREPARATION

Since the Investigation deals with finding missing function rules, a good preparation for the lesson would be to play the guess-the-function-rule game as a warm-up activity.

INVESTIGATION

The Investigation introduces the main idea of the lesson—finding special products and quotients involving multiples of 10 and 100. Most students will be able to find the missing function rules quickly. Students who are slow in discovering the rules are likely to need extra help with the ideas of this lesson.

DISCUSSION

Much of this material will be review for most of your students, so you will want to adjust the time and emphasis on this material according to the abilities of your class. A rapid review should be adequate for most students; however, you will find that you may need to proceed more slowly with others.

In Exercise 1, be sure to emphasize that students must be able to multiply by 10, 100, and 1000 *mentally*. In Exercise 2, you can explain each step as follows:

$30 \cdot 40 = (3 \cdot \mathbf{10}) \cdot (4 \cdot \mathbf{10})$ Multiplication fact

$= (3 \cdot 4) \cdot (\mathbf{10} \cdot \mathbf{10})$ Associativity and Commutativity

$= 12 \cdot 100 = \mathbf{1200}$ Multiplication facts

The flow chart in Exercise 3 gives the shortcut for finding products like $20 \cdot 60$. Stress the inverse relationship between multiplication and division when discussing Exercise 4.

Special Products and Quotients

Investigating the Ideas

Study the function tables below.

A

Function Rule [redacted]	
n	$f(n)$
2	20
7	70
67	670

B

Function Rule [redacted]	
n	$f(n)$
6	600
13	1300
145	14 500

C

Function Rule [redacted]	
n	$f(n)$
30	3
150	15
490	49

Can you discover a function rule for each table?

List a few more number pairs for each table.

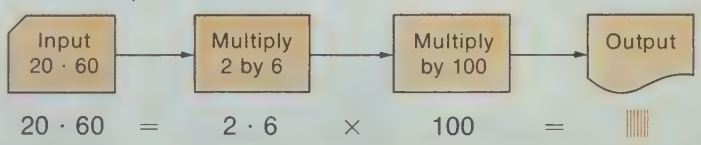
Discussing the Ideas

1. Many special products and quotients involving multiples of 10, 100, and 1000 can easily be found by using the basic multiplication facts, the basic principles, and the inverse relation between multiplication and division.
   - **A** Explain how to find the product of any whole number and 10 mentally.
   - **B** How can you multiply any number by 100? by 1000?

2. The diagram shows some steps in finding the product of 30 and 40. Give the missing number in each step.

$$30 \cdot 40 = (3 \cdot \boldsymbol{a}) \cdot (4 \cdot 10)$$
$$= (3 \cdot 4) \cdot \boldsymbol{b}$$
$$= \boldsymbol{c}$$

3. The flow chart shows a shortcut for multiplying $20 \cdot 60$. What is the product?



$20 \cdot 60 = 2 \cdot 6 \times 100 =$ [redacted]

4. If you know that $20 \cdot 80 = 1600$, explain how you can find these quotients. $1600 \div 20$ and $1600 \div 80$

Using the Ideas

1. Find the products.

A $5 \cdot 10$	C $8 \cdot 100$	E $100 \cdot 6$	G $9 \cdot 10$	I $100 \cdot 7$	K $9 \cdot 1000$
B $10 \cdot 7$	D $2 \cdot 10$	F $3 \cdot 100$	H $10 \cdot 4$	J $5 \cdot 100$	L $1000 \cdot 40$

2. Find the number for a. Then find the number for b.

A $10 \cdot 10 = a \rightarrow 7 \cdot 10 \cdot 10 = b$

B $7 \cdot 10 \cdot 10 = a \rightarrow 70 \cdot 10 = b$

C $10 \cdot 100 = a \rightarrow 4 \cdot 10 \cdot 100 = b$

D $4 \cdot 10 \cdot 100 = a \rightarrow 40 \cdot 100 = b$

3. Find the products.

A $40 \cdot 10$	D $100 \cdot 70$	G $4 \cdot 300$	J $120 \cdot 20$	M $200 \cdot 20$
B $10 \cdot 30$	E $20 \cdot 40$	H $60 \cdot 40$	K $60 \cdot 1000$	N $90 \cdot 300$
C $60 \cdot 100$	F $30 \cdot 50$	I $50 \cdot 40$	L $70 \cdot 30$	O $800 \cdot 60$

4. Find the quotients.

A Since $60 \cdot 30 = 1800$, we know that $1800 \div 30 = a$.

B Since $40 \cdot 20 = 800$, we know that $800 \div 20 = c$.

C Since $70 \cdot 40 = 2800$, we know that $2800 \div 40 = x$.

D Since $80 \cdot 20 = 1600$, we know that $1600 \div 20 = m$.

E Since $50 \cdot 300 = 15\,000$, we know that $15\,000 \div 300 = r$.

5. Think about the missing factor. Then give the quotient.

A $? \cdot 30 = 600$ → $600 \div 30 = n$

B $? \cdot 40 = 2400$ → $2400 \div 40 = a$

C $? \cdot 50 = 1500$ → $1500 \div 50 = t$

D $r \cdot 60 = 1800$
$1800 \div 60 = r$

E $m \cdot 8 = 1600$
$1600 \div 8 = m$

F $s \cdot 300 = 1200$
$1200 \div 300 = s$

6. Find the quotients.

A $400 \div 8$	C $1200 \div 60$	E $63\,000 \div 700$
B $6000 \div 3$	D $2700 \div 90$	F $280\,000 \div 4000$

★**7.** Find the products and quotients. Write your answers in scientific notation.

A $3 \cdot 10^3 \cdot 2 \cdot 10^4$	E $(7 \cdot 10^7) \div 7$	I $500 \cdot 10^2 \cdot 20 \cdot 10^3$
B $5 \cdot 10^4 \cdot 4 \cdot 10^5$	F $50 \cdot 10^3 \cdot 10^2$	J $3\,000\,000 \div (30 \cdot 10^3)$
C $4 \cdot 10^6 \div 10^3$	G $60\,000 \div (2 \cdot 10^2)$	K $(2 \cdot 10^{10}) \div (10 \cdot 10^2)$
D $(3 \cdot 10^8) \div 10^8$	H $100\,000 \div (5 \cdot 10^4)$	L $10 \cdot 10^4 \cdot 4 \cdot 10^{10}$

More practice, page S-6, Set 10

UTILIZATION

Assign the exercises according to the needs and abilities of your students. If feasible, you might prefer to have students do most of the exercises orally, since mental computation should be stressed in this lesson.

EXTENSION

For additional practice problems, see Supplementary Exercise Set 10 on page S-6, *Workbook* page 15, and *Duplicator Masters* page 12.

Remedial: Provide students with more practice problems like the following. (Answers are shown in *italics*.)

1. $3 \cdot 100$ *300*	6. $4000 \div 50$ *80*
2. $10 \cdot 67$ *670*	7. $70 \cdot 80$ *5600*
3. $1200 \div 20$ *60*	8. $600 \cdot 10$ *6000*
4. $800 \div 2$ *400*	9. $5600 \div 8$ *700*
5. $100 \cdot 80$ *8000*	10. $12\,000 \div 40$ *300*

Enrichment: To extend the ideas presented in exercise 7, provide additional exercises like those that follow.

1. Find the products and quotients. Write your answers in *scientific notation*.

	Answers
A. $3 \cdot 10^3 \cdot 2 \cdot 10^4$	$6 \cdot 10^7$
B. $5 \cdot 10^4 \cdot 4 \cdot 10^5$	$2 \cdot 10^{10}$
C. $4 \cdot 10^6 \div 10^3$	$4 \cdot 10^3$
D. $3 \cdot 10^8 \div 10^8$	$3 \cdot 10^0$
E. $7 \cdot 10^7 \div 7$	$1 \cdot 10^7$
F. $50 \cdot 10^3 \cdot 10^2$	$5 \cdot 10^6$

2. Write the numbers in scientific notation. Then give the product or quotient in scientific notation.
   A. 10 000 000 · 300 000
   B. 600 000 000 ÷ 30 000
   C. 4 000 000 000 ÷ 200 000
   D. 400 000 000 · 2 000 000

   Answers
   A. $(1 \cdot 10^7) \cdot (3 \cdot 10^5) = 3 \cdot 10^{12}$
   B. $(6 \cdot 10^8) \div (3 \cdot 10^4) = 2 \cdot 10^4$
   C. $(4 \cdot 10^9) \div (2 \cdot 10^5) = 2 \cdot 10^4$
   D. $(4 \cdot 10^8) \cdot (2 \cdot 10^6) = 8 \cdot 10^{14}$

Assignments

Minimum 1–3, oral; 4, 5.
Average 1–3, oral; 4–6.
Maximum 1–6, oral; 7.

Objective

The student can compute products using factors having 2 or more digits by using the multiplication algorithm.

PREPARATION

Review orally some problems involving multiplying with multiples of 10 and 100, as in the previous lesson. Then proceed to the Investigation.

INVESTIGATION

The purpose of the Investigation is to examine the steps one must use in finding the product by the usual multiplication algorithm. Most students will be successful in finding the three products shown, but some may not be able to relate the sum of the three products to the multiplication problem 329×6. Do not be overly concerned about this in the Investigation stage, since the purpose of the Discussion is to clarify and elaborate upon the ideas presented in the Investigation.

DISCUSSION

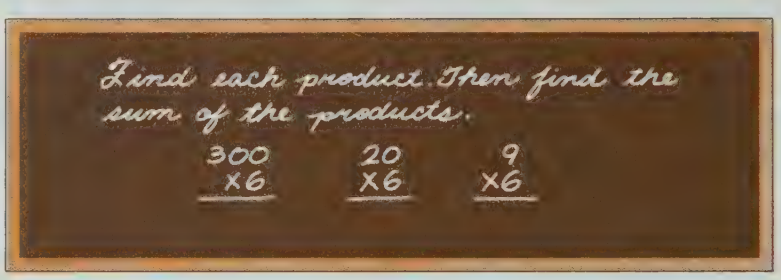
The distributive principle is the key to understanding the multiplication algorithm. Relate the steps in the algorithm in Exercise 2 to the work of Exercise 1. Note that the partial products are the same but then we find their sum as we multiply. Exercises 3 and 4 cover more complicated multiplication examples. In step 2 of Exercise 3, the product $50 \cdot 23 = 1150$ is found. In step 3 the sum of the partial products $138 + 1150$ is found.

The amount of emphasis given to the details of the multiplication algorithm will depend, of course, upon the abilities of your students. For some students, a close look at the algorithm may be enlightening. For students who have good computational skill in multiplication, such emphasis will not be necessary.

In general, you should adjust the amount of work assigned in this lesson according to the needs and abilities of your students.

Computing Products

Investigating the Ideas



Can you find a number such that when multiplied by 6 will have a product equal to the sum above?

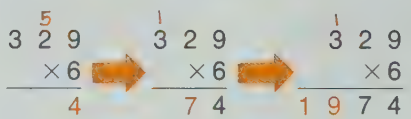
Discussing the Ideas

1. The distributive principle can help you find the number in the Investigation.

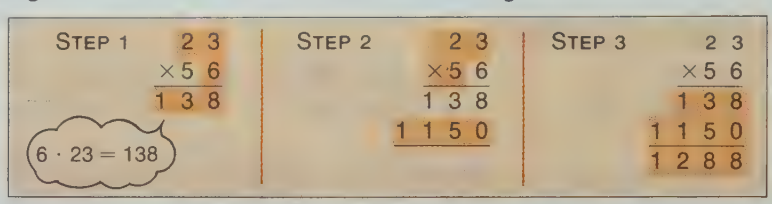
   $6 \cdot 300 + 6 \cdot 20 + 6 \cdot 9 = 6 \cdot (300 + 20 + 9) = 6 \cdot n$

   What is the number for n?

2. Explain the steps in this shortcut method for finding products.



3. The distributive principle is also applied in the shortcut method of multiplying when factors with more than one digit are used. Explain what you do in steps 2 and 3 to complete the work.



4. Tell which two factors give the partial product covered by each row of ▥. Then give this partial product. Find the total product.

THE HUMAN BODY

1. Your heart, under normal conditions, beats about 72 times per minute. About how many times does your heart beat during
   - **A** 1 hour?
   - **B** a 24-hour day?
   - **C** a 365-day year?
   - **D** a 70-year lifetime?

2. We breathe (inhale and exhale) at a rate of about 18 times per minute. About how many times do we breathe in an hour at this rate?

3. An adult man weighs 75 kilograms. An adult whale can weigh 1250 times as much. How many kilograms can an adult whale weigh?

4. A person uses about 275 joules of energy per hour while sleeping. How many joules are used by a person who sleeps from 9:30 P.M. to 7:30 A.M.?

5. A student uses about 550 joules of energy per hour while sitting at a desk in the classroom. How many joules does the student use during a 6-hour school day?

6. The brain of a baby weighs about 340 grams. The weight of a baby is 12 times his brain weight. About how many kilograms does the baby weigh?

7. An adult weighs about 35 times his brain weight. If a man's brain weight is about 2 kilograms, what is his body weight?

8. Find the products.

A 32 × 43	**B** 32 × 54	**C** 27 × 58	**D** 352 × 46	**E** 257 × 84	**F** 368 × 29
G 423 × 514	**H** 367 × 425	**I** 378 × 496	**J** 3524 × 86	**K** 4763 × 584	**L** 8732 × 4658

★9. Rewrite the following multiplication problems replacing each ▒ by a digit so that the answers are correct.

A
```
  ▒▒
 ×5 4
 ----
 ▒▒▒
▒▒▒
-----
3▒3 4
```

B
```
  9 9
 ×▒▒
 ----
 ▒▒▒
▒▒▒
-----
2 2▒
```

C
```
  ▒▒▒
 ×7 7
 ----
 ▒▒▒
▒▒▒▒
-----
2 7 6▒
```

D
```
   ▒▒▒
 ×9 9 9
 -------
  ▒▒▒
 ▒▒▒
▒▒▒
-------
4 7▒▒ 4
```

More practice, page S-7, Set 11

UTILIZATION

If some students need to improve their multiplication speed skills, they should be asked to do most of the parts of Exercise 8. Students who are already proficient in multiplication should probably concentrate on Exercises 1 through 7, which are designed to improve problem-solving skills.

EXTENSION

Additional practice is provided by Supplementary Exercise Set 11 on page S-7, *Workbook* page 16, and *Duplicator Masters* page 13. *Arithmetic Skill Cards* W-5 and W-6 are also appropriate for use with this lesson.

Enrichment: A method called "casting out nines" can be an aid in checking multiplication problems. First, define the *digital sum* of a number as the sum obtained by adding the digits in the number until a sum of 9 or less is reached, as in the example below.

NUMBER: 2188 →
$2 + 1 + 8 + 8 = 19 = 1 + 9 = 10 = 1 + 0 = 1$
DIGITAL SUM

Checking a multiplication problem by this method involves finding the product of the digital sums of the two factors and showing that this product has the same digital sum as the digital sum of the product in the example, as shown below.

	DIGITAL SUMS
626	5
×84	×3
2504	15 → 1 + 5 → 6
5008	
52584 → 5 + 2 + 5 + 8 + 4 = 24 → 6	Digital sums are the same.

Students can quickly master this technique, but it should be noted that the method does not detect such errors as transposition of digits or misalignment of the partial products.

Assignments

Minimum 1, 2, 8A–D. Average 1–4, 8A–H. Maximum 1–9.

Objective

Given a multiplication problem, the student can round factors to the nearest multiple of 10, 100, or 1000 to estimate the product.

PREPARATION

Give some oral problems of multiplication involving easy multiples of 10, 100, and 1000, such as 3×400, 20×40, 6×1000, 50×400, etc.

INVESTIGATION

Each of the estimation problems suggested in the Investigation will probably require the student to round some numbers and then multiply.

The estimation problems could be assigned to pairs or small groups of students since checking the estimates involves measuring and recording the results.

DISCUSSION

Allow students to describe how they found their estimates. Some students may strive for needlessly precise answers. For instance, a student who is 13 years old may claim that his age in seconds is exactly 409 968 000 seconds. Point out to such a student that unless he or she made the computation on the same day of the year and at precisely the same time of day that he or she was born, the result is not exact.

Exercise 2 reviews the process of rounding to appropriate multiples of 10, 100, or 1000 as an aid in estimating.

It is useful to ask students whether their estimate is greater or smaller than the actual product. In Exercise 3A the estimate is clearly greater than the actual product since one factor was rounded up. In Exercise 3B the estimate is clearly smaller than the actual product since both factors were rounded down. However, in Exercise 3C one number has been rounded up while the other number has been rounded down, so the student would have some difficulty in deciding whether the estimate is greater or smaller than the actual product.

Estimating Large Products

Investigating the Ideas

Try one or more of these estimation problems.

A. Estimate your age in seconds.

B. Estimate your height in millimetres.

C. Estimate your total mass in grams.

Can you find a way of checking your estimate?

Discussing the Ideas

1. **A** Explain how you made your estimates.
B Were your estimates too large or too small?

2. In estimation problems, you can often choose the nearest multiples of 10, 100, or 1000 to represent the numbers in the problems.
Tell which of the two multiples of 10, 100, or 1000 is nearer to the number given in color.

A	70, 78, 80	**C**	100, 193, 200	**E**	100, 149, 200	**G**	6000, 6281, 7000
B	30, 31, 40	**D**	600, 629, 700	**F**	800, 850, 900	**H**	5000, 5706, 6000

3. Study the example for estimating products in the table below.
Give the missing **estimated** products for parts B and C.

	Product	We think	We say
A	78×9	$80 \cdot 9 = 720$	$78 \cdot 9$ is approximately 720
B	815×31	$800 \cdot 30 =$ ▮	$815 \cdot 31$ is approximately ▮
C	629×193	$600 \cdot 200 =$ ▮	$629 \cdot 193$ is approximately ▮

4. Give an estimate for each product.

A	$69 \cdot 7$	**B**	$209 \cdot 8$	**C**	$72 \cdot 81$	**D**	$427 \cdot 29$

Using the Ideas

1. Round the first factor to the nearest multiple of 10. Then give an estimate of the product.

A $78 \cdot 4$ **B** $92 \cdot 8$ **C** $31 \cdot 6$ **D** $84 \cdot 5$ **E** $59 \cdot 9$

2. Round the first factor to the nearest multiple of 100. Then give an estimate of the product.

A $621 \cdot 8$ **B** $978 \cdot 6$ **C** $409 \cdot 7$ **D** $894 \cdot 5$ **E** $719 \cdot 9$

3. Round each factor to the nearest multiple of 10. Then give an estimate of the product.

A $23 \cdot 41$ **B** $79 \cdot 52$ **C** $67 \cdot 19$ **D** $45 \cdot 81$ **E** $83 \cdot 58$

4. Give an estimate for each part.

A A man drove his car at an average speed of 78 kilometres per hour (km/h) for 9 hours. About how far did he travel?

B An airplane travels for 5 hours at an average speed of 921 kilometres per hour. About how far does it travel?

C Ball point pens cost 39¢. About how much money would 31 students have to pay for ball point pens?

D Terry had to take 1267 steps to walk one kilometre. About how many steps would she have to take to walk four kilometres?

E In one section of a football stadium there are 27 rows with 63 seats in each row. About how many seats are in this section?

F Sound travels about 332 metres per second in air. About how far does it travel in a minute?

5. From the set of numbers {10, 20, 30, 40, 50, 60, 70, 80, 90}, select the largest number that will make the sentence true.

A $n \cdot 6 < 307$ **C** $n \cdot 9 < 555$ **E** $n \cdot 4 < 281$ **G** $n \cdot 7 < 644$

B $n \cdot 8 < 658$ **D** $n \cdot 5 < 127$ **F** $n \cdot 3 < 179$ **H** $n \cdot 6 < 444$

6. Find the largest whole number for n that will make the sentence true.

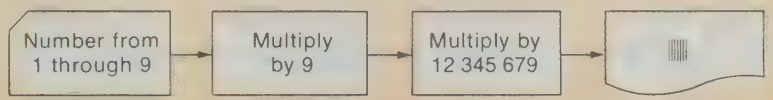
A $n \cdot 32 < 137$ **D** $n \cdot 83 < 681$ **G** $n \cdot 72 < 584$

B $n \cdot 57 < 381$ **E** $n \cdot 91 < 508$ ☆**H** $n \cdot 714 < 2347$

C $n \cdot 23 < 118$ **F** $n \cdot 48 < 380$ ☆**I** $n \cdot 579 < 3827$

think

Find the output number for this flow chart.



UTILIZATION

Most students will benefit from a discussion of the word problems in Exercise 4. Have the student tell which rounded factors they chose to estimate the products.

Exercises 5 and 6 are important because they illustrate the kind of estimation problems students encounter in determining quotients in long division problems.

Do not insist on one particular rounding and estimation process. There are a number of different ways of rounding that may result in fairly good estimates.

EXTENSION

The exercises on page 17 of the *Workbook* and on page 14 of the *Duplicator Masters* provide further practice with the concepts of this lesson.

Think Solution

The output number will be a nine-digit number whose digits will be the input number repeated. For example, if the number 4 is the input, then the flow chart requires computing the product $36 \cdot 12\,345\,679$, which is 444 444 444.

Assignments

Minimum 1–3, 5A–D, 6A–C.
Average 1–3, 4A–C, 5A–F, 6A–F.
Maximum 1–6.

Objective

Given a division problem with a one-digit divisor, the student can compute the quotient.

PREPARATION

It may be helpful to review the concept of division as the inverse of multiplication. Give some simple division problems that students can do mentally, including some in which the dividends are multiples of ten.

INVESTIGATION

The Investigation might be used as a kind of diagnostic test to determine how much review or reteaching of division by one-digit divisors is appropriate for your students. The coded answers provide some additional challenge as well as immediate reinforcement.

DISCUSSION

The purpose of Discussion Exercise 1 is to review the basic ideas underlying the division algorithm. The algorithm consists of repeatedly subtracting multiples of the divisor until a remainder smaller than the divisor occurs. Then the total number of times the divisor has been subtracted is counted, in this case 236.

Discussion Exercise 2 presents the simplest form of the division algorithm, usually called "short division," in a flow chart format. Relate the steps of this method to the various steps shown in the example accompanying Exercise 1. You may find it worthwhile to ask students to work through some additional problems similar to those in parts A through D, depending on the needs of the students.

Dividing

Investigating the Ideas

How well can you divide when the divisor has only one digit?

1. $3\overline{)87}$	**2.** $7\overline{)126}$	**3.** $8\overline{)234}$
4. $9\overline{)4621}$	**5.** $6\overline{)7041}$	**6.** $5\overline{)8003}$

Can you find the quotients and remainders without getting more than one incorrect answer?

Check your answers by using the code below.

CODE: 1 = △, 2 = □, 3 = ⌓, 4 = ⏢, 5 = ☆, 6 = ○, 7 = ●, 8 = ◎, 9 = ◎, 0 = ▯

Coded Answers: **1.** □◎ **2.** △◎ **3.** □◎, R□

4. ☆△⌓, R⏢ **5.** △△●⌓, R⏢ **6.** △○▯▯, R⏢

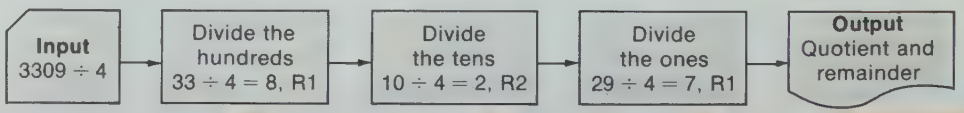
Discussing the Ideas

1. Refer to the example at the right to answer the questions.

- **A** How many sevens were subtracted the first time?
- **B** How many sevens were subtracted the second time? third time?
- **C** All total, how many sevens were subtracted?
- **D** How does the remainder compare in size to the divisor?
- **E** What was done to check to see if the answer was correct.

Divisor		Quotient		CHECK
		236 ←		
7	1656	Dividend		236
	−1400	← 200 · 7		× 7
	256			1652
	−210	← 30 · 7		+ 4
	46			1656
	−42	← 6 · 7		
	4	Remainder		

2. When the divisor is a number less than 10, you can use a shortcut method called "short division." Study the flow chart and the example given below it.



$4\overline{)3309}$ $\quad$ $\overset{8}{4\overline{)33^{1}09}}$ $\quad$ $\overset{82}{4\overline{)330^{2}9}}$ $\quad$ $\overset{827,\ R1}{4\overline{)330^{2}9}}$ $\quad$ $\overset{827,\ R1}{4\overline{)3309}}$

Try this method on these problems. Check your work.

A $6\overline{)2491}$ **B** $8\overline{)6213}$ **C** $9\overline{)4239}$ **D** $7\overline{)6905}$

Using the Ideas

1. Use "short division" to find the quotients and remainders.

A $8\overline{)386}$	E $9\overline{)4105}$	I $3\overline{)12\,345}$	M $5\overline{)36\,628}$
B $9\overline{)314}$	F $7\overline{)18\,488}$	J $4\overline{)81\,624}$	N $2\overline{)3\,584\,286}$
C $6\overline{)4455}$	G $9\overline{)61\,407}$	K $9\overline{)2300}$	O $9\overline{)111\,111\,111}$
D $5\overline{)3646}$	H $7\overline{)1001}$	L $4\overline{)70\,771}$	P $3\overline{)186\,001}$

2. On a field trip 4 buses were used to transport 216 students. Each bus contained the same number of students. How many students were on each bus?

3. The Jameson family traveled 865 kilometres in 9 hours. About how many kilometres per hour did they average?

4. John worked in a supermarket. He had to put 9 oranges into each produce bag. How many bags would 1000 oranges fill? How many oranges would be left over?

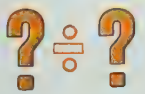
★5. If the remainder is zero, what is the ones' digit of each dividend? What is the quotient?

A $7\overline{)482\,35}$ B $8\overline{)34\,15}$ C $6\overline{)31\,22}$ D $9\overline{)347\,91}$

★6. If the remainder is zero, there are two possibilities for the ones' digit in each dividend. Give both possibilities.

A $6\overline{)31\,51}$ B $4\overline{)71\,43}$ C $6\overline{)3\,00}$ D $7\overline{)68\,71}$

★7. What division fact is shown by the repeated subtraction?



5313
−4600
713
−690
23
−23
0

think

What are the digits in the division problem? Each letter stands for a single digit.

$$\begin{array}{r} b\ b \\ a\overline{)\,b\ c\ c} \\ \underline{a} \\ b\ c \\ \underline{a} \\ b \end{array}$$

More practice, page S-7, Set 12

UTILIZATION

Keep in mind the ability level of your students and adjust the assignment to meet their needs. Students who are proficient in dividing need not be assigned a great many exercises. Instead they should concentrate on improving their word problem solving skill.

Solution, Exercise 7

The numbers 4600, 690, and 23 must be multiples of the divisor. Since 23 is prime, the divisor must be 23. The number must have three digits, since there are three subtraction steps in the algorithm.

$$23\overline{)5313}\quad 231$$

EXTENSION

For further practice with the concepts of this lesson, assignments may be made from Supplementary Exercise Set 12 on page S-7, page 18 of the *Workbook*, and page 15 of the *Duplicator Masters*. For remedial help, *Arithmetic Skill Card* W-7 is recommended.

Think Solution

In the division problem, we first know that $b = 1$, because b in the quotient times divisor a must equal a.

$$\begin{array}{r} 11 \\ a\overline{)1cc} \\ \underline{a} \\ 1c \\ \underline{a} \\ 1 \end{array}$$

Then $1c - a = 1$, and we know that $c = 0$ and $a = 9$ because $10 - 9$ is the only possible combination that has a difference of 1. Thus, $a = 9$, $b = 1$, $c = 0$.

Assignments

Minimum 1A–H, 2, 4. Average 1A–K, 2, 3, 4. Maximum 1–7.

Objective

Given a division problem with a divisor of more than one digit, the student can use the standard long division algorithm to compute the quotient.

PREPARATION

This lesson reviews the algorithm for long division. You may prefer to introduce this lesson by giving a short quiz involving divisors having at least two digits. The results of the quiz would then aid you in determining the amount of emphasis to give the lesson.

DISCUSSION

Exercise 1 is designed to help the student understand the reasoning behind the division algorithm shown in Steps 1 and 2 of the displayed example.

Exercise 2 introduces rounding the divisor in order to estimate the partial quotients. Exercise 3 gives some further practice in estimating quotients. Note that in Exercise 3C if the student uses 70 as a trial divisor, then he would estimate the tens' digit of the quotient to be 7. But, since $7 \cdot 74 = 518$ and $518 > 514$, the estimate of 7 is too large.

Similarly, in Exercise 3D, if 30 is used as a trial divisor, the tens' digit would be estimated to be 2. But $2 \cdot 27 = 54$ and when this product is subtracted from 82 the remainder is larger than the divisor. Thus,

3C.
$$\begin{array}{r} 7 \\ 74\overline{)5143} \\ -518 \\ \end{array}$$
← This product is too large.

3D.
$$\begin{array}{r} 2 \\ 27\overline{)821} \\ -54 \\ \hline 28 \end{array}$$
← This product is too small.
← Remainder is greater than divisor.

Using Larger Divisors

Discussing the Ideas

1. You can find the quotient for $67\overline{)1594}$ by finding one-digit quotients for $67\overline{)159}$ and $67\overline{)254}$.

- **A** What are these quotients?
- **B** Explain how the one-digit quotients are used in the two steps of the division problem at the right.
- **C** What is the quotient?
- **D** What is the remainder?
- **E** What would you do to check the division?

Step 1 Dividing tens	Step 2 Dividing ones
$\begin{array}{r} \quad 2 \\ 67\overline{)1594} \\ -134 \\ \hline \quad 25 \end{array}$	$\begin{array}{r} \quad 23 \\ 67\overline{)1594} \\ -134 \\ \hline \quad 254 \\ -201 \\ \hline \quad 53 \end{array}$

2. The example shows how you can think about rounding a divisor to estimate partial quotients. Complete the division.

Problem	Think	Write
$29\overline{)747}$	$30\overline{)747}$ (2, 60)	$29\overline{)747}$ (2, 58)

3. Estimate the tens' digit in each quotient.

- **A** $91\overline{)4186}$
- **B** $37\overline{)1234}$
- **C** $74\overline{)5143}$
- **D** $27\overline{)821}$

EXAMPLE: $48\overline{)2762}$ (5)

THINK: Round 48 to 50. Then 276 ÷ 50 gives 5 as the tens' digit.

4. **A** Find the quotients in Exercise 3 above.

- **B** In which parts was your estimate of the tens' digit too large? Why do you think so?
- **C** In which parts was your estimate too small? Why?

5. Estimate the first digit of the quotient. Round the divisor to the nearest hundred to make your estimate.

- **A** $832\overline{)190\,851}$
- **B** $419\overline{)322\,498}$

Using the Ideas

1. Find the quotients and remainders.

A $19\overline{)564}$	**D** $68\ 235 \div 47$	**G** $64\overline{)2139}$	**J** $36\overline{)3024}$
B $52\overline{)7922}$	**E** $1147 \div 19$	**H** $77\overline{)61\ 256}$	**K** $75\overline{)5101}$
C $86\overline{)2395}$	**F** $49\ 221 \div 53$	**I** $91\overline{)8333}$	**L** $43\overline{)1307}$

2. Find the quotients and remainders.

$500\overline{)23\ 849}$ $700\overline{)46\ 464}$ $900\overline{)38\ 574}$

A $512\overline{)23\ 849}$	**B** $681\overline{)46\ 464}$	**C** $902\overline{)38\ 574}$
D $777\overline{)111\ 111}$	**E** $832\overline{)190\ 851}$	**F** $429\overline{)312\ 498}$
G $676\overline{)829\ 166}$	**H** $964\overline{)72\ 841}$	**I** $2375\overline{)643\ 809}$

Short Stories

1. 396 students. 9 school buses. Same number of students in each bus. How many students in each bus?

2. Strawberries cost $10.80 a crate. 24 boxes in a crate. How much for one box?

3. Trip: 747 kilometres. Time: 9 hours. What was the average speed?

4. Three boys on a scale. Total mass: 105 kilograms. What is the average mass?

5. Each lawn mowed for $1.75. Earnings: $14. How many lawns were mowed?

6. A satellite in orbit goes 28 000 kilometres per hour. How far in 15 hours?

7. 400-metre dash. Time: 44 seconds. About how many metres per second?

8. The Dodgers played 73 baseball games. Scored 366 runs. About how many runs averaged per game?

More practice, page S-8, Set 13

UTILIZATION

These exercises should be assigned selectively, according to the needs and abilities of your students. Students who are able to divide efficiently should not be burdened with excessive repetitive work. Other students, however, may require considerable work in order to attain adequate skill in using this particular process.

The short stories present problem-solving situations with a minimum use of words.

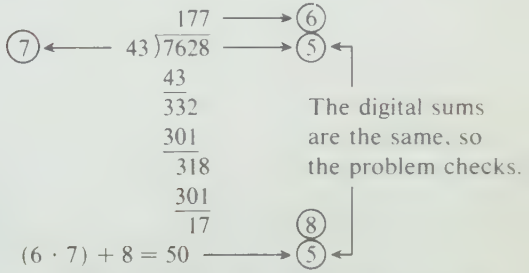
EXTENSION

Further practice can be assigned from Supplementary Exercise Set 13 on page S-8, *Workbook* page 19, and *Duplicator Masters* page 16. *Arithmetic Skill Cards* W-8, W-9, W-10, and W-13 are also appropriate for extending the work of this lesson.

Enrichment: The method of casting out nines, which was discussed in the commentary for page A-49 as a way to check multiplication, can be adapted for use in checking division as well. The usual check for division is

divisor · quotient + remainder = dividend.

Casting out nines, we need to use only the digital sums of the numbers in the equation above, as in the example below.



Remind students that this method of checking is not completely foolproof. If the check is not successful, a careful reworking of the problem will usually reveal a mistake.

This would be an opportune time to encourage some students to undertake Research Projects B and C on page A-57.

Assignments

Minimum 1A–F, 2A–C. Average 1A–F, 2A–F, Short stories 1–4. Maximum All problems.

Objective

The student will demonstrate the ability to work with the ideas presented in this module.

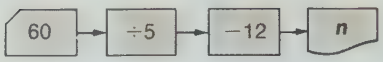
UTILIZATION

Assign the exercises according to the needs of your students. The exercises could be completed by each student independently or by small groups of students working together, discussing their results, and checking their answers.

The exercises on page 20 of the *Workbook* may be assigned as additional review of the ideas presented in this module.

REVIEWING THE IDEAS

1. Find the output number n in the flow chart.



2. In the statement following, give the other subtraction equation: Since $6 + 7 = 13$, we know that $13 - 7 = 6$ and $13 - \blacksquare = \blacksquare$.

3. Write two division equations that are related to the multiplication equation $13 \cdot 8 = 104$.

4. Give the missing numbers for each screen in the function tables.

	Function Rule n 8	
	n	$f(n)$
A	5	▒
B	7	▒
C	9	▒

	Function Rule $(5 \cdot n) + 4$	
	n	$f(n)$
D	3	▒
E	9	▒
F	0	▒

5. For each equation, give the number of the basic principle.

A $a + 0 = a$

B $(6 + 7) + 9 = 6 + (7 + 9)$

C $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

D $a + 7 = 7 + a$

E $7 \cdot (n + 6) = (7 \cdot n) + (7 \cdot 6)$

F $586 \cdot 78 = 78 \cdot 586$

G $5280 \cdot 1 = 5280$

1. Commutative principle (+)
2. Commutative principle (×)
3. Associative principle (+)
4. Associative principle (×)
5. Zero principle
6. One principle
7. Distributive principle

6. Solve the equations.

A $8 \cdot (7 + 12) = x$

B $(63 \div 7) \cdot 9 = y$

C $(8 \cdot 4) + (8 \cdot 3) = z$

7. Find the sums.

A
$$\begin{array}{r} 829 \\ 642 \\ +\,189 \\ \hline \end{array}$$

B
$$\begin{array}{r} 86\,288 \\ 127\,429 \\ 5\,766 \\ 62\,981 \\ \hline \end{array}$$

8. Find the differences.

A
$$\begin{array}{r} 2801 \\ -\,1977 \\ \hline \end{array}$$

B
$$\begin{array}{r} 51\,742 \\ -\,36\,955 \\ \hline \end{array}$$

9. Find the products and quotients.

A $7 \cdot 20$

B $50 \cdot 60$

C $360 \div 6$

D $7200 \div 80$

10. Choose the best estimate for each product.

A
$$\begin{array}{r} 82 \\ \times\,37 \\ \hline \end{array}$$
Estimates: 2400, 3200, 3600

B
$$\begin{array}{r} 609 \\ \times\,31 \\ \hline \end{array}$$
Estimates: 180, 1800, 18 000

11. Find the products.

A
$$\begin{array}{r} 78 \\ \times\,59 \\ \hline \end{array}$$

B
$$\begin{array}{r} 288 \\ \times\,543 \\ \hline \end{array}$$

12. Find the quotients and remainders.

A $39\overline{)7272}$

B $265\overline{)38\,742}$

13. Find the number for n.

A $17\overline{)\,n\,}$ $\quad = 326$

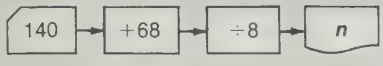
B $n\overline{)473}$ $\quad = 11$

C $61\overline{)33\,855}$ $\quad = n$

D $n\overline{)33\,453}$ $\quad = 81$

TEST YOURSELF

1. Find the output number in the flow chart.



2. Write two subtraction equations using the numbers in the addition equation $9 + 7 = 16$.

3. Write two division equations using the numbers in the multiplication equation $8 \cdot 4 = 32$.

4. Complete the function table.

Function Rule: $4 \cdot n - 3$

	Input	Output
A	2	
B	5	
C	10	

5. Solve the equation $7 \cdot (39 \div 3) = n$.

6. Solve each equation. Think of the basic principles.

A $14 \cdot x = 14$ B $84 + 0 = n$

C $27 + 89 = 89 + y$

D $(12 \cdot 31) \cdot 28 = 12 \cdot (31 \cdot w)$

E $8 \cdot (20 + 6) = 8 \cdot r + 8 \cdot 6$

7. Find the sum and difference.

A $\begin{array}{r} 2356 \\ +1098 \\ \hline \end{array}$ B $\begin{array}{r} 8003 \\ -2896 \\ \hline \end{array}$

8. Solve the equations.

A $30 \cdot 600 = n$ B $500 \cdot n = 40\,000$

C $3500 \div 50 = n$

9. Find the products.

A $\begin{array}{r} 261 \\ \times 75 \\ \hline \end{array}$ B $\begin{array}{r} 742 \\ \times 386 \\ \hline \end{array}$

10. Find the quotients and remainders.

A $61\overline{)7488}$ B $304\overline{)77\,777}$

RESEARCH PROJECTS

A The English mathematician John Napier invented a device called **Napier's rods** to multiply whole numbers. Find out how the rods were made and learn how to use them in multiplication. (See *The Giant Golden Book of Mathematics* by Irving Adler; New York: Golden Press, 1960 available from Whitman Goldman, Ltd.)

B Learn how to operate a desk calculator. Make up several addition, subtraction, multiplication, and division problems and then solve them yourself. Check your answers using the desk calculator.

C A common method of dividing large numbers in the Middle Ages was the **galley** or **scratch method**. Find out how this method works and then try to use it with some division problems. (See *History of Mathematics* by D. E. Smith; New York: Dover Publications, 1953, Vol. II available from General Publishing Co., Ltd.)

D Construct your own computer. (See *I Can Learn About Calculators and Computers* by Raymond Kenyon; Evanston, Illinois: Harper and Row, 1961, pp. 25–104 available from Fitzhenry & Whiteside, Ltd.)

TEST YOURSELF

These self-evaluation test problems are designed to cover the objectives stated at the beginning of the module. If students have difficulty with a specific problem, they should reread the objective for that problem and then restudy the pages covering the material in question. Note the annotated correlation between the test items and the objectives which introduced the module.

RESEARCH PROJECTS

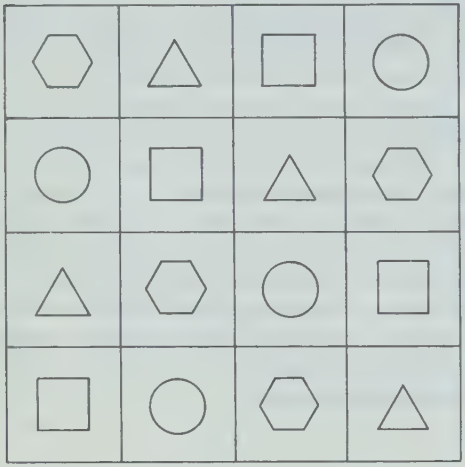
Project A would be suitable for use as a large-group project. Cardboard models of Napier's rods would provide durable models.

If desk calculators are available, students should be given the opportunity to learn how to use them to perform the four basic operations, as suggested in Project B. Department stores now sell miniature calculators which operate on small batteries or AC current. Many models sell for considerably less than $100, and some students may have such a computer at home. If so, they might ask their parents' permission to bring the computer to school for a class demonstration.

MATHEMATICAL RECREATION

This puzzle will provide an excellent challenge for your students. If feasible, encourage some students to make a permanent model of the puzzle. The base could be made of wood and the pieces cut from colored plastic.

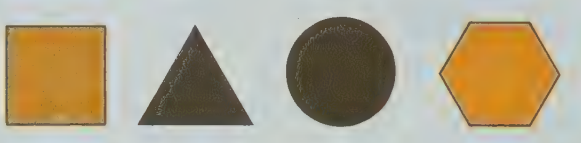
One solution of the puzzle is shown below. As a hint for the solution, you might suggest that it is helpful to try to work on the two diagonals first.



MATHEMATICAL RECREATIONS

A Four Color and Shape Attribute Puzzle

Make four copies of each of the geometric shapes shown below. (Sixteen pieces in all). Color the four pieces of each shape in four different colors; for example, red, green, yellow and blue.



The object of the puzzle is to try to place the 16 pieces on a 4 by 4 square in such a way that no two pieces that have the same shape or the same color are in the same row, same column, or in either diagonal of the grid.

MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. 26 **2.** $16 - 9 = 7$, $16 - 7 = 9$ **3.** $32 \div 4 = 8$, $32 \div 8 = 4$

4. A 5; B 17; C 37 **5.** 91 **6.** A 1; B 84; C 27; D 28; E 20

7. A 3454; B 5107 **8.** A 18 000; B 80; C 70

9. A 19 575; B 286 412 **10.** A 122 R 46; B 255, R 257

UNIT A: THE ARITHMETIC OF WHOLE NUMBERS

Module 3: Problem Solving

General Objectives

To improve the student's skill in writing and solving simple linear equations.

To focus attention upon the various aspects of solving word problems.

	Performance Objectives	Pupil Text	Reteach-Reinforce	Related Activities
20	**RED** Given a flow chart for an equation, the student will be able to write the equation.	A-60, 61	**WB** 21	**PD** f-1, 2
21	**RED** Given an appropriate word problem, the student can write an equation for the problem.	A-60, 61	**WB** 21	**PD** f-1, 2, 3, 4
22	**RED** Given a linear equation, the student will be able to solve the the equation by using inverse operations.	A-62, 63	**WB** 22 **DM** 17	**PD** f-1, 2, 3, 4 **SWM 2** 178, 179 **SWM 3** 50, 51
23	**RED** Given an appropriate word problem, the student can estimate the answer to the problem.	A-68, 69	**WB** 24	**SWM 1** 178, 179
24	**YELLOW** Given an appropriate word problem, the student can write and solve an equation for the problem.	A-64, 65 66, 67	**WB** 23	
25	**YELLOW** Given appropriate word problems, the student can use estimation and problem-solving techniques to solve the problems.	A-65, 66, 67		**SWM 2** 182
26	***GREEN*** *Given appropriate word problems, the student can use diagrams, logical reasoning, or other problem-solving techniques to solve the problems.*	A-70, 71 A-72, 73	**WB** 25 **DM** 18	**SWM 2** 180, 181
	Reviewing the Ideas	A-74	**WB** 26	

MATHEMATICS

Since this module deals with applications of mathematics to problem solving, there are no new mathematical concepts introduced in this module.

TEACHING THE MODULE

Vocabulary

equation	solution
inverse flow chart	variable

In this module several facets of problem solving are examined:

(1) choosing correct operations
(2) writing and solving related equations
(3) finding the significant information in problems
(4) estimating answers
(5) using drawings to help solve problems
(6) reasoning in problem solving

Careful treatment of these facets should help students to become better problem solvers.

In your efforts to attain the objectives of this chapter, you should keep in mind that considerable research supports the contention that the ability to do rote computation and the ability to solve problems are relatively independent abilities. Thus, it is possible for a student to be quite proficient in arithmetic computations but quite weak in problem solving. It is equally true that some students may be good problem solvers but poor computers.

In teaching the module, it is also important to

realize that some students have an aversion to word problems in mathematics. Such students are usually those who have some verbal difficulties, and a special effort must be made to improve these students' attitude toward solving word problems. The basic teaching strategy of this series – Investigation, Discussion, Utilization – should be a great help in this effort.

Probably the most important phase of the teaching strategy for the module is *discussion*, not just in the regular Discussion stage, but in discussion of the methods of attack on the problems at the Utilization stage as well.

We encourage you to allow students to work in small groups rather than individually in this module so that ideas may be freely exchanged and discussed.

You might also consider covering the module by weaving one or two lessons at a time in with work on later modules. The lessons could be presented in this manner at any time following coverage of Module B

In general, try to motivate students with the challenge of problem solving, but do not overemphasize the material to the extent that students lose interest.

Lesson Schedule

Although each lesson is intended to constitute about one day's work, you may find that some lessons will require more time for some students. Plan to allow about 8 days with faster students and up to 10 or 12 days with slower students.

Evaluation

Probably the best way to assess students' progress in this module is through a day-by-day evaluation based on observing and talking to your students as they work through it.

Any teacher-made test for the module should be designed with the realization that most students require considerable time to read, interpret, and solve word problems. The self-evaluation test (Test Yourself) could serve as a reasonable model for a teacher-made test, or you might prefer to use the Module Achievement Test available from the publisher as part of the Evaluation Program for this series.

RESOURCES FOR ACTIVE LEARNING

Activities

ESS: Attribute Games and Problems, Webster, McGraw-Hill

Experiments and Puzzles in Logic, Midwest Publications

Let's Play Games in Mathematics, Vol. 7, "Baseball Problem," pp. 56–57, National Textbook Co.

Madison Project: *Explorations*, Part 2, Addison-Wesley

Nuffield Project: *Logic*, Wiley

Problem Solving in the Mathematics Laboratory, Prindle, Weber & Schmidt

Manipulative Devices

Colored cubes (Addison-Wesley)

Flowchart template (Addison-Wesley)

RelationShapes (Addison-Wesley)

Games and Puzzles

Beeline (Selective Educational Equipment)

Chess, checkers, 3-D Tic-tac-toe (school supplier)

GO (Mind/Matter; Sigma)

Helix (Sigma)

Kalah (Mancala) (Creative Publications; Mind/Matter)

MEM (Selective Educational Equipment)

Mudcrack (Creative Publications)

Nine-Men-Morris (Mill Game) (Creative Publications)

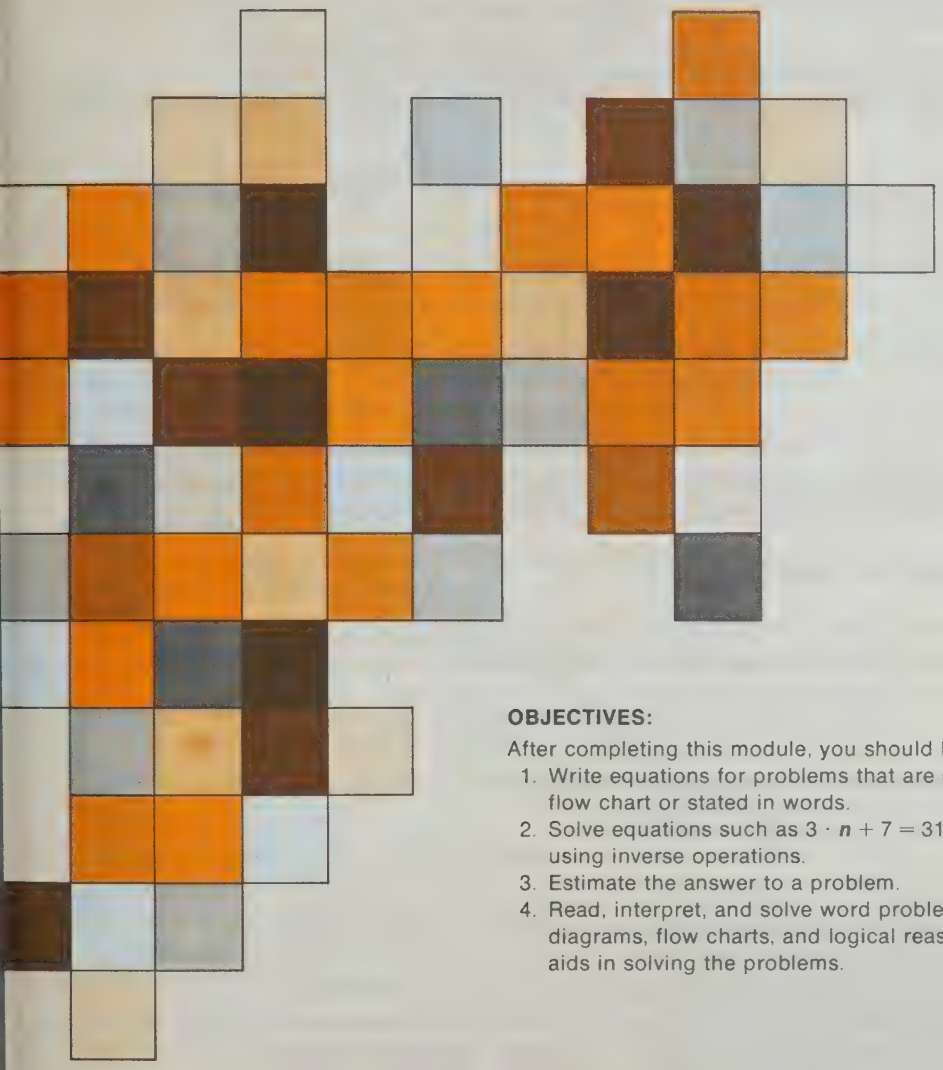
Score Four (Cuisenaire Co.)

Tac-Tickle (Wff'N Proof)

Vector (Cuisenaire Co.)

Wff'N Proof (Cuisenaire Co.; Wff'N Proof)

MODULE 3: *Problem Solving*



OBJECTIVES:

After completing this module, you should be able to:

1. Write equations for problems that are given by a flow chart or stated in words.
2. Solve equations such as $3 \cdot n + 7 = 31$ by using inverse operations.
3. Estimate the answer to a problem.
4. Read, interpret, and solve word problems using diagrams, flow charts, and logical reasoning as aids in solving the problems.

Objective

Given a flow chart which suggests an equation, the student can write the equation.

PREPARATION

Review the use of flow charts. On the chalkboard or overhead projector display some flow charts involving one or two operations and have students mentally compute the output numbers for input numbers suggested by other students or by you.

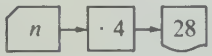
INVESTIGATION

The purpose of the Investigation is to provide a step-by-step procedure for writing simple equations. You should not ask students to solve the equations at this time. Solution of equations will be considered in later lessons. Encourage students to observe and discuss the fact that the equations for the last two flow charts require the use of parentheses.

DISCUSSION

Exercise 1 introduces the word *variable* to the student. The flow chart in Exercise 2 differs from those in the Investigation in that the operations are denoted by the standard symbols rather than spelled out.

Plan to spend time reading through and discussing the word problems in Exercise 3 carefully; the problem of translating an English sentence into a mathematical equation requires careful guidance. Illustrate each sentence with a flow chart to help students see the steps.

3A. 

$4 \cdot n = 28$

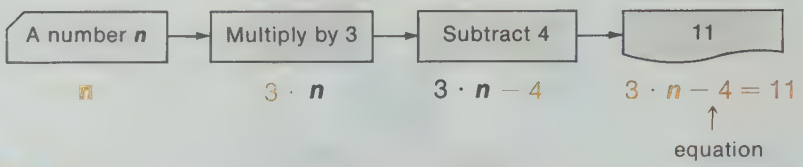
3B.

$3 \cdot n - 1 = 11$

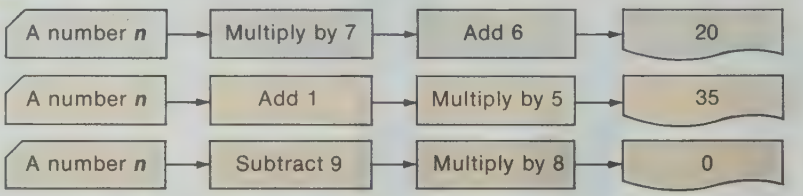
Writing Equations

Investigating the Ideas

The flow chart below suggests an equation.



Can you write an equation for each of the flow charts below?

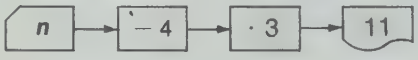


Discussing the Ideas

1. The letters used in equations are called **variables**. A variable is used to represent any number. Any letter may be used as a variable. What is the variable in each of these equations?

   A $y + 7 = 13$ B $2 \cdot x + 19 = 37$ C $14 - t = 6$

2. The flow chart below suggests an equation.

   Which equation do you think it is?

   A $3 \cdot n - 11 = 4$ B $3 \cdot n - 4 = 11$ C $(n - 4) \cdot 3 = 11$

3. The equation $n + 3 = 12$ is a mathematical way of writing the sentence "If three is added to a certain number, the sum is 12."

   The flow chart for the equation is:

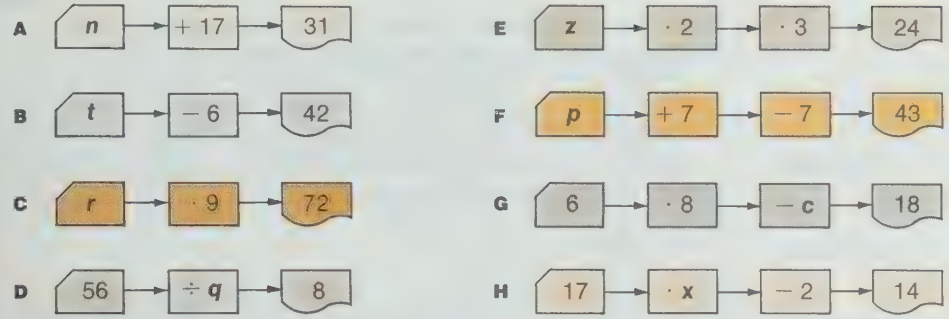
   Write an equation for each of these statements. (A flow chart may help you think about the equation.)

   A When a certain number is multiplied by 4, the product is 28.

   B If a certain number is multiplied by 3 and then 1 is subtracted, the result is 11.

Using the Ideas

1. Write an equation for each flow chart.



2. Choose a variable for the unknown number in each sentence and then write an equation for each sentence.
   - **A** When seven is added to a certain number, the sum is 19.
   - **B** When 38 is subtracted from a certain number, the difference is 29.
   - **C** When a certain number is multiplied by 8, the product is 96.
   - **D** When a certain number is multiplied by 2 and then 17 is added to the product, the result is 35.

3. Sometimes parentheses must be used when writing equations.

   EXAMPLE: When the sum of a certain number and 7 is multiplied by 3, the result is 21.

   EQUATION: Let n represent the "certain number." $3 \cdot (n + 7) = 21$

   Write an equation for each of these sentences.
   - **A** When 4 is subtracted from a certain number and the difference is multiplied by 5, the result is 15.
   - **B** When a certain number is added to 9 and the sum is multiplied by 6, the result is 120.
   - **C** When the sum of 19 and 12 is subtracted from a certain number, the result is 11.

think

"If you had been born in the year of 1980 then you would be 45 years old in the year 45^2 or 2025 A.D." said Jean. "That is true" replied Carlos, "But my grandmother was n years old in the year n^2 and she is still living!" In what year was Carlos' grandmother born?

More practice, page S-8, Set 14

UTILIZATION

Keep in mind that the objective of the lesson is to enable students to write equations rather than to solve equations.

The equations for the flow charts in Exercise 1 should be readily found by most students, but the equations for the sentences in Exercise 2 and 3 are likely to be more difficult. In discussing these problems, a flow chart for each equation might be a helpful teaching tool.

EXTENSION

Remedial: Provide additional practice exercises involving flow charts and equations. (Also see Supplementary Exercise Set 14 on page S-8 and page 21 of the *Workbook*.)

Enrichment: Ask students to find the numbers for the variables in the flow charts for Exercise 1.

Think Solution

Carlos' grandmother was born in 1892. Note that $44^2 = 1936$, and if Carlos' grandmother was 44 years old in the year 1936, she was born in the year 1892. If we try 43, we see that $43^2 = 1849$, which would mean a birth year of 1806. A person born in 1806 is not likely to be living today!

Assignments

Minimum 1A–D, G–H, 2. Average 1–3. Maximum 1–3.

Objective

Given an equation or a flow chart for an equation, the student can find the solution for the equation by using an inverse flow chart.

PREPARATION

Prepare a brief review of flow charts and equations derived from flow charts.

INVESTIGATION

The purpose of the Investigation is to give the student a technique for solving certain simple linear equations. Rather than use the more elaborate axiomatic approach to solving equations that would be used in a high school algebra class, we simply help the student think about using inverse operations (which "undo" the operations that were performed on the number) to find a solution.

Have the students read through the Investigation slowly. The important idea here is not so much to solve the equation, but to relate the problem, its equation, its flow chart, and the inverse flow chart to each other.

DISCUSSION

The logic that underlies the solving of equations is beyond the scope of most students at this level. Briefly, when we solve an equation by using a sequence of steps such as

$12 \cdot n + 15 = 111 \rightarrow 12 \cdot n = 96 \rightarrow$
$n = 96 \div 12 = 8,$

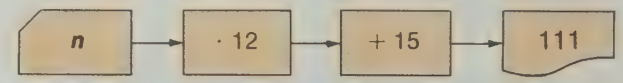
we have shown that *no number other than 8 could* be a solution to the equation. However, we have not shown that 8 is a solution. It is necessary to substitute 8 for n in the equation and carry out the computation to show $12 \cdot 8 + 15 = 111$. This step is usually called a "check" of the solution. It is an important step in more advanced algebraic work with equations where some numbers we find *as possible solutions* do not always turn out to be solutions to the equation in question.

In discussing Exercise 5, try to get students to solve the equation by thinking about inverse operations. Thus $(18 - 11) \cdot 2 = n$, hence $n = 14$.

Solving Problems with Flow Charts

Investigating the Ideas

The flow charts below may help you to understand how to solve equations for certain types of problems.

Problem	A certain number is multiplied by 12, and 15 is added to this product. The result is 111. What was the original number?
Equation	Let ***n*** represent the number. $12 \cdot n + 15 = 111$
Flow chart for the equation	A 
Inverse flow chart	B

Can you find the output number ***n*** for flow chart B?	What is the input number ***n*** for flow chart A?

Discussing the Ideas

1. How do the instructions for flow chart A compare with those for flow chart B?

2. Explain why flow chart B is called the **inverse flow chart** of A.

3. The number for the variable in an equation that will form a true statement is called a **solution** for the equation. How can you check to see if the output number for flow chart B is a solution to the equation $12 \cdot n + 15 = 111$?

4. You can think of flow chart B as "undoing" the operations in flow chart A. What would the inverse flow chart be for this one?

5. What is the solution to the equation $(n \div 2) + 11 = 18$? Explain how to solve the equation?

Using the Ideas

In Exercises 1, 2, and 3, study flow chart A. Then give the instructions and numbers that go with each part of inverse flow chart B.

1. A: n → · 36 → + 24 → 1500

   B: → → → →

   A B C D

2. A: n → − 20 → ÷ 7 → 2

   B: → → → →

   A B C D

3. A: n → − 48 → + 14 → · 7 → 245

   B: → → → → →

   A B C D E

4. Write an equation for flow chart A in Exercises 1, 2, and 3. Then use flow chart B to help you solve the equations.

5. Solve each equation. Use flow charts if necessary.

   A $(n - 7) \div 4 = 14$

   B $n \cdot 9 + 6 = 33$

   C $(n + 17) \cdot 14 - 209 = 71$

   D $\left(\frac{n}{2} - 9\right) \cdot 7 = 14$

6. A Construct a flow chart for the following problem:
   If 427 is added to a certain number n and the sum is multiplied by 12, the result will be 5760.

   B Construct the inverse flow chart for part A and then find the output number for this flow chart.

UTILIZATION

Exercises 1 through 4 provide practice in constructing inverse flow charts and writing equations associated with the flow charts.

For Exercise 5, encourage students to construct flow charts for the equations that present difficulties. Then use an inverse flow chart to solve the equations. If this work is covered quickly by the students, they can proceed to the following page for additional practice.

Solutions, Exercise 4

1. $n \cdot 36 + 24 = 1500$; $n = 41$
2. $(n - 20) \div 7 = 2$; $n = 34$
3. $(n - 48 + 14) \cdot 7 = 245$; $n = 69$

EXTENSION

Page 22 of the *Workbook* and page 17 of the *Duplicator Masters* offer exercises that may be assigned to provide further practice with the ideas presented in this lesson.

Remedial: Have the students find solutions to the equations in Exercise 1 on page A-61.

Enrichment: Have the students find solutions for the equations they wrote for Exercises 2 and 3 on page A-61.

Assignments

Minimum 1–4. Average 1–6. Maximum 1–6.

Objective

Given a flow chart for an equation, the student can write and solve an equation for it.

UTILIZATION

This page provides additional practice with the type of equations introduced in the previous lesson. Work through the solution to Exercise 1A with the students before assigning the other exercises. You should expect that most students will be able to complete these problems independently. Encourage your students to check their solutions by substituting each particular solution for the input number in the related flow chart.

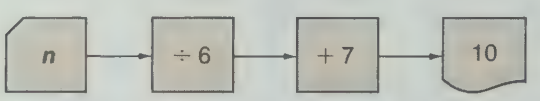
Assignments

Minimum 1.
Average: Even-numbered exercises.
Maximum All exercises.

Writing and Solving Equations

1. Write an equation for each flow chart. Then solve the equation. Think about inverse operations and inverse flow charts to help you solve the equations. Part A is completed as an example.

A

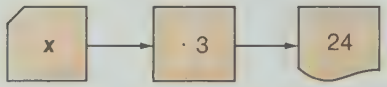


EQUATION: $(n \div 6) + 7 = 10$

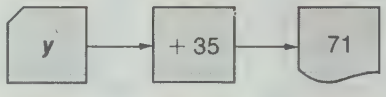
SOLUTION: If $(n \div 6) + 7 = 10$, then $n \div 6 = 10 - 7$. And if $n \div 6 = 3$, then $n = 6 \cdot 3 = 18$

CHECK: $(18 \div 6) + 7 = 10$

B

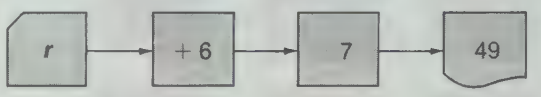


C

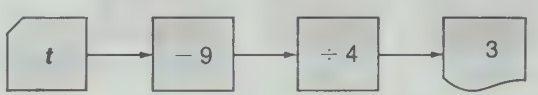


D

E



F



G

2. Solve the equations. Check your answers.

A $y + 35 = 71$

B $13 \cdot z = 91$

C $r - 12 = 99$

D $3 \cdot a - 7 = 11$

E $21 - 7 \cdot b = 0$

F $14 \cdot t + 8 = 120$

G $(x \div 2) + 5 = 9$

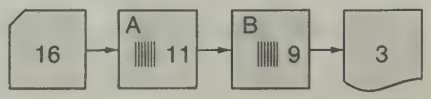
H $(n - 4) \div 6 = 2$

I $(24 \cdot w) \div 9 = 8$

J $135 - 9 \cdot t = 81$

think

The operation signs in the instruction boxes A and B have been covered. What are the two operations?



Solving Word Problems With Equations

Write an equation for each word problem, then solve the equation.
The first exercise is completed as an example.

1. A certain number is multiplied by 8 and then 11 is subtracted from the product. The result is 13. What is the number?

SOLUTION: Let n represent the number.

$$8 \cdot n - 11 = 13$$
$$8 \cdot n = 13 + 11 = 24$$
$$n = 24 \div 8 = 3$$
$$n = 3$$

CHECK: $8 \cdot 3 - 11 = 13$

2. Diane is 153 centimetres (cm) tall. She is 19 cm taller than Elaine. How tall is Elaine?

3. A man averaged 63 kilometres (km) per hour on a trip. If his trip was 441 km, how long did it take him to make the trip?

4. Bill said, "I am thinking of a number. If you multiply it by 10 and then subtract 10, you get 100." What was Bill's number?

5. Karen had a recipe for a small number of cookies but she wanted to make more. First she doubled the amount of flour and then used another extra portion of flour. Altogether she used 7 portions. How many portions of flour did the original recipe call for?

6. Janice had 23 pieces of candy. She ate three pieces and then divided the remaining pieces equally among 4 of her friends. How many pieces of candy did each friend get?

7. If 6 is added to a certain number and the sum is multiplied by 4, the result is 40. What is the number?

8. Larry scored the same number of points in each of 4 games of a card game. In the next game he lost 47 points so that his final score was only 1 point. How many points did he score in each of the first 4 games?

9. Sarah measured the distance around her classroom and found that it was 40 metres. The length of the room was 12 metres. What was the width of the room?

10. David had a certain amount of money. He spent half of it for a notebook and then spent 30 cents for a milk shake. Finally, he had 14 cents left. How much money did he have in the beginning?

think

A man needs to take a fox, a goose, and a sack of corn across a river. He has a boat in which he cannot take more than one across at a time. He cannot leave the fox and the goose together or the goose and the corn, for one will eat the other. How can he get all three across the river, taking one at a time with him in the boat? Draw a diagram to show your solution.

Objective

Given a word problem, the student can write and solve an equation for the problem.

UTILIZATION

Plan to spend some time discussing Exercise 1 and its solution before assigning the remainder of the exercises.

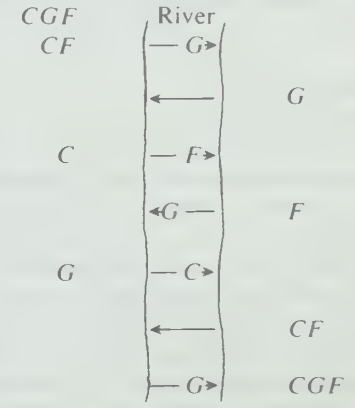
EXTENSION

Page 23 of the *Workbook* provides problems that would be appropriate for further practice or remedial work.

Think Solution

The key to this problem is to take the goose back to the original side of the river after having taken the fox across.

F = fox, G = goose, C = corn



Assignments

Minimum 2–5. Average 2–8
Maximum 2–10.

Objective

Given word problems, the student can recognize the necessary information and solve the problem.

PREPARATION

No special preparation is necessary before students undertake the Investigation.

INVESTIGATION

Have the students begin immediately on the Investigation. The first problem may be easily solved by most students, but they may think it strange that, unlike most problems in mathematics texts, it contains superfluous information. The second problem may prove even more puzzling, because it cannot be solved on the basis of the information given. Do not allow students to spend excessive time on the Investigation; some of them might become frustrated in their efforts to solve the second problem.

DISCUSSION

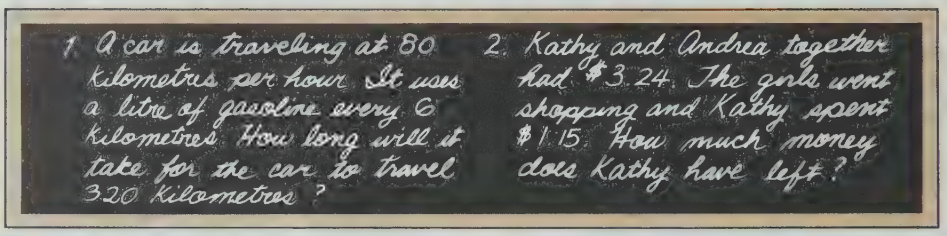
Exercises 1 and 2 should bring out the purpose of the lesson, which is to stress the importance of reading word problems carefully and selecting the information pertinent to the solution of the problem.

Exercises 3 and 4 are also significant Discussion Exercises. In everyday living we often encounter situations in which we have available a mass of data from which we must select only the information relevant to the solution of a given problem. At other times, however, we may be faced with situations in which we lack sufficient information to solve a given problem. Thus, the Discussion Problems parallel our real-life experiences. Provide more examples if needed.

Finding Important Information in Problems

Investigating the Ideas

Study the problems shown on the chalkboard.


1. A car is traveling at 80 kilometres per hour. It uses a litre of gasoline every 6 kilometres. How long will it take for the car to travel 320 kilometres?
2. Kathy and Andrea together had $3.24. The girls went shopping and Kathy spent $1.15. How much money does Kathy have left?

?	Can you solve either one or both of these problems?	Give your solutions and be able to explain them.

Discussing the Ideas

1. A Which problem contained some numerical information that you did not need in order to solve the problem?
   B What was the information?

2. A Which problem did not contain enough information for you to solve the problem?
   B What other information would you need in order to solve the problem?

3. None of these problems contains enough information to solve the problem. What extra facts are needed?
   A There are 90 students in the seventh grade at Centre Junior High. They plan to use buses on a field trip. How many will they need?
   B Students in the seventh-grade class sold 180 tickets for the class play. Adult tickets were $1.00 and student tickets were 50¢. How much did they earn for the class?

4. In each of these problems, information that is not needed is given. Pick out the unnecessary information.
   A A set of five Canadian coins has a value of 51¢. There is only one penny in the set. What are the other coins?
   B A number is multiplied by 5 and to this product 4 is added. The original number is an odd number less than 10 and the final result is 39. Find the original number.

Using the Ideas

Try to solve each of the problems. Some may not contain enough information while others contain more than enough.

1. Jill is in the seventh grade. Her brother is 14 years old. Jill added 365 to her age in years, subtracted 52, divided by 13, and ended with a result of 25. How old is Jill?

2. The distance between two cities is 2351 kilometres. The larger city has a population of 129 740. A jet plane has already flown 854 kilometres from the larger city toward the smaller city. How far is the plane from the smaller city?

3. In five days of vacation driving the Smith family drove 425, 513, 524, 411, and 387 kilometres. They used 294 litres of gasoline. If their odometer reading was 23 496 when they left home, what was it after the five days of driving?

4. There are 1512 trees in a large orchard with 36 trees in each row. 432 of the trees are apricot trees. How many rows of trees are there?

5. From a board that was 3 metres long, Dave cut off a piece 1 metre 26 centimetres long. How long was the piece of board that was left?

6. Mrs. Charles saw some small boxes of strawberries priced at 59¢ a box. How much would she have to pay for a crate of strawberries?

7. Carla found that the average length of one of her paces was 55 centimetres. About how many paces would she have to walk to go a distance of one kilometre?

8. A box of a certain kind of cereal contains 380 grams of cereal. How many kilograms of cereal would there be in a carton of cereal?

9. If you save $2.00 per day, how many weeks would you have to save until you would have enough money to buy a bicycle that costs $112?

10. Sue and Jane together had $2.95. Jane spent 98 cents. How much money did Jane have left?

★ 11. There are 463 students in grades 7, 8, and 9 at Jordan Junior High School. There are 31 more students in the eighth grade than in the seventh grade. There are 17 more ninth graders than eighth graders. How many students are there in each grade?

think

Cathy has 10 coins. None of the coins is a half dollar. The value of the coins is 59 cents. Can you determine which coins she must have?

UTILIZATION

Assign the exercises according to the needs and abilities of your students. The exercises are grouped so that Exercises 1–4 contain more information than is needed. Exercises 5, 7, and 9 can be solved, but students must use some information not explicitly stated in the problem, such as the number of centimetres in a metre, the number of metres in a kilometre, and the number of days in a week. In Exercises 6, 8, and 10, however, the information given is not sufficient to yield reliable solutions.

Solutions, Exercise 11

The solution to starred Exercise 11 can be found as follows:

$$\begin{aligned} \text{Let } x &= \text{Grade 7 students.} \\ x + 31 &= \text{Grade 8 students} \\ x + 31 + 17 &= \text{Grade 9 students} \end{aligned}$$

Then $x + (x + 31) + (x + 31 + 17) = 463$.

$$\begin{aligned} 3x + 79 &= 463 \\ 3x &= 384 \\ x &= 128 \text{ (Grade 7)} \\ x + 31 &= 159 \text{ (Grade 8)} \\ x + 31 + 17 &= 176 \text{ (Grade 9)} \end{aligned}$$

Think Solution

Of the 10 coins, 4 are necessarily pennies since the value of the coins is 59¢. The problem, then, is to find 6 coins such that their value is 55¢. Trial and error readily reveal that the other 6 coins must be either 1 quarter, 1 dime, and 4 nickels, or 5 dimes and 1 nickel.

Assignments

Minimum 1–6. Average 1–9. Maximum 1–11.

Objective

Given a word problem, the student can use rounding to give an estimate of its solution.

PREPARATION

A good preparation for this lesson would be to review rounding of numbers (pages A-12 and A-13) and mental computation of sums, differences, products, and quotients involving the rounded numbers (pages A-46 and A-47).

INVESTIGATION

This lesson can be extremely important in helping students form good habits for estimating answers. While your students may have already had considerable experience in estimation in a strictly computational context, they must now be encouraged to apply these experiences to problem-solving situations in which they are asked to give estimates of the solution before they actually solve the problem.

Allow the students 3 or 4 minutes to study and record their estimates for the three problems in the Investigation.

DISCUSSION

Student estimates for the Investigation problems will probably vary, especially for problems 1 and 3.

If students rounded the population of Thunder Bay to 100 000, their estimate for the population of Toronto would be 600 000. However, some students may round the Thunder Bay population to 110 000, in which case their estimate of the Toronto population would be 660 000.

For problem 2, students should round 4 hours and 53 minutes to 5 hours and round 3974 kilometres to 4000 kilometres, which would give an estimated average speed of 800 km/h.

In discussing Exercise 3, students should observe that the possible extreme expenditures would be $6 and $9.90, but if they base their estimates on the average of the most expensive and the least expensive items, the likelihood is that their estimate will be closer to the actual cost than if they base their estimate on either extreme.

Estimating Answers

Investigating the Ideas

In many mathematical problems, you can come very close to the correct answer by making a careful estimate.

Can you give an estimate for each problem below?

1. The population of Thunder Bay, is 115 000. The population of Toronto is more than six times the size of Thunder Bay. What is the approximate population of Toronto?

2. A non-stop flight between two large cities took 4 hours and 53 minutes. If the distance between cities is 3974 kilometres, what was the average rate of speed?

3. Maryann bought ten items at the supermarket. Every item cost between 59¢ and $1.00. What is the total cost of the items?

Discussing the Ideas

1. Explain the reasoning you used in estimating the answer for problem 1 in the Investigation.

2. Explain how you made your estimate for problem 2.

3. What was your estimate for problem 3 and how did you make it?

4. Why do you think it is important to make estimates of answers to problems?

5. A factory worker earns $132.50 per week. His weekly deductions for taxes, insurance, and so forth, amount to $23.67. What do you estimate is his take-home pay for a year?

Using the Ideas

After each problem, three possible answers are given. None of them are exact answers but one of them will be quite close to the correct result. Use estimation to pick out this answer.

1. A basketball team scored between 45 and 65 points in each of 19 games. About how many points did they score in all of their games? (ANSWERS: 1500; 1100; 700)

2. For every 4 students at a track meet, there were 3 adults. If 124 student tickets were sold, how many adult tickets were sold? (ANSWERS: 40; 90; 150)

3. The distance from the earth to the sun is about 149 000 000 kilometres, and that from the earth to the moon is about 382 000 kilometres. About how many times as far is it from the earth to the sun as from the earth to the moon? (ANSWERS: 4 000 000; 4000; 400)

4. What is the sum of the whole numbers from 1 to 100? (ANSWERS: 500; 5000; 50 000)

5. Cris counted the number of times the letter "e" occurred in each of 10 lines of type. Here is the data.

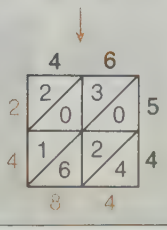
Line	1	2	3	4	5	6	7	8	9	10
Number of e's	5	8	6	8	2	6	5	7	5	7

About how many times, on the average, did "e" occur in each line? (ANSWERS: 4; 6; 8)

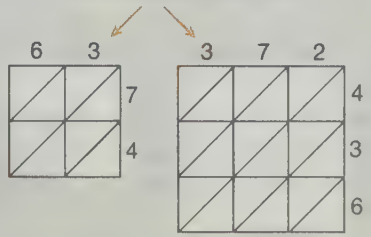
★6. Suppose that a rocket ship can travel from the earth to Mars in 260 days. It takes 45 years and 273 days to reach the planet Pluto. Approximately how many round trips could one rocket ship make between the earth and Mars while a second ship was traveling to Pluto? (ANSWERS: 30; 80; 6000)

think

The diagram shows an old method of multiplying **46** and **54** to get the product 2484



Using this method, find these products.



UTILIZATION

It is important to discuss these exercises after your students have completed them, so that a number of students have a chance to explain their methods for arriving at the estimates. This may enable other students to see alternative methods of thinking about a given problem and estimating the answer.

EXTENSION

See page 24 of the *Workbook* for problems that might be assigned for further practice.

Remedial: Provide the following problems with these directions: "Round the numbers in each problem and give an estimate for the answer."

1. What is the product of 987 and 864? ($1000 \cdot 900 = 900\,000$)
2. How much are 315 m^3 of soybeans worth at a price of \$2.89 per m^3? ($300 \cdot \$3 = \$900$)
3. A jet airplane travels at 680 km/h on a non-stop flight from Chicago to Miami. If the airline distance is 1973 km, about how long does the flight take? ($2000 \div 700 \approx 3$ hours)

Enrichment: Have students carry out an activity based on problem 3 of the Investigation. Ask them to write the numerals 60 through 99 on small cards or slips of paper and put them in a box. Then have them draw out ten cards, record the numbers, and find the sum. Have them repeat the experiment several times and then ask: "What is the average sum obtained by this method?"

Think Solution

The final product is obtained by starting in the lower right-hand corner and finding the sums between the diagonals that run from upper right to lower left. The carrying process occurs as in normal addition.

Notice that the idea of place value is operating essentially in the same way as in the normal multiplication algorithm. Historically, this is called the "lattice" or a "jalousie" method of multiplication.

Assignments

Minimum 1–3. Average 1–5. Maximum 1–6.

Objectives

The student can make a sketch to aid in solving a problem.

The student can use a given sketch to help solve a problem.

PREPARATION

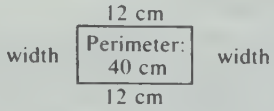
You may find it helpful to review some of the problem-solving techniques already covered in this module, such as reading carefully to find the relevant information and making estimates of answers.

INVESTIGATION

Many problems that students find difficult will seem simpler to understand and solve if they make a simple sketch, drawing, or diagram of the significant information of the problem. In the Investigation, the student is expected to discover that drawing B best illustrates the facts presented in the problem and should use this drawing to help solve the problem.

DISCUSSION

In discussing Exercise 1, many students may explain that they used a process of elimination in choosing the correct drawing. Drawing A shows the board divided in *half*, which does not reflect the stipulation that one piece is 3 times as long as the other. Nor does drawing C illustrate the given facts, because it shows that one piece is twice as long as the other. Therefore, the correct drawing is B. Student explanations may vary but a simple explanation is that from the problem and the drawing one can see that the board is 4 times as long as the shorter piece. Hence, the short piece is $200 \div 4 = 50$ cm long, and the longer piece is $3 \cdot 50$ cm $=$ 150 cm long. The diagram for Exercise 3 might look like this:

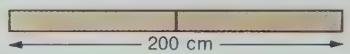
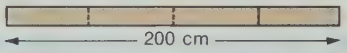
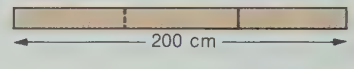


For Exercise 3C, the response should be something like this: Since $P = 2(l + w)$ and $\frac{P}{2} = 20$ cm, then $l + w = 20$ cm. We know that $l = 12$, so $w = 20 - 12$, or 8 cm.

Using Drawings to Solve Problems

Investigating the Ideas

A drawing or sketch of a problem may often help you solve a problem. Study the problem and then decide which drawing best illustrates the problem.

PROBLEM	DRAWINGS
A board 200 centimetres long is cut into two pieces. One piece is 3 times as long as the other. How long is each piece?	A  B  C 

Can you use the correct drawing to help you solve the problem?

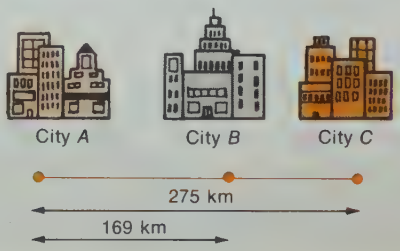
Discussing the Ideas

1. How did you decide which drawing was the correct one?

2. Explain how you used the drawings to solve the problem.

3. A Draw a diagram for this problem:
   The perimeter (distance around) of a rectangle is 40 centimetres (cm). The rectangle is 12 cm long. How wide is the rectangle?
   B What is the distance halfway around the rectangle?
   C If half of the perimeter is 20 cm, how can you use this fact to find the width of the rectangle?
   D Can you find the width in another way?

4. A Tell what facts are shown by the sketch at the right.
   B Make up a problem about the sketch and explain how to solve your problem.



Using the Ideas

In Exercises 1 through 5, make a drawing to help you analyze each problem. Then solve the problem.

1. A board is cut into two pieces. The shorter piece is 28 centimetres long and the longer piece is twice as long. How long was the board before cutting?

2. The perimeter of a rectangle is 50 metres. If it is 8 metres wide, how long is the rectangle?

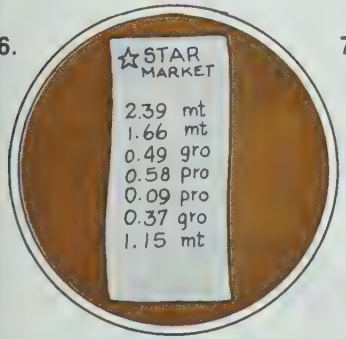
3. A triangle has two sides of equal length. The perimeter of the triangle is 20 cm. The length of one of the two sides of equal length is 8 cm. How long is the third side?

4. There are two rows of cards on a table. One row contains 12 cards. If we moved 5 cards from this row to the second row, the second row would have twice as many cards as the first. How many cards are in the second row?

★ 5. Two cars were 100 kilometres apart. They drove toward each other and met in 2 hours at a point which was 40 kilometres from the starting point of the slower car. How fast was each car travelling?

Each sketch or drawing in Exercises 6 through 11 suggests a problem. Write and solve your own problem for each.

6.



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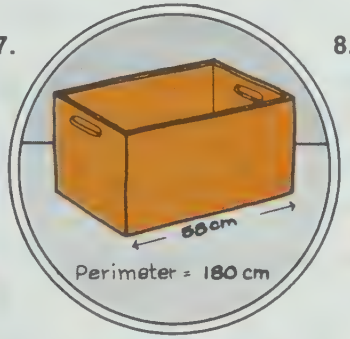

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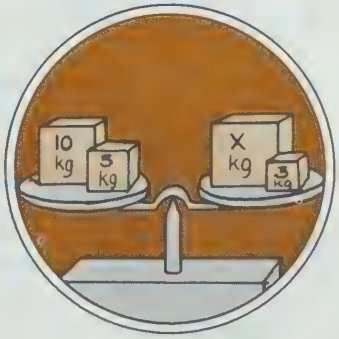

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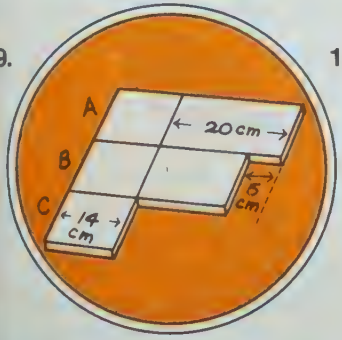

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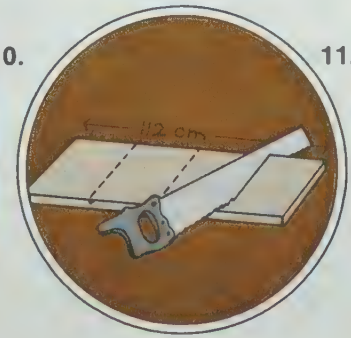

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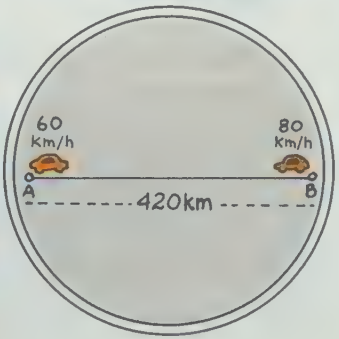


More practice, page S-9, Set 15

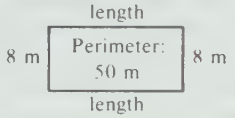
UTILIZATION

An important part of the teaching strategy for these problems is to allow ample time for discussion of these exercises. Students should be encouraged to make chalkboard sketches of the facts in each problem and then explain how their sketches were used to help interpret the problems.

Solutions, Exercises 1–5

1.

2.



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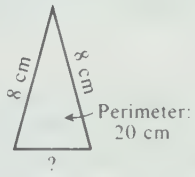

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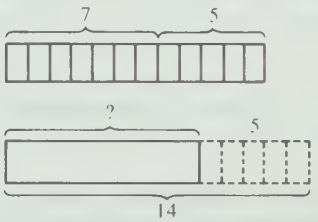

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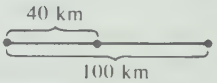


EXTENSION

Further practice with the ideas presented in this lesson is provided by Supplementary Exercise Set 15 on page S-9, page 25 of the *Workbook*, and page 18 of the *Duplicator Masters*.

Assignments

Minimum 1–3, 6, 7, 10. Average 1–3, 6–10. Maximum 1–11.

Objective

Given a word problem with a set of multiple choice answers, the student can use logical reasoning to choose the correct answer.

PREPARATION

Briefly discuss the necessity for careful reading of word problems before proceeding with the Investigation.

INVESTIGATION

The Investigation problem is a type of problem that is often incorrectly solved because the solver does not read carefully enough or fails to reason logically in reaching a solution.

Suggest that students write down their choice of solution. Then conduct an oral poll of the class and make a tally of the number of students who selected solution A, B, C, D, or E. Then begin discussion of the problem.

DISCUSSION

In dicussing Exercises 1 and 2, bring out the fact that the fastest boy can wax the car alone in 3 hrs, which would rule out choices A, B, C, D. The two boys together could wax the car in less than 3 hours, so 2 hours is a reasonable estimate and, in fact, the exact answer. (One boy can wax $\frac{1}{3}$ of the car in an hour, the other can wax $\frac{1}{6}$ of it in an hour, so together they can wax $\frac{1}{2}$ of the car in an hour, or all of it in 2 hours.)

Exercise 3 gives some practice with the type of problem found on the Using the Ideas page. Have students explain their choices. Some may be embarrassed to find that the answer for 3A is given in the question itself.

Investigating the Ideas

Careful thinking and reasoning are important in problem solving. Read the problem below carefully.

Jim can wax a car in 6 hours. Bill can wax a car in 3 hours. If they work together, how long will it take them to wax one car?

Which of these times is the best estimate for the answer to the problem?

A 9 hours **B** 6 hours **C** $4\frac{1}{2}$ hours **D** 3 hours **E** 2 hours

Discussing the Ideas

1. What is wrong with the reasoning of a student who thought that "9 hours" was the answer to the Investigation problem?

2. Explain the reasoning you used to select your estimate.

3. Unless you carefully read a problem and use logical reasoning, you may easily miss simple story problems. In each problem below, several answers are given but only one is correct. Find the correct answer and explain your reasoning for each problem.

A Harry had 20 hamsters. All but 7 died. How many were left?
ANSWERS: 27 13 20 7

B How many hours are there in 24 days?
ANSWERS: 12 24 144 576

C A baseball team has played 150 games. It has won 30 more games than it has lost. How many games has it lost?
ANSWERS: 120 60 90 180

Using the Ideas

In Exercises 1 through 4, choose the best answer for each problem. Think carefully!

1. John had 45 baseball cards. He gave Rick 8 of them. Now John and Rick have the same number of cards. How many do they have together?

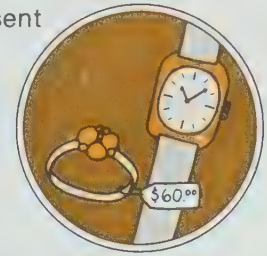
A 90 **B** 53 **C** 74 **D** 16

2. Mary spent half of her money for a birthday gift and $1.50 for lunch. She has $1.00 left. How much money did she have in the beginning?

A $5.00 **B** $2.50 **C** $10.00 **D** $1.25

3. Six years ago Sam was 6 years old and Mike was 18. What is the number by which you must multiply Sam's present age in order to get Mike's present age?

A 4 **B** 3 **C** 2

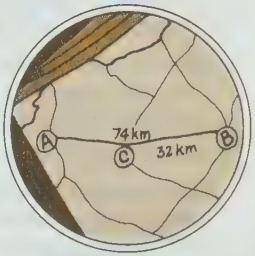


4. A ring cost $60. A watch cost $40 more than the ring. What was the total cost of the watch and ring?

A $20 **B** $100 **C** $160

In Exercises 5 through 8, solve the problems.

5. It is 74 kilometres from town A to town B. It is 32 kilometres from town B to town C, and town C is on the road between A and B. How far is it from town A to town C?



6. Joan is now 6 years old. Ed is twice as old as Joan. When Joan is 12, how old will Ed be?

7. Beth had $20. She spent $2.00 in one store and half of the remaining money in a second store. How much money did she have left?

★**8.** One pipe can fill a tank in 15 minutes. A second pipe can fill the tank in 12 minutes. If the first pipe is used for 10 minutes, then shut off and the tank is then filled using the second pipe, how long will it take in all to fill the tank?

think

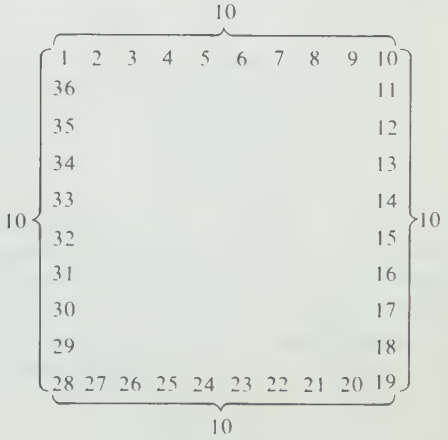
A man fenced his garden so that the fence had the form of a square. When he finished, there were 10 fence posts on each side. How many posts did he use?

UTILIZATION

Perhaps the most important phase of this lesson is the discussion that should follow after the students have completed the exercises. The alternate choices of answers for each exercise are answers that the student could easily make if he simply read the problem too quickly or his rationale was faulty. Give your students a chance to explain their reasoning and encourage a variety of explanations to establish the fact that there is often more than one way to interpret and solve a problem.

Think Solution

Students who use faulty logic will probably think there are 40 posts in all since $4 \cdot 10 = 40$. This kind of reasoning fails to take into account the fact that each corner post is common to two sides. Actually, there are $40 - 4$ posts, or 36 posts in all.



Assignments

Minimum 1–5, oral. Average 1–3, oral; 4–7. Maximum 1–8.

Objective

The student will demonstrate the ability to work with the ideas presented in this module.

UTILIZATION

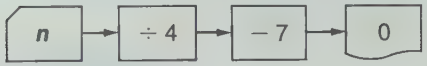
The review problems could be completed independently by each student or serve as a basis for small-group discussions by students. Students should be encouraged to refer back to those lessons pertaining to a given problem if it causes difficulty.

EXTENSION

Additional review problems for this module are provided on page 26 of the *Workbook*.

REVIEWING THE IDEAS

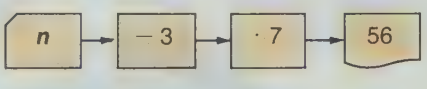
1. Write an equation for the flow chart.



2. Make the inverse flow chart for the one in Exercise 1.

3. Find the output number for the flow chart in Exercise 2.

4. Write and solve an equation for the flow chart.

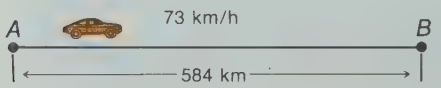


5. Solve the equations.
   - A $3 \cdot n + 7 = 19$
   - B $(n \div 3) + 1 = 6$
   - C $(n - 12) + 7 = 9$

6. 236 is added to 928, and then the sum is doubled. If this result is divided by 6, what is the quotient?

7. Which of the three answers is correct? The average life of a man is about 2 000 000 000 seconds. The giant turtle lives $\frac{1}{2}$ again as long as man. How many seconds is the average life of the giant turtle?
   - A 1 000 000 000
   - B 1 500 000 000
   - C 3 000 000 000

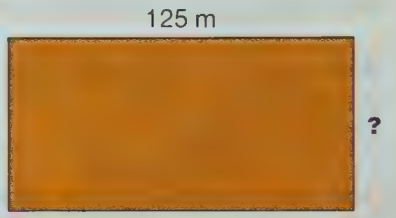
8. Write and solve your own problem for this picture.



9. Estimate the answers to the following following problems.
   - A A cubic centimetre of gold weighs about 19 grams. About how much does 43 cubic centimetres of gold weigh?
   - B What is the product of 987 and 864?
   - C How much are 315 kilograms of soybeans worth if the market price is $0.89 per kilogram?

10. Draw a diagram that will show the information in each problem. Then solve the problem.
    - A The length of a rectangle is 4 centimetres longer than its width. The perimeter of the rectangle is 56 centimetres. Find the length and width of the rectangle.
    - B A piece of wire 36 metres long is cut into two pieces. One piece is twice as long as the other. How long is each piece?

11. A rectangular lot is 125 metres long. If the distance around is 380 metres, how wide is the lot?



12. Janet is 11 years old. Her grandfather is five times as old as she is now. How old will her grandfather be when Janet is 20 years old?

TEST YOURSELF

1. Write the equation suggested by the flow chart.

2. Make the inverse flow chart for this flow chart.

3. What is the number for n in Exercise 2?

4. Solve the equation. Check your answer.

$$6 \cdot n - 15 = 39$$

5. Write an equation for this sentence:

   If 7 is added to a certain number and the sum is multiplied by 4, the result is 48.

6. Solve the equation in Exercise 5.

7. Estimate the answer to this problem:
   A jet airplane travels at 800 km/h. If it flies nonstop for a distance of 1592 km, about how long will the flight take?

8. Draw a diagram that will help you understand the problem. Then solve the problem.
   The perimeter of a triangle is 42 centimetres (cm). Two of the sides have the same length and the third side is 6 cm longer than each of the other two. How long is each side?

9. Which answer is correct?
   A pencil cost 10¢. A pen costs one dollar more than the pencil. How much will they cost together?
   A 90¢ B $1.10 C $1.20

10. Barry took 28 shots at the basket during a basketball game. He missed 3 times as many baskets as he made. How many baskets did he make?

RESEARCH PROJECTS

A Read about some of the famous mathematicians of the past. Write or give a report to your class about some of the interesting things about the lives of the mathematicians. Try to find what their main contributions to mathematics were. You may find that one or more of them may be of interest to you in other fields also.

Fermat	Euler	Archimedes
Gauss	Euclid	Newton

(see *Men of Mathematics* by E. T. Bell; New York: Simon and Schuster, 1962.) available from Musson Book Co.)

B Suppose you had 12 round pies that fit exactly into a shallow rectangular tray when they are arranged as in the diagram below.

Could you design a differently shaped tray that would hold these pies but take up less area than the rectangular one? How much less area would your tray take?

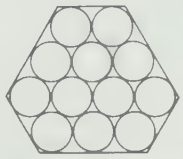
TEST YOURSELF

Keep in mind that development of skill in solving word problems is a long-range, continuing objective in mathematics. Satisfactory achievement of the objectives for this module cannot be assumed to insure future maintenance of word-problem skills.

Observe that the annotations on the student text page include correlations to the objectives listed at the beginning of the module.

RESEARCH PROJECTS

For Research Project B, some students will find it helpful actually to cut out twelve circular disks and experiment with various arrangements such as these:



Then the students can measure and compare the areas of the trays that would hold the various arrangements. This can lead to learning how to compute areas of different geometric shapes and the value of π, and to many other geometric ideas.

MATHEMATICAL RECREATION

This recreation is a probability game based on the *product* of the numbers coming up on the dice, rather than the sum. The scores assigned in the table are such that products that are equally likely to occur are assigned the same score. Thus, the products 6 and 12 have the same score value because they are equally likely to occur. A product of 6 occurs if one of the pairs $(3,2)$, $(2,3)$, $(1,6)$, or $(6,1)$ comes up. A product of 12 occurs if $(6,2)$, $(2,6)$, $(3,4)$, or $(4,3)$ occurs.

The probability for each set of events (products) is given below.

$P\ (4) = \frac{3}{36} = \frac{1}{12}$

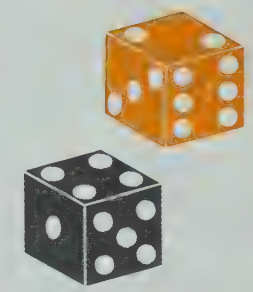
$P\ (6, 12) = \frac{8}{36} = \frac{2}{9}$

$P\ (1, 9, 16, 25, 36) = \frac{5}{36}$

P (all other products) $= \frac{20}{36} = \frac{5}{9}$

MATHEMATICAL RECREATIONS

PROBABILITY PRODUCT GAME



This is a game for two or more players. The players take turns tossing a pair of dice. Using the product of the numbers tossed, the player scores the number shown on the chart below. For example, if a 4 and a 2 come up on a toss of the dice, the product is 8. According to the chart, your score 5 points!

PRODUCT	SCORE
4	20
6 or 12	10
1, 9, 16, 25, or 36	15
2, 3, 5, 8, or 10	8
15, 18, 20, 24, 30	5

The first player to score a total of 100 points wins the game.

MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. $6 \cdot n + 2 = 44$ **2.**

3. 12

4. 9 **5.** $(n + 7) \cdot 4 = 48$ **6.** 5 **7.** 2 hours **8.** 12 cm, 12 cm, 18 cm

9. c $1.20 **10.** 7 baskets

A–76

UNIT A: THE ARITHMETIC OF WHOLE NUMBERS

Module 4: Number Theory

General Objectives

To review factors and multiples of numbers.
To provide experiences with prime and composite numbers.
To develop further skills in working with exponents and powers of numbers.
To review and strengthen understanding of the greatest common factor and the least common multiple of two numbers.

	Performance Objectives	Pupil Text	Reteach-Reinforce	Related Activities
27	**RED** Given a number less than 100, the student can list all of its factors.	A-80, 81	**WB** 27 **DM** 19	**SP** g-1
28	**RED** Given two numbers, the student can determine if one number is a factor of the other.	A-80, 81	**WB** 27 **DM** 19	
29	**RED** The student can identify prime and composite numbers.	A-82, 83		**SP** g-1 **SWM 1** 152, 153 **SWM 3** 20, 21
30	**RED** Given appropriate composite numbers, the student can find their prime factorizations.	A-84, 85	**WB** 28 **DM** 20	**SP** g-1
31	**RED** Given two numbers, the student can find their greatest common factor.	A-88, 89	**WB** 30 **DM** 21	**ASC** W-11 **SP** g-2
32	**RED** Given two numbers, the student can find their least common multiple.	A-90, 91	**WB** 31 **DM** 22	**ASC** W-12 **SP** g-3
33	**YELLOW** The student can use exponential notation for the prime factorization of a number.	A-86, 87	**WB** 29 **DM** 20	**SP** g-4
34	***GREEN*** *Given three numbers, the student can find the least common multiple of the numbers.*	A-91		**SP** g-3
	Reviewing the Ideas	A-92	**WB** 30	

MATHEMATICS

Consider the following definition.

If a, b, and c are whole numbers such that $a \cdot c = b$, then a and c are whole-number *factors* of b.

With this definition, we can define *prime number*.

The set of prime numbers is the set of all whole numbers x such that x has exactly two factors.

Every number greater than 1 that is not prime is a product of one and only one collection of prime numbers. Such numbers are called *composite numbers*. For example, $6 = 2 \cdot 3$; $8 = 2 \cdot 2 \cdot 2$; $9 = 3 \cdot 3$; $100 = 2 \cdot 2 \cdot 5 \cdot 5$; and so on. Expressing a composite number as a product of primes is called the *complete factorization of a number*. The fact that each composite number can be expressed as a product of primes in exactly one way (without regard to order) is called the Fundamental Theorem of Arithmetic.

Closely connected to the idea of the factors of a number is the idea of the *common factors* of two numbers and the *greatest common factor* of two numbers. As far as its use in arithmetic is concerned, the most important application of greatest common factor is in changing fractions to lowest terms. That is, if we have a fraction which is not in lowest terms, we can change this fraction to lowest terms by

dividing numerator and denominator by the greatest common factor of the two numbers. We define greatest common factor simply as the largest number in the set of factors common to both numbers.

Consider the following definition of the *multiples* of a number.

> If a and b are whole numbers and c is the product of a and b, then c is said to be a multiple of a (and of b).

Note that zero is a multiple of every whole number. For this reason, when we speak of the least common multiple of two numbers, we mean the smallest number other than zero that is a multiple of both numbers. The concept of least common multiple is valuable in the study of fractional numbers because the least common multiple of two denominators is the least common denominator for two fractions, and we use a common denominator to find the sum or difference of two fractional numbers.

TEACHING THE MODULE

Materials

Centimetre ruler, graph paper, scissors, colored strips, adding machine tape.

Vocabulary

common factor
common multiple
composite number
even number
factor
factor tree
greatest common factor (GCF)
intersection of two sets
least common multiple (LCM)
multiple
odd number
prime factor
prime factorization
prime number
union of two sets

Most of the topics in this module will have been covered in earlier books, so you should adjust your teaching to your students' level of understanding. If they have a good grasp of the ideas, this is an excellent module in which to provide considerable enrichment material. The teaching commentary accompanying each lesson often recommends topics for enrichment. Also, call the students' attention to the Research Projects at the end of the module.

Elementary number theory can be a very motivating topic for many students. It provides opportunity for practice in computation in a context which is interesting to students. Many of the ideas explored in basic number theory are applicable to other topics. The concepts of greatest common factor and least common multiple, for example, are especially useful in work with fractional numbers. Many students find the ideas about prime numbers quite fascinating, and extra time devoted to this topic may well yield dividends of heightened student interest.

Lesson Schedule

You should plan to spend about 9 or 10 days on the module. However, if you provide considerable enrichment or a more in-depth study of the material, you may need as much as 15 days to complete the module.

Evaluation

Among the skills that students should possess at the end of this module is the ability to find the greatest common denominator and the least common multiple of two numbers. They should also be able to express a number as a product of primes and use exponential notation. Use day-to-day observations as a guide in helping you evaluate students. Encourage them to use the self-evaluation test at the end of the module as an aid in analyzing their abilities in the work of this module.

A cumulative review of the complete learning unit is found at the end of this module. We recommend that you provide a thorough review before giving an achievement test covering the complete unit. Module and Unit Achievement Tests are available from the publisher as part of the Evaluation Program designed for use with this series.

RESOURCES FOR ACTIVE LEARNING

Activities

Aftermath 2, "Goldbach's Conjecture," p. 49, Creative Publications
Let's Play Games in Mathematics, Vol. 7, "Factors," pp. 46–47, National Textbook Co.
Modern Math Games, "Two for the Primes," p. 12; "Prime Factorization," pp. 49–50, Fearon
Nuffield Project: *Number Patterns 2; Problems – Red Set*, Nos. 4, 4A, Wiley

Manipulative Devices

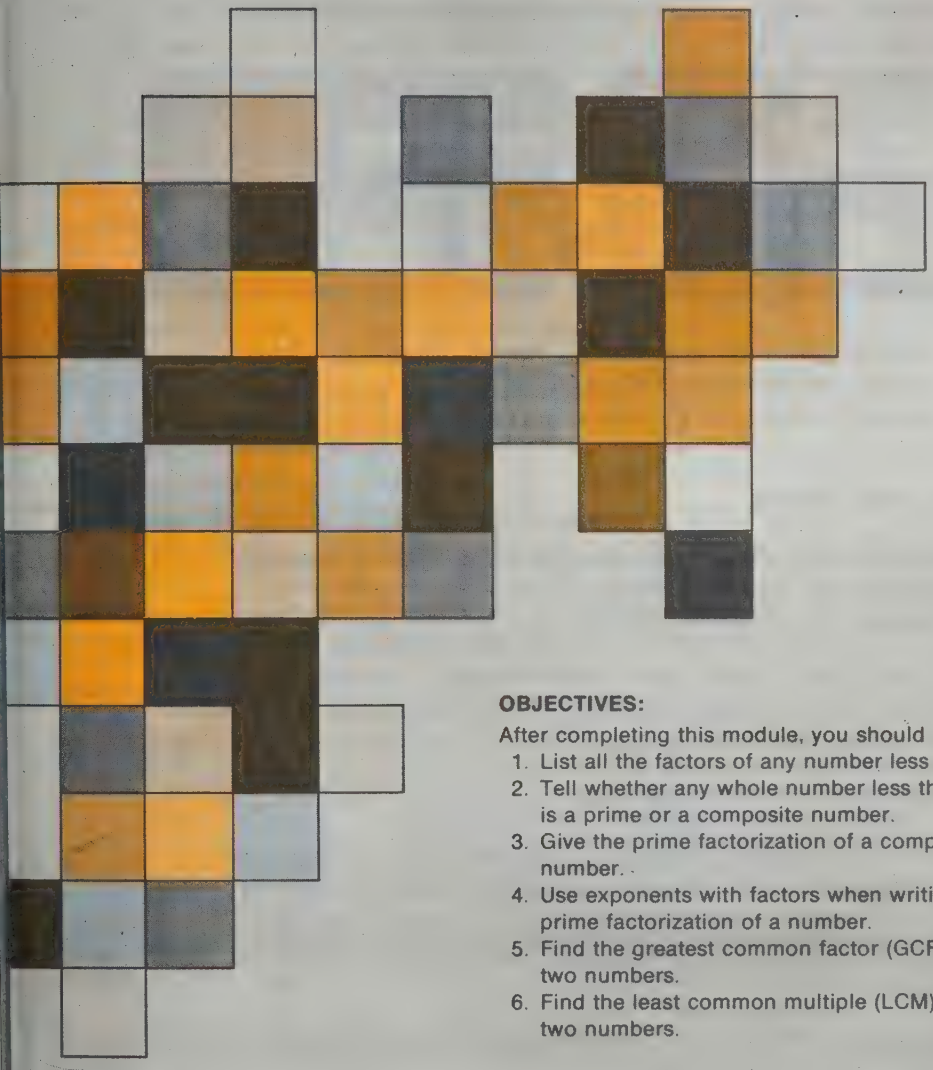
Spinner Kit (Houghton Mifflin)

Games and Puzzles

Domino Number Games (Heath)
Multifacto/Producto (Scott Foresman)
Operational Systems Games (Webster, McGraw-Hill)
Prime Drag (Creative Publications)
Prime Factor (Creative Publications)

UNIT A: *The Arithmetic of Whole Numbers*

MODULE 4: *Number Theory*



OBJECTIVES:

After completing this module, you should be able to:

1. List all the factors of any number less than 100.
2. Tell whether any whole number less than 100 is a prime or a composite number.
3. Give the prime factorization of a composite number.
4. Use exponents with factors when writing the prime factorization of a number.
5. Find the greatest common factor (GCF) of any two numbers.
6. Find the least common multiple (LCM) of any two numbers.

Objective

The student can find multiples of a given whole number and decide whether one number is a multiple of another number.

PREPARATION

A good preparation would be a short session of the guess-the-function rule game.

INVESTIGATION

The purpose of the Investigation is to review the idea of even numbers (multiples of 2) and odd numbers by means of function machines.

Encourage students to record the results of the Investigation in tabular form. This will help to emphasize the pattern of input and output numbers.

Input A	Output A	Output B
0	0	1
1	2	3
2	4	5
3	6	7
.	.	.
.	.	.
.	.	.

DISCUSSION

Exercises 1, 2, and 3 review the ideas of *even* and *odd* numbers and multiples of 2. Other multiples are discussed in the remaining exercises. The amount of time you should spend on these ideas will depend largely on the background and abilities of your students.

In discussing Exercise 5, be sure that students understand that for larger numbers, they can use division to determine whether the given number is a multiple of another number. This process stems from the definition of a multiple of a number:

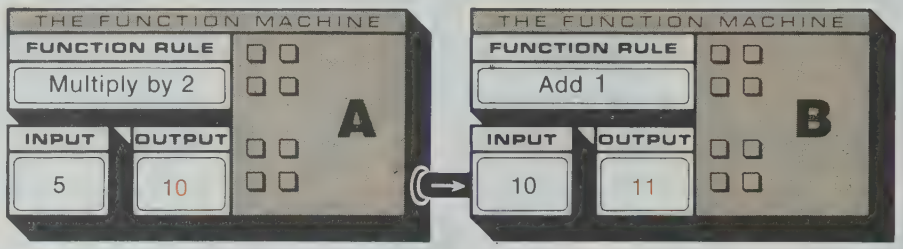
m is a multiple of a number *n* if there is a whole number *w* such that $n \cdot w = m$.

Therefore, to check if *m* is a multiple of *n*, one divides *m* by *n* to see if there is a whole number quotient *w* and a remainder of zero.

Whole Numbers and Multiples

Investigating the Ideas

The two function machines are connected together so that the output number for A is always the input number for B. The input for A is any whole number from the set of whole numbers {0, 1, 2, 3, 4, 5, . . .}.



Choose ten whole numbers as input numbers for function machine A. Can you find the output numbers for both function machines? Record your answers.

Discussing the Ideas

1. A number that is *2 times a whole number* is an **even number** or a **multiple of 2**. Which function machine produced only even numbers for outputs?

2. A number that is *1 more than an even number* is called an **odd number**. Will the output numbers for function machine B always be an odd number?

3. How can you decide whether a number is even or odd?

4. If each whole number is multiplied by 3, the resulting numbers are **multiples of 3**. Give the missing multiples of 3 for each ▒.

Whole numbers	0	1	2	3	4	5	6	7	8	9	. . .	*n*	. . .
	×3	×3	×3	×3	×3	×3	×3	×3	×3	×3		×3	
Multiples of 3	0	3	6	▒	▒	▒	▒	21	▒	▒	. . .	$3 \cdot n$	. . .

5. Is 51 a multiple of 3? How can you decide?

6. What are the first four multiples of 5? of 7?

Using the Ideas

1. For each number listed, tell whether it is an **even** or an **odd** number.

A 17	**C** 79	**E** 0	**G** 1007	**I** 12 004
B 42	**D** 217	**F** 358	**H** 2393	**J** 237 986

2. Write **even** or **odd** to complete each sentence.

- **A** The sum of any two even numbers is ? .
- **B** The sum of any two odd numbers is ? .
- **C** The sum of any even number and any odd number is ? .
- **D** The product of any two odd numbers is ? .
- **E** The product of any two even numbers is ? .
- **F** The product of any even number and any odd number is ? .

3. **A** List the first ten multiples of 3.

- **B** Choose any two of the multiples of 3. Is their sum a multiple of 3?
- **C** Find the product of any two multiples of 3. Is the product also a multiple of 3?

4. **A** List the first ten multiples of 5.

- **B** What is the smallest number, other than 0, that is a multiple of both 3 and 5?

5. The two examples below show how you can use division to decide whether one number is a multiple of another.

EXAMPLE 1: Is 91 a multiple of 7?

$$7\overline{)91}\quad 13,\ 7,\ 21,\ 0$$

Since the remainder is 0, $7 \cdot 13 = 91$, 91 **is a multiple of** 7.

EXAMPLE 2: Is 274 a multiple of 14?

$$14\overline{)274}\quad 19,\ 14,\ 134,\ 126,\ 8$$

The remainder is not 0. Therefore, 274 is **not** a multiple of 14.

Use division to answer these questions.

- **A** Is 442 a multiple of 17?
- **B** Is 1008 a multiple of 9?
- **C** Is 551 a multiple of 23?
- **D** Is 165 a multiple of 15?
- **E** Is 1001 a multiple of 143?
- **F** Is 2486 a multiple of 36?

UTILIZATION

Students are expected to find the answers to Exercises 2 and 3 by trying some examples and observing the results.

You may find it useful to give some other examples like those for Exercise 5 before assigning parts A through F as independent work.

EXTENSION

Remedial: Ask students to find the first ten multiples of each whole number from 1 through 9.

Enrichment: Suggest to the students that they do some arithmetic of even and odd numbers. The problems below are examples.

1. Copy and complete the addition and multiplication tables for "Evens" and "Odds."

+	E	O	
E	E		O
O			E
	O		

×	E	O	
E	E		E
O			O
	E		

2. Write *odd* or *even* for each blank.
   - A. The sum of any three odd numbers is ? . *(odd)*
   - B. The sum of any ten odd numbers is ? . *(even)*
   - C. The product of any ten odd numbers is ? . *(odd)*
   - D. The sum of an even number of odd numbers is ? . *(even)*
   - E. The sum of an odd number of odd numbers is ? . *(odd)*
3. A student adds 20 numbers. Seven are odd and the rest are even. He gives 4366 as his answer. What can you say about his work? *(He is incorrect.)*
4. Work the problems in the "Arithmetic of Odds and Evens."
   - A. $E + E + O + E + E =$ ▨ (O)
   - B. $O + E + O + E + O =$ ▨ (O)
   - C. $O + O + O + O + O + O =$ ▨ (E)
   - ★ D. $O \cdot (E + O) + O \cdot (O + O) =$ ▨ (O)

Assignments

Minimum 1 and 2, oral; 3; 5 A–C.
Average 1–3, 5. Maximum 1–5.

Objectives

Given a number less than 100, the student can list the factors of the number.

Given two numbers, the student can decide if one of the numbers is a factor of the other.

PREPARATION

Review the idea of factors and products by naming some products and one factor of each such product and asking students to name the second factor.

INVESTIGATION

The Investigation introduces the meaning of "the set of factors of a number." You may wish to give further examples before having the students try the open-ended Investigation question.

For part A, any of the numbers in {2, 3, 5, 7, 11, 13, . . .} have exactly two factors. Of course, these are prime numbers.

For part B, any of the numbers in {4, 9, 25, 49, 121, . . .} have exactly three factors. These numbers are squares of prime numbers.

In part C, encourage students to try to find numbers less than 100 that have more than 6 factors. Some typical responses might be 24 (8 factors), 36 (9 factors), 64 (7 factors), 60 (12 factors).

DISCUSSION

During the Discussion, be sure to have someone show (1) that 1 is a factor of every number, (2) that every whole number is a factor of zero, and (3) that every number except 1 has at least two factors. You might also find it helpful to have students build several more factor trees like the one in Exercise 2. It is important to point out that several different factor trees may be made for a given composite number. Demonstrate this with the number 36 by having your students construct different factor trees for it.

In Exercise 5, make sure students understand why division can be used to see if one number is a factor of the other. Ask them to think about inverse operations and the meaning of *factor*.

Factors

Investigating the Ideas

Since $6 \cdot 3 = 18$, we know that { 6 is a factor of 18. / 3 is a factor of 18.

Since $9 \cdot 2 = 18$, we know that { 9 is a factor of 18. / 2 is a factor of 18.

Since $18 \cdot 1 = 18$, we know that { 18 is a factor of 18. / 1 is a factor of 18.

There are no other pairs of whole numbers whose product is 18.
The set of all factors of 18 is {1, 2, 3, 6, 9, 18}.

	Can you find an example of each of the numbers described below?	Record your examples.

A A number with only two factors.

B A number with exactly three factors.

C A number with more than six factors.

Discussing the Ideas

1. Every whole number, except the number 1, has at least two factors. Also, since $0 = 0 \cdot 0 = 1 \cdot 0 = 2 \cdot 0 = \ldots$, every whole number is a factor of zero.

$1 = 1 \cdot 1$ The only factor of 1 is 1

$2 = 1 \cdot 2$ The only factors of 2 are 1 and 2

$6 = 1 \cdot 6 = 2 \cdot 3$ The factors of 6 are 1, 2, 3, and 6

What are the factors of the following numbers?

A 7 **B** 8 **C** 9 **D** 10 **E** 11 **F** 13

2. An interesting way to find the factors of a number is to build "factor trees" for numbers. What are the missing factors in the factor tree for 40?

3. What number, other than 1, is a factor of every even number?

4. Explain how the equation below can help you list all the factors of 48.

$$48 = 1 \cdot 48 = 2 \cdot 24 = 3 \cdot 16 = 4 \cdot 12 = 6 \cdot 8$$

5. Is 8 a factor of 136? Explain how you can use division to help you decide.

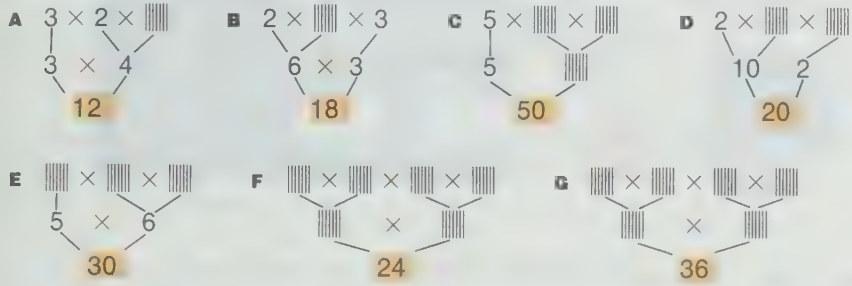
$$8\overline{)136}\ ?$$

Using the Ideas

1. Use the equations to help you list all the factors for each number.

   A $25 = 1 \cdot 25 = 5 \cdot 5$
   B $77 = 1 \cdot 77 = 7 \cdot 11$
   C $143 = 1 \cdot 143 = 11 \cdot 13$
   D $27 = 1 \cdot 27 = 3 \cdot 9$
   E $24 = 1 \cdot 24 = 2 \cdot 12 = 3 \cdot 8 = 4 \cdot 6$
   F $100 = 1 \cdot 100 = 2 \cdot 50 = 4 \cdot 25 = 5 \cdot 20 = 10 \cdot 10$

2. Copy the factor trees and supply the names for each ▨.



3. List all the factors of these numbers.

A	12	**C**	40	**E**	26	**G**	30	**I**	27	**K**	29
B	15	**D**	18	**F**	6	**H**	16	**J**	50	**L**	64

4. Which number less than 10 have exactly two factors?

5. Use division to help you answer each question.

   A Is 9 a factor of 297?
   B Is 8 a factor of 1000?
   C Is 13 a factor of 211?
   D Is 26 a factor of 130?
   E Is 18 a factor of 556?
   F Is 37 a factor of 999?

think

The three examples below will help you understand the meaning of **abundant**, **deficient**, and **perfect** numbers.

A The factors of 12: {1, 2, 3, 4, 6} $1 + 2 + 3 + 4 + 6 = 16$ ← Sum of the factors
(except 12) Since $16 > 12$, 12 is an **abundant** number.

B The factors of 10: {1, 2, 5} $1 + 2 + 5 = 8$ ← Sum of the factors
(except 10) Since $8 < 10$, 10 is a **deficient** number.

C The factors of 6: {1, 2, 3} $1 + 2 + 3 = 6$ ← Sum of the factors
(except 6) Since $6 = 6$, 6 is a **perfect** number.

Tell what kind of number each of these are: **abundant, deficient,** or **perfect.**

A 18 **B** 26 **C** 28 **D** 30 **E** 16 **F** 32 **G** 45

UTILIZATION

Have the students work on the exercises independently, but allow time for discussion afterward. Observe that in all the factor trees except the one in 2G there is only one possible pair of factors in the first row above the number because of the given pattern, but in 2G there are two possibilities.

Answers, Exercise 3

A. 1, 2, 3, 4, 6, 12
B. 1, 3, 5, 15
C. 1, 2, 4, 5, 8, 10, 20, 40
D. 1, 2, 3, 6, 9, 18
E. 1, 2, 13, 26
F. 1, 2, 3, 6
G. 1, 2, 3, 5, 6, 10, 15, 30
H. 1, 2, 4, 8, 16
I. 1, 3, 9, 27
J. 1, 2, 5, 10, 25, 50
K. 1, 29
L. 1, 2, 4, 8, 16, 32, 64

EXTENSION

Appropriate supplementary exercises are provided on page 27 of the *Workbook* and on page 19 of the *Duplicator Masters*.

Remedial: Ask students to do these exercises.

1. Find all the factors of each number.
   A. 10 B. 14 C. 20 D. 32 E. 52
2. Construct factor trees for each number.
   A. 16 B. 30 C. 40 D. 27

Enrichment: Ask students to find how many different factor trees can be constructed for 100. (Answer: 6; any two factor trees are different if the factors on the same row are different.)

Think Solution

Every whole number is either abundant, deficient, or perfect. The next perfect number greater than 6 is 28. The next greater perfect number is 496. The next is 8128, followed by 33 550 336 and 8 589 869 056. Only 23 perfect numbers have so far been found.

To find out more about perfect numbers, students should investigate Research Project C on page A-93.

Assignments

Minimum 1, 2, 3A–F. Average 1–4, 5A–C. Maximum 1–5.

Objective

Given a whole number greater than 1 and less than 100, the student can state whether it is prime or composite.

PREPARATION

Write 8, 13, 15, 18, and 25 on the chalkboard. Have the students list all the factors of these numbers. Discuss which number has the greatest number of factors and which has the fewest.

INVESTIGATION

You will want to ascertain that students understand the function rule before they proceed with the Investigation. Allow sufficient time for everyone to complete the Investigation. Encourage fast workers to find the factors of some numbers greater than 20.

Number	Number of Factors	Number	Number of Factors	Number	Number of Factors
1	1	8	4	15	4
2	2	9	3	16	5
3	2	10	4	17	2
4	3	11	2	18	6
5	2	12	6	19	2
6	4	13	2	20	6
7	2	14	4		

DISCUSSION

In discussing Exercise 1, students can refer to the table they have completed for the Investigation. The definition of *prime* number in Exercise 2 is preferred to the one sometimes given by students: "A prime number has only itself and 1 as factors." This definition does not make it clear whether 1 is a prime or not.

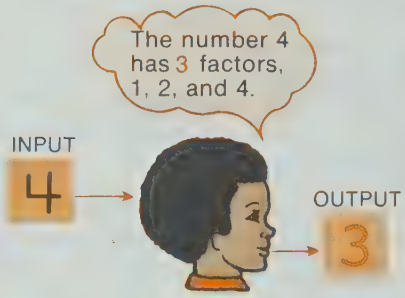
The fact that 2 is the only even prime number should be emphasized in Exercise 3.

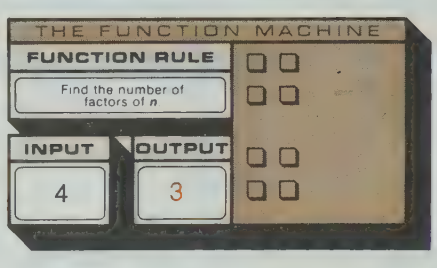
Composite numbers are defined in Exercise 4. Note that composite numbers are greater than 1; hence, 0 and 1 are neither composite nor prime. The number 0 has infinitely many factors, while 1 has only one factor.

Prime and Composite Numbers

Investigating the Ideas

William thought of an interesting function rule. Study his rule and the way he thought about it.





Can you make a function table for the function machine and give the output numbers for each input number from 1 through 20?

Discussing the Ideas

1. A Which number from 1 through 20 has the fewest factors?
   B What is the greatest number of factors that any of the numbers from 1 through 20 has?

2. A whole number which has exactly two factors is called a **prime** number. Which numbers from 1 through 20 are prime numbers?

3. Are any of the even numbers prime numbers? How many?

4. Numbers greater than 1 which have more than two factors are called **composite** numbers.
   A What is the smallest composite number?
   B Is 39 a prime number or a composite number?

5. A Is 1 a prime number, a composite number, or neither? Explain your answer.
   B Is 0 a prime number, a composite number, or neither? Explain your answer.

Using the Ideas

1. List all the prime numbers less than 100. (If you find all of them, there should be 25 numbers in your list.)

2. Tell how many composite number there are
   - **A** less than 10.
   - **B** less than 100.

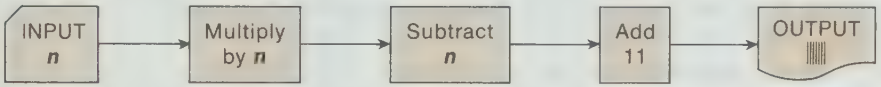
3. How many of the prime numbers are even numbers?

4. There is just one pair of prime numbers whose difference is 1. What is the pair?

5. The smallest pair of prime numbers with a difference of 2 is 3 and 5. Such numbers are called **twin primes**. Find six other pairs of twin primes.

6. **A** Find two prime numbers whose difference is 3.
   B Do you think there is any other such pair?

7. The set of **all** factors of a certain number is {1, 17, 289}. Another number has the set of factors {1, 151}.
   - **A** Which number is prime?
   - **B** What are the two numbers?

8. 

   - **A** Find the output number for the flow chart using the numbers 1 through 10 as input numbers.
   - **B** What kind of numbers does the flow chart produce?
   - **C** Try 11 as an input number for the flow chart. Is the output a prime number?

think

No even number after 2 is a prime number. But every even number larger than 2 seems to be the sum of two primes. No one has ever been able to prove that this is so, but most mathematicians believe it.

1. Check this for the even numbers up to 50.

2. Express 200 as the sum of two primes.

$4 = 2 + 2$
$6 = 3 + 3$
$8 = 3 + 5$
$10 = 3 + 7$
$12 = 5 + 7$
$14 = 3 + 11$
$16 = 5 + 11$
$18 = 7 + 11$
$20 = 7 + 13$
.
.
.

UTILIZATION

Exercise 1 will take most students considerable time. You may find it advisable to assign only this exercise initially. Also, you might wish to have students use the Sieve of Eratosthenes (see Extension) to find the primes less than 100. Then, after the primes are found, assign the other exercises.

The flow chart in Exercise 8 gives the equation $n^2 - n + 11 =$ output. This interesting equation is an example of a "prime rich" equation because when n is replaced by any whole number less than 11, a prime number results as the output. However, for 11, we have $11^2 - 11 + 11 = 121$, and 121 is *not* a prime. This example illustrates that we must be careful not to draw hasty conclusions! The fact that the equation produced ten primes in a row does not mean that it will always produce primes.

EXTENSION

Enrichment: The expression $n^2 - n + 41$ is another example of a "prime rich" formula. The expression yields prime numbers for all whole number replacements from 1 through 40 but fails for 41. Some students might be interested in checking to see that this is so.

Think Solution

In most cases, more than one combination of addends is possible Sample combinations are given below.

1. $22 = 11 + 11$, $24 = 11 + 13$, $26 = 13 + 13$, $28 = 11 + 17$, $30 = 11 + 19$, $32 = 3 + 29$, $34 = 17 + 17$, $36 = 13 + 23$, $38 = 19 + 19$, $40 = 11 + 29$, $42 = 11 + 31$, $44 = 13 + 31$, $46 = 23 + 23$, $48 = 11 + 37$, $50 = 13 + 37$
2. $200 = 97 + 103 = 3 + 197 = 37 + 163$, etc.

Assignments

Minimum 1–4, 7. Average 1–4, 7, 8.
Maximum 1–8.

Objective

Given an appropriate composite number, the student can give its prime factorization.

PREPARATION

Review the meaning of prime numbers. Students should be able to name the prime numbers that are less than 20.

INVESTIGATION

The aim of the Investigation is for the student to discover that regardless of the factor tree he constructs for the number 60, the prime factors in the top row, except for the order in which they appear, are always the same. The fact that each prime number has a unique set of prime factors is called the Fundamental Theorem of Arithmetic or, sometimes, the Unique Factorization Theorem. (See the Mathematics section at the beginning of this module.)

It is not necessary that students find *all* the different factor trees possible for 60, only that they construct 2 or 3 more trees so that some comparison can be made. Other choices for the first two factors of 60 are $2 \cdot 30$, $6 \cdot 10$, and $5 \cdot 12$.

DISCUSSION

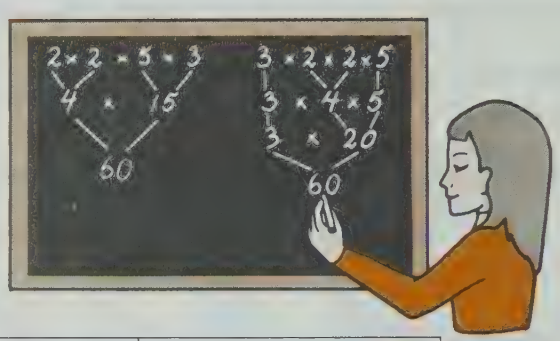
In Exercise 1, be sure students understand that the Investigation shows that when 60 is expressed as a product of prime numbers there is only one set of prime factors: $60 = 2 \cdot 2 \cdot 3 \cdot 5$. The terms *prime factorization* and *factored completely* (discussed in Exercise 2) are both used to indicate that a number is expressed as a product of prime numbers.

For Exercise 3, students should find the partial product of designated factors and pairs of factors in order to find the composite number made up of those factors.

Prime Factorization

Investigating the Ideas

In order to find which prime numbers can be multiplied together to give a product of 60, Kathy made some **factor trees**.



How many more factor trees can you build for the number 60?	Do not use 1 as one of the factors.

Discussing the Ideas

1. What do you notice about the "top row" of all of your factor trees for 60?

2. When a composite number is written as a product of primes, we say that we have **factored the number completely** or have given the **prime factorization** of the number. If the order of the factors is disregarded, each composite number can be expressed as a product of primes in one and only one way.

   $12 = 2 \cdot 2 \cdot 3 \rightarrow$ No primes other than 2, 2, and 3 have a product of 12.
   $70 = 2 \cdot 5 \cdot 7 \rightarrow$ No primes other than 2, 5, and 7 have a product of 70.
   $30 = 2 \cdot 3 \cdot 5 \rightarrow$ No primes other than 2, 3, and 5 have a product of 30.

   A What is the prime factorization of 60?
   B What is the prime factorization of 24?

3. The top row of a factor tree for a number gives the prime factorization of the number. What number has been factored completely in each one of these factor trees?

   A $2 \times 3 \times 3 \times 7$ B $3 \times 3 \times 3 \times 11$ C $7 \times 11 \times 13$

Using the Ideas

1. Copy and complete the factor trees. The top row of factors is the prime factorization of the bottom number of the tree.

A $7 \times 2 \times \square$ / 7×4 / 28

B $2 \times \square \times \square$ / 6×4 / 24

C $5 \times \square \times \square$ / $5 \times \square$ / 45

D $2 \times \square \times \square$ / $10 \times \square$ / 30

E $\square \times \square \times \square$ / $\square \times 6$ / 66

F $\square \times \square \times \square \times \square$ / $10 \times \square$ / 90

G $\square \times \square \times \square \times \square$ / $\square \times 14$ / 84

2. Copy and complete the equations so that you have the prime factorization of each given number.

A $10 = 2 \cdot d$

B $12 = 3 \cdot 2 \cdot e$

C $14 = 2 \cdot f$

D $16 = 2 \cdot 2 \cdot 2 \cdot h$

E $18 = 2 \cdot 3 \cdot i$

F $20 = 2 \cdot 2 \cdot j$

G $24 = 2 \cdot 3 \cdot 2 \cdot m$

H $27 = 3 \cdot 3 \cdot n$

I $28 = 2 \cdot 2 \cdot p$

3. Factor each of the following numbers completely.

A 33 **B** 35 **C** 36 **D** 38 **E** 40 **F** 42 **G** 50 **H** 81 **I** 100 **J** 250 **K** 231 **L** 126

4. Give the number for each prime factorization shown.

A $2 \cdot 5 \cdot 5 \cdot 2$

B $2 \cdot 3 \cdot 7$

C $2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$

D $5 \cdot 7 \cdot 13$

E $2 \cdot 11 \cdot 11 \cdot 17$

F $3 \cdot 3 \cdot 3 \cdot 3 \cdot 5$

G $29 \cdot 31$

H $2 \cdot 3 \cdot 13$

I $5 \cdot 5 \cdot 5 \cdot 5 \cdot 5$

J $3 \cdot 19 \cdot 23$

K $2 \cdot 37 \cdot 61$

L $2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7$

★ **5.** Give the prime factorization of 1024.

think

1. How many factors has 7?
   How many has $2 \cdot 7$?
2. How many factors has 9?
   How many has $2 \cdot 9$?
3. How many factors has 15?
   How many has $2 \cdot 15$?
4. How many factors has 45?
   How many has $2 \cdot 45$?
5. How many factors has 10?
   How many has $2 \cdot 10$?
6. How many factors has 14?
   How many has $2 \cdot 14$?
7. An **odd** number N has 100 factors.
   $\{1, \ldots, N\}$
   How many factors has $2 \cdot N$?

More practice, page S-10, Set 16

UTILIZATION

The exercises progress in order of difficulty. In Exercise 1 factor trees aid the student in finding prime factorizations. In Exercise 2, partial factorizations are given. In Exercise 3 the student must complete all of the factorization. In this problem encourage students to make a factor tree if they have difficulty finding the prime factors.

EXTENSION

See Supplementary Exercises, Set 16, page S-10, and page 28 of the *Workbook* for additional practice.

Remedial: Suggest that students build factor trees to find the prime factorization of these numbers.

A. 16 B. 28 C. 48 D. 50
E. 100 F. 80 G. 56 H. 27

Enrichment: Have the students complete two function tables, using the rule $4 \cdot n - 1$ for one table and $4 \cdot n + 1$ for the other table and the numbers 1 to 15 as input numbers for each table. Have them check those output numbers which are prime numbers. Then ask: Do the output numbers for these functions include all prime numbers less than 62, except 2?

Think Solution

If an odd number has a certain number of factors and you multiply it by 2, you will have a 2 to associate with each of the original factors. This will give you twice as many different factors. The number chosen must be odd, for if you used an even number such as 8, the factors would be $\{1, 2, 4, 8\}$. When you consider $2 \cdot 8$, the original factors $\{1, 2, 4, 8\}$ are united with any new factors that are generated by doubling each of the original ones; for example, $\{2, 4, 8, 16\}$ gives $\{1, 2, 4, 8, 16\}$ as factors of $2 \cdot 8$.

Assignments

Minimum 1, 2, 3A–F. Average 1, 2, 3, 4A–H. Maximum 1–5.

Objective

Given repeated factors of a number, the student can express the number as a power of the repeated factor.

PREPARATION

Plan a short warmup in which students build factor trees for two or three composite numbers.

INVESTIGATION

The first two factor trees in the Investigation will be easy for most students, but the third tree is a bit harder.

If some students finish quickly, you might challenge them to find some other numbers whose factor trees have top rows that are composed of all the same prime factors. The repeated factors in the factor trees will be related to the exponential notation in the Discussion section.

DISCUSSION

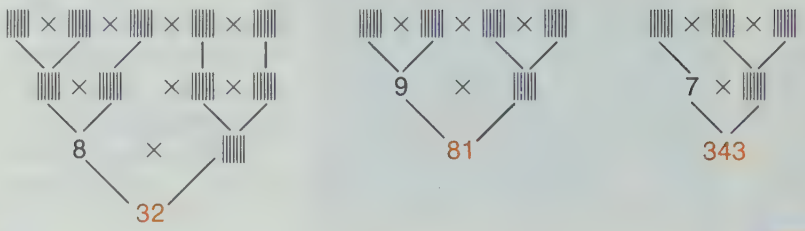
Use the results of the Investigation to motivate a discussion of exponents. Since exponents have been used previously with 10 as the base number (page A-8), the only new idea here is the use of exponents with other bases.

Extend the discussion of Exercise 3 to include a review of the zero exponent. That is, $a^0 = 1$, where $a \neq 0$. (See the Utilization commentary relative to zero exponents, Module 1, page A-9.)

Exponents and Factors

Investigating the Ideas

These factor trees are unusual.



Can you complete them to find out why they are unusual?

Discussing the Ideas

1. What did you find out about the factor trees that made them unusual?

2. When the same factor is repeated several times, you can use exponents just as you did for powers of ten.

Symbol	We think	We say
8^2	$8 \cdot 8$ Two factors of 8	8 squared or 8 to the second power
3^3	$3 \cdot 3 \cdot 3$ Three factors of 3	3 cubed or 3 to the third power
11^5	$11 \cdot 11 \cdot 11 \cdot 11 \cdot 11$ Five factors of 11	11 to the fifth power

How would you represent each number in the Investigation using exponents?

3. The prime factorization of some numbers can be simplified by using exponents.

EXAMPLES: $36 = 2 \cdot 2 \cdot 3 \cdot 3 = 2^2 \cdot 3^2$
$200 = 2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 = 2^3 \cdot 5^2$

How would you write the prime factorization of these numbers using exponents?

A 27 **B** 100 **C** 90 **D** 64 **E** 225

Using the Ideas

1. Write each product using exponents.

EXAMPLE: $7 \cdot 7 \cdot 7 = 7^3$

A $4 \cdot 4$ C $10 \cdot 10 \cdot 10$ E $2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$

B $9 \cdot 9 \cdot 9 \cdot 9 \cdot 9$ D $15 \cdot 15 \cdot 15 \cdot 15 \cdot 15 \cdot 15$ F $5 \cdot 4 \cdot 5 \cdot 4 \cdot 5 \cdot 4 \cdot 5$

2. Write each number in expanded form and find the results.

EXAMPLE: $8^5 = 8 \cdot 8 \cdot 8 \cdot 8 \cdot 8 = 32\,768$

A 16^2 C 2^8 E 3^4 G 13^3 ★I $4^0 \cdot 3^0 \cdot 2^0$

B 4^4 D 9^2 F 17^1 H $2^3 \cdot 3^2 \cdot 4^1$ ★J $4^3 \cdot 3^2 \cdot 2^1 \cdot 1^0$

3. Give the prime factorization of each number. Use exponents.

A 80 B 108 C 243 D 1000 E 144

4. Solve the equations.

A $5^3 = n$ D $2^6 = a$ G $9^3 = t$ J $1^{15} = m$

B $3^3 = h$ E $13^2 = r$ H $8^4 = b$ K $23^1 = w$

C $5^5 = y$ F $4^5 = c$ I $4^8 = s$ ★L $4^3 = 2^n$

5. Find the products.

A $2^3 \cdot 3^2$ C $7 \cdot 3^3$ E $10^3 \cdot 10^2$ G $7^1 \cdot 11^1 \cdot 13^1$

B $5^2 \cdot 10^2$ D $2^4 \cdot 2^3$ F $2^2 \cdot 3^2 \cdot 5^2$ H $8^2 \cdot 11^2$

★**6.** For each number below, a factorization is given. Decide whether or not it is a prime factorization. If not, factor the number completely.

A $72 = 2 \cdot 2 \cdot 2 \cdot 9$ C $713 = 23 \cdot 31$ E $1331 = 11 \cdot 121$ G $1001 = 77 \cdot 13$

B $216 = 8 \cdot 3 \cdot 9$ D $627 = 11 \cdot 57$ F $360 = 2^3 \cdot 45$ H $6923 = 161 \cdot 43$

★**7.** Use the facts $2 \cdot 5 = 10$, $2^2 \cdot 5^2 = 100$, $2^3 \cdot 5^3 = 1000$, and $2^4 \cdot 5^4 = 10\,000$ to find the products quickly.

A $2 \cdot 7 \cdot 5$ D $2^3 \cdot 11 \cdot 5^2$ G $2^4 \cdot 23 \cdot 5^4$ J $2^5 \cdot 7 \cdot 11 \cdot 5^4$

B $2^2 \cdot 13 \cdot 5^2$ E $2^2 \cdot 12 \cdot 5^3$ H $2^4 \cdot 71 \cdot 5^3$ K $2^4 \cdot 3 \cdot 7 \cdot 5^4$

C $2^2 \cdot 45 \cdot 5^2$ F $2^3 \cdot 147 \cdot 5^3$ I $2^3 \cdot 101 \cdot 5^4$ L $2^6 \cdot 3^2 \cdot 5^6$

★**8.** Use the fact that $7 \cdot 11 \cdot 13 = 1001$ to find the products quickly.

A $7^2 \cdot 11 \cdot 13$ C $7 \cdot 11 \cdot 13^2$ E $7 \cdot 11 \cdot 13 \cdot 17$ G $7 \cdot 11^2 \cdot 13 \cdot 23$

B $7 \cdot 11^2 \cdot 13$ D $7^2 \cdot 11^2 \cdot 13$ F $7 \cdot 11 \cdot 13 \cdot 53$ H $7^2 \cdot 11^2 \cdot 13^2$

UTILIZATION

In assigning the exercises, consider the fact that, although there are but 8 exercises, there are nearly 70 responses required of the students. Therefore some selectivity is probably necessary.

In working Exercise 4, some of the students may develop computational shortcuts such as:

C. $y = 5^5 = 5^2 \cdot 5^3$
$= 25 \cdot 125 = 3125$

F. $c = 4^5 = 4^2 \cdot 4^3$
$= 16 \cdot 64 = 1024$

H. $b = 8^4 = (8^2)^2$
$= 64^2 = 4096$

I. $s = 4^8 = (4^2)^4 = 16^4$
$= (16^2)^2 = 256^2 = 65\,536$

EXTENSION

Further practice exerciseses are provided on page 29 of the *Workbook* and page 20 of the *Duplicator Masters*.

Remedial: Ask students to write each of the following using exponents.

1. $3 \cdot 3 \cdot 3$ 2. $5 \cdot 5$ 3. $6 \cdot 6 \cdot 6 \cdot 6$
4. $7 \cdot 7 \cdot 7 \cdot 7 \cdot 7$ 5. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$
6. $10 \cdot 10 \cdot 10$ 7. $9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9$

Enrichment: Have students work some exercises like the ones below.

1. Give the correct symbol ($<$ or $>$) for each ▒.

A. 2^3 ▒ 3^2 ($<$) C. 5^4 ▒ 8^3 ($>$)

B. 9^3 ▒ 5^4 ($>$) D. 2^{10} ▒ 10^3 ($>$)

2. Give the number for each ▒.

A. $2^2 + 3^2 + 6^2 =$ ▒^{2} (7^2)

B. $3^2 + 4^2 + 12^2 =$ ▒^{2} (13^2)

C. $4^2 + 5^2 + 20^2 =$ ▒^{2} (21^2)

D. $5^2 + 6^2 + 30^2 =$ ▒^{2} (31^2)

Assignments

Minimum 1A–D, 2A–F, 3, 5A–D.
Average 1, 2A–H, 3, 4A–F, 5.
Maximum 1D–F, 2G–J, 3, 4G–L, 5, alternate parts of 6, 7, 8.

Objective

Given two whole numbers, the student can find the greatest common factor of the numbers.

PREPARATION

Materials: 1cm graph paper (*Duplicator Masters*, page 83).

Write the numbers 24 and 30 on the chalkboard. Have students find the set of all factors of each number.

INVESTIGATION

Distribute graph paper to the students. Emphasize that they are to find the largest possible squares that the 18-by-12 rectangle can be divided into. Of course students are expected to discover that the length of the side of the largest square is the greatest common factor of 18 and 12. This fact need not be brought out until in the Discussion stage. If this Investigation proceeds rapidly, you may have students try some other rectangles, such as 6 by 15 (GCF is 3) and 9 by 14 (GCF is 1).

DISCUSSION

In Discussion Exercise 1 the fact that the rectangle can be divided into squares 3 units on a side indicates that 3 *is a common factor* of 12 and 18 but not necessarily the *greatest* common factor.

Exercise 2 illustrates one method of determining the greatest common factor of two numbers.

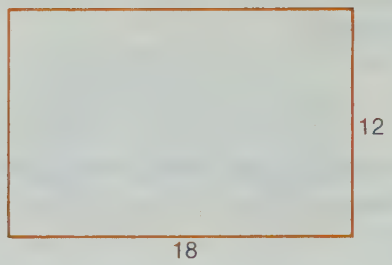
The flow chart in Exercise 3 gives a method of finding the GCF of two numbers that is based on prime factorization of the numbers.

Give your students a chance to explain this method, encouraging such explanations as the following: "By writing the prime factorization, I can then choose the common factors from each of the factorizations in order to get the GCF for the two numbers."

Greatest Common Factor (GCF)

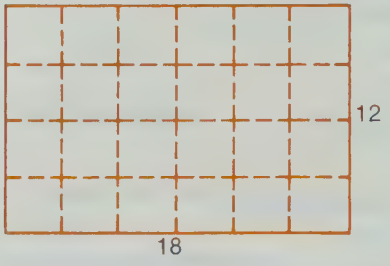
Investigating the Ideas

Draw a rectangle 18 units long and 12 units wide on graph paper.



Can you divide the entire rectangular region into square regions so that the squares are all the same size and as large as they can possibly be?

Discussing the Ideas



1. Tam divided the rectangular region into squares that were 3 units on each side. Did she find the **largest possible** square that could be used?

2. John said, "To find the largest sided square, I thought about the greatest number that is a factor of both 12 and 18."

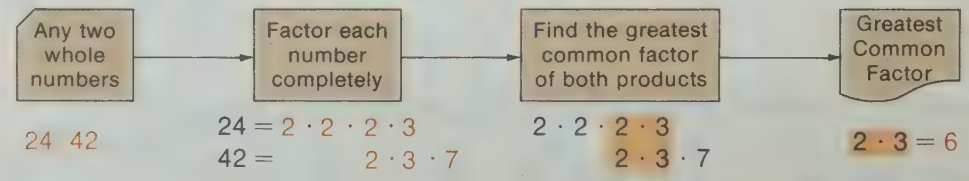
   What is that number?

Set of factors of 18 → $A = \{1, 2, 3, 6, 9, 18\}$
Set of factors of 12 → $B = \{1, 2, 3, 4, 6, 12\}$
Set of common factors of 18 and 12 is the **intersection** of sets A and B.

$$A \cap B = \{1, 2, 3, 6\}$$

The **greatest common factor** (GCF) of 18 and 12 is ▒.

3. Study the flow chart to learn how prime factorizations can be used to find the GCF of two numbers. Use the flow chart to help you find the GCF of 70 and 105


Any two whole numbers
Factor each number completely
Find the greatest common factor of both products
Greatest Common Factor
24 42
24 = 2 · 2 · 2 · 3
42 = 2 · 3 · 7
2 · 2 · 2 · 3
2 · 3 · 7
2 · 3 = 6

Using the Ideas

1. **A** List the factors of 12. **C** List the common factors of 12 and 20.
B List the factors of 20. **D** What number is the GCF of 12 and 20?

2. **A** List the factors of 28. **C** List the common factors of 28 and 35.
B List the factors of 35. **D** What number is the GCF of 28 and 35?

3. Find the GCF of each pair of numbers. The complete factorization is given and may help you. The answer to Part **A** is 14.

A $42 = 2 \cdot 3 \cdot 7$, $70 = 2 \cdot 5 \cdot 7$
B $70 = 2 \cdot 5 \cdot 7$, $30 = 2 \cdot 3 \cdot 5$
C $55 = 5 \cdot 11$, $15 = 3 \cdot 5$
D $12 = 2 \cdot 2 \cdot 3$, $18 = 2 \cdot 3 \cdot 3$
E $10 = 2 \cdot 5$, $70 = 2 \cdot 5 \cdot 7$
F $18 = 2 \cdot 3 \cdot 3$, $15 = 3 \cdot 5$
G $42 = 3 \cdot 7 \cdot 2$, $24 = 3 \cdot 2 \cdot 2 \cdot 2$
H $24 = 3 \cdot 2^3$, $6 = 2 \cdot 3$
I $40 = 5 \cdot 2^3$, $10 = 5 \cdot 2$
J $90 = 2 \cdot 3^2 \cdot 5$, $60 = 2^2 \cdot 3 \cdot 5$
K $105 = 3 \cdot 5 \cdot 7$, $225 = 3^2 \cdot 5^2$
L $385 = 5 \cdot 7 \cdot 11$, $175 = 5^2 \cdot 7$

4. Find the GCF of each pair of numbers. (Sometimes it is 1.)

A 10, 25
B 6, 18
C 8, 15
D 24, 36
E 40, 100
F 8, 27
G 8, 18
H 15, 25
I 6, 8
J 12, 16
K 11, 12
L 14, 35
M 100, 60
N 75, 125
O 48, 72
P 85, 68
Q 45, 36
R 90, 120
S 150, 270
T 450, 360

★**5.** The letter N represents a number between 50 and 60. The GCF of N and 16 is 8. Find the number.

★**6.** Thirteen is not a factor of N. What is the GCF of 13 and N?

★**7.** The GCF of A and B is 5. What is the GCF of $2 \cdot A$ and $2 \cdot B$?

★**8.** Two numbers have a GCF of 12. One number is 5 times as large as the other. Find the two numbers.

think

Use the whole numbers from 1 through 20 as input numbers. Then find the output numbers.

How many of the output numbers are prime numbers?

More practice, page S-11, Set 17

UTILIZATION

Assign the exercises according to the abilities and needs of your students. Exercises 5 through 8 are intended to challenge your more able students.

EXTENSION

To provide additional practice, make assignments from Supplementary Exercise Set 17, page S-11; *Workbook* page 30; and *Duplicator Masters* page 21. *Arithmetic Skill Card* W-11 would also be appropriate for use here.

Enrichment: Some students may be interested in learning a method of finding the GCF of some larger numbers. The method shown below is called the Euclidean Algorithm. (See Research Project D, page A-93.) Example: Find the GCF of 182 and 259.

Step 1
Divide the larger number by the smaller.

$$\begin{array}{r} 1 \\ 182\overline{)259} \\ \underline{182} \\ 77 \end{array}$$

Step 2
Divide the divisor by the remainder.

$$\begin{array}{r} 2 \\ 77\overline{)182} \\ \underline{154} \\ 28 \end{array}$$

Step 3
Repeat Step 2 with new remainder and divisor.

$$\begin{array}{r} 2 \\ 28\overline{)77} \\ \underline{56} \\ 21 \end{array}$$

Step 4
Repeat Step 2.

$$\begin{array}{r} 1 \\ 21\overline{)28} \\ \underline{21} \\ 7 \end{array}$$

Step 5
Repeat Step 2.

$$\begin{array}{r} 3 \\ 7\overline{)21} \\ \underline{21} \\ 0 \end{array}$$

Step 6
When zero remainder is reached, the last divisor is the GCF—in this case 7.

$$182 = 7 \cdot 2 \cdot 13$$
$$259 = 7 \cdot 37$$

Assignments

Minimum 1, 2, 3A–C, 4A–E.
Average 1, 2, 3A–F, 4A–O.
Maximum 1, 2, 3G–L, 4K–T, 5–8.

Objective

Given two whole numbers, the student can find the least common multiple of the numbers.

PREPARATION

Materials: Centimetre rulers: scissors: adding machine tape, long strips of newsprint, or string.

Since the Investigation requires active involvement with materials, it may take more time than usual; therefore it would be best to begin immediately with the Investigation.

INVESTIGATION

Distribute the needed materials to the students. This investigation will lend itself well to students working in pairs or in small groups. You may need to give some students help to be sure that they understand the nature of the Investigation. You might ask: "If you cut off a strip of paper 16 cm long, you could divide it into 2 pieces 8 cm long, but could you also divide a 16 cm strip into pieces 14 cm long?" "If you cut off a strip 28 cm long, it could be divided into 14-cm strips, but could it be divided into 8-cm strips?"

If others have difficulty, suggest that they mark off lengths of 8 cm along one edge of the paper strip and mark off lengths of 14 cm along the other edge. Thus it will be easy to see when the marks first coincide after the start, and that the desired length is 56 cm.

DISCUSSION

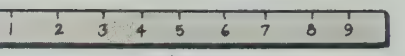
Exercise 1 introduces the term *least common multiple* in terms of the activity in the Investigation. Exercise 2 shows one method of finding the LCM of two numbers.

The least common multiple is the smallest nonzero multiple of two numbers.

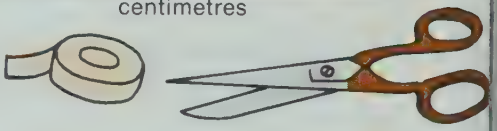
Exercise 3 gives a shorter method of finding the LCM, which many students will prefer to use. One simply thinks about successive multiples of the smaller number until one reaches a multiple of the larger number. You may want to point out that, if two numbers have 1 as their greatest common factor, the LCM is simply the product of two numbers.

Least Common Multiple (LCM)

Investigating the Ideas



You will need some adding machine tape or strips of paper, a centimetre ruler, and a pair of scissors for this Investigation.



What is the shortest length of paper that can be cut so that

A you can cut it into pieces 8 centimetres long?

B you could cut it into pieces 14 centimetres long and have nothing left over in either case?

Discussing the Ideas

1. The number that is the shortest length of a strip of paper that could be cut either into 8 centimetre (cm) pieces or 14 cm pieces is the **least common multiple** (LCM) of 8 and 14. What is the LCM of 8 and 14?

2. Thinking about the multiples of a pair of numbers can help you choose the LCM of the pair.

EXAMPLE: Multiples of 4 → $A = \{0, 4, 8, 12, 16, 20, 24, 28, 32, 36, \ldots\}$

Multiples of 6 → $B = \{0, 6, 12, 18, 24, 30, 36, 42, \ldots\}$

Common nonzero multiples of 4 and 6 → $\{12, 24, 36, \ldots\}$

The LCM of 4 and 6 is 12.

After studying the example, try this problem.

A List the multiples of 3 to 36.

B List the multiples of 4 to 36.

C List the common multiples of 3 and 4 to 36.

D Pick out the LCM of 3 and 4.

3. You may find it easier just to list multiples of the larger number until you find a number that is a multiple of the smaller. EXAMPLE: Find the LCM of 8 and 6.

Multiples of 8 → $\{0, 8, 16, 24\}$.

The LCM of 8 and 6 is 24.

Use this method to find the LCM of each pair of numbers.

A 6, 9 **B** 4, 14 **C** 12, 20 **D** 3, 8 **E** 5, 7

Using the Ideas

1. **A** List the multiples of 2 that are less than 25.
B List the multiples of 3 that are less than 25.
C List the common multiples of 2 and 3 that are less than 25.
D What is the LCM of 2 and 3?

2. For each pair of numbers, find the LCM by finding the smallest multiple of the larger number that is also a multiple of the smaller number.

EXAMPLE: 8, 10; Multiples of 10 are {10, 20, 30, 40}. ← 40 is also a multiple of 8.

A 3, 5	**D** 6, 12	**G** 5, 6	**J** 10, 11	**M** 4, 6
B 10, 4	**E** 8, 12	**H** 1, 15	**K** 15, 25	**N** 40, 60
C 2, 7	**F** 8, 3	**I** 7, 10	**L** 16, 24	**O** 12, 15

3. If you were working these addition problems, what common denominator would you choose? Think about the LCM of the denominators.

A $\frac{1}{2}+\frac{3}{4}$	**C** $\frac{3}{5}+\frac{1}{2}$	**E** $\frac{7}{12}+\frac{1}{8}$	**G** $\frac{7}{10}+\frac{3}{8}$	**I** $\frac{7}{12}+\frac{1}{16}$
B $\frac{2}{3}+\frac{1}{2}$	**D** $\frac{3}{8}+\frac{5}{6}$	**F** $\frac{4}{3}+\frac{2}{5}$	**H** $\frac{5}{8}+\frac{3}{16}$	**J** $\frac{4}{15}+\frac{3}{20}$

4. To find the LCM of 4, 6, and 10, we can first find the LCM of 6 and 10. It is 30. Now find the LCM of 30 and 4. It is 60. Thus 60 is the LCM of 4, 6, and 10. Find the LCM of the following groups of numbers.

A 2, 3, 5	**C** 2, 4, 6	**E** 4, 6, 12	**G** 3, 4, 8	**I** 10, 20, 30
B 2, 3, 4	**D** 2, 4, 8	**F** 2, 3, 6	**H** 5, 10, 15	**J** 8, 12, 15

★**5.** You can use prime factorizations to find the LCM of two numbers. Study the example, then use this method to find the LCM of each of these pairs of numbers.

EXAMPLE: $12 = 2 \cdot 2 \cdot 3$
$20 = 2 \cdot 2 \cdot 5$
The LCM of 12 and 20 is $2 \cdot 2 \cdot 3 \cdot 5 = 60$.

A 10, 15 **B** 6, 20 **C** 12, 14 **D** 9, 24 **E** 18, 12

★**6.** Find the LCM of each number pair.

A 2^2 and 3^2	**C** 2^3 and $2 \cdot 3^2$	**E** $2^2 \cdot 3$ and $3^2 \cdot 5$
B 3^3 and 5^2	**D** $2^2 \cdot 5$ and $3 \cdot 5^2$	**F** $3^2 \cdot 5^3 \cdot 7$ and $3^3 \cdot 5 \cdot 11$

★**7.** Two cars are driving around a 2-km track. They start off together. Car A makes one lap every 80 seconds. Car B makes one lap every 60 seconds.
A How long will it take the faster car to gain a lap on the slower one?
B How long will it take the faster car to gain a lead of 10 laps?

More practice, page S-11, Set 18

UTILIZATION

Notice that even though Exercise 3 involves work with fractions, the task at hand is simply to find the LCM for a pair of whole numbers. That is, students need to consider only the denominators of these fractions. Students should understand that they are not required to add the fractions.

Exercise 5 shows a method of using prime factorization to determine the LCM of two numbers. Although this exercise is meant primarily for more able students, you may find other students interested in learning this method.

Solution, Exercise 7

A. This exercise involves finding the least common multiple of 60 and 80, which is 240. After 240 seconds the slower car (80 s/lap) will have made 3 laps, but the faster car (60 s/lap) will have made 4 laps, thus having gained one lap on the slower car.
B. Since the faster car gains one lap every 240 seconds, it will gain 10 laps in 2400 s, or 40 minutes.

EXTENSION

For appropriate additional practice, see Supplementary Exercise Set 18 on page S-11, *Workbook* page 31, *Duplicator Masters* page 22, and *Arithmetic Skill Card* W-12.

Enrichment: Suggest exercises like these for the students.

1. Find the greatest common factor of 24 and 40. (8)
2. Find the least common multiple of 24 and 40 (120)
3. Find the product of the GCF and the LCM of 24 and 40 (960)
4. Find the product of $24 \cdot 40$ (960)
5. Compare your answers to C and D. Try some other pairs of numbers. Does this always work?

Assignments

Minimum 1, 2A–I, 3A–D.
Average 1, 2A–I, 3, 4A–F.
Maximum 1, 2J–P, 3, 4E–J, 5–7.

Objective

The student can demonstrate the ability to work with the concepts presented in this module.

UTILIZATION

You may assign the review exercises for independent study or use them as a basis for small-group or large-group discussion. The review problems should be excellent preparation for the self-evaluation test (Test Yourself), the module achievement test accompanying the series, or for a teacher-made test.

Answers, Review Exercises 1, 2, 11

1 A. Any number that is 2 times a whole number is an even number.
B. Any number that is 1 more than an even number is an odd number.

2 A. 14: {1, 2, 7, 14}
B. 60: {1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60}
C. 210: {1, 2, 3, 5, 6, 7, 10, 14, 15, 21, 30, 35, 42, 70, 105, 210}

11 A. Set $A = \{1, 3, 5, 9, 15, 27, 45, 135\}$
B. Set $B = \{1, 5, 7, 25, 35, 175\}$
C. Set $(A \cap B) = \{1, 5\}$

EXTENSION

Additional review exercises are provided on page 32 of the *Workbook*.

REVIEWING THE IDEAS

1. Give a definition for
A an even number. B an odd number.

2. List all the factors of
A 14 B 60 C 210

3. Show that 48 is a factor of 5808.

4. A A prime number has exactly ? factors.
B A ? number has more than two factors.
C Is 1 a prime number?
D Is 0 a composite number?

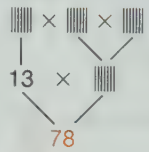
5. Tell whether or not each number is prime.

A 2	E 25	I 81
B 3	F 35	J 83
C 13	G 51	★ K 121
D 18	H 63	★ L 441

6. Complete the factor trees.

A

45

B 

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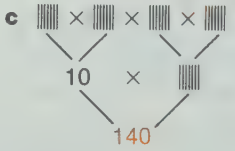
C 

140

D

225

7. Replace each factorization by a prime factorization.

A $72 = 8 \cdot 9$	C $100 = 4 \cdot 25$
B $144 = 16 \cdot 9$	D $675 = 5 \cdot 135$

8. Give the prime factorization of 84.

9. Write each product using exponents.

A $2 \cdot 2 \cdot 2 \cdot 2$	B $23 \cdot 23$
C $7 \cdot 7 \cdot 7 \cdot 7 \cdot 7$	D $6 \cdot 6 \cdot 6 \cdot 4 \cdot 4$
E $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$	

10. Find the prime factorization of each number. Use exponents to express the factorization.

A 27	C 48	E 400
B 60	D 120	F 72

11. If set A contains the factors of 135 and set B contains the factors of 175, list all the numbers in
A set A. B set B. C set $A \cap B$.

12. Find the greatest common factor (GCF) of these pairs of numbers.

A 8, 16	C 12, 29	E 60, 18
B 8, 15	D 14, 49	F 26, 182

13. Use the prime factorizations to find the GCF of each pair of numbers.

A $15 = 3 \cdot 5$	$25 = 5 \cdot 5$
B $44 = 2^2 \cdot 11$	$52 = 2^2 \cdot 13$
C $77 = 7 \cdot 11$	$143 = 11 \cdot 13$
D $81 = 3^4$	$72 = 2^3 \cdot 3^2$

14. A List the first 5 multiples of 6.
B List the first 5 multiples of 8.
C What is the least common multiple of 6 and 8?

15. Find the least common multiple (LCM) of 6 and
A 3 B 9 C 8 D 4 E 5

16. Find the LCM of each group of numbers.

A 4, 8, 12	C 4, 6, 9
B 3, 6, 10	D 10, 12, 20

TEST YOURSELF

1. List all the factors of 40.

2. Is 23 a factor of **431** or **437**?

3. Complete the sentence:

   If a number has exactly two different factors, then it is a ? number.

4. Which numbers in the set are prime number?

   {4, 5, 21, 29, 39, 47, 87}

5. Complete the factor tree.

6. If the prime factorization of n is $2 \cdot 5 \cdot 11$, what number is n?

7. Give the prime factorization of 54.

8. Write each product using exponents.
   - A $3 \cdot 3 \cdot 3 \cdot 3$
   - B $5 \cdot 5 \cdot 7 \cdot 7 \cdot 13 \cdot 13 \cdot 13$
   - C $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$

9. Which number is larger? 2^5 or 5^2?

10. What is the greatest common factor (GCF) of each pair of number?
    - A 10, 25
    - B 42, 77

11. What is the least common multiple (LCM) of each pair of numbers?
    - A 8, 12
    - B 20, 24

RESEARCH PROJECTS

A The **Seive of Eratosthenes** is a method that can be used to find prime numbers. Look up this method in a mathematics reference book and use it to find all the prime numbers less than 200. (See *A Panorama of Numbers* by Robert J. Wisner; Glenview, Illinois: Scott, Foresman and Company, 1970 available from Gage Educational Publishing, Ltd.)

B How can you tell when a number is exactly divisible by another number? Find out about some **divisibility rules**. In particular, find out how to tell when a number is divisible by 3, 4, 7, 9, 11, or 13. (See *Enrichment Mathematics for the Grades, Twenty-seventh Yearbook*; Washington, D.C.: National Council of Teachers of Mathematics, 1963.)

C Find out more about **perfect numbers**. How many perfect numbers less than 10 000 are known? Is there a formula for perfect numbers? (See *The Enjoyment of Mathematics* by Hans Rademacher and Otto Toelitz; Princeton, New Jersey: Princeton University Press, 1957.)

D The **Euclidean Algorithm** is a method of finding the greatest common factor of two numbers. It is particularly useful if both numbers are large. Find out how to use this method and use it to find the GCF of some pairs of large numbers such as 806 and 1333. (See *Fundamental Concepts of Elementary Mathematics* by Clarence H. Heinke; Belmont, California: Dickenson Publishing Company, 1970, pp.164–165.)

TEST YOURSELF

Encourage students to check the objective related to any exercise they find difficult. (See annotated correlations on the student page.) They should then refer to the lesson (or lessons) that treat that objective and restudy it.

RESEARCH PROJECTS

Research Project A offers students an opportunity to extend their experiences with prime numbers through work with the Sieve of Eratosthenes.

Project B, which is related to the lessons on multiples and factors (pages A-78 through A-81) may appeal to a number of students. Among the divisibility rules that they might find particularly interesting is the one for 11, which can be stated as follows:

> A number n is divisible by 11 if the units' digit minus the tens' digit plus the hundreds' digit minus the thousands' digit, and so on, is divisible by 11. For example, 5918 is divisible by 11 because $8 - 1 + 9 - 5 = 11$, which is of course divisible by 11.

Those students who worked the Think problem on page A-81 might find Research Project C appealing, for it invites them to learn more about perfect numbers.

Objective

The student will be able to demonstrate a basic understanding of the whole-number concepts presented in Unit A.

UTILIZATION

Since there are a large number of problems in this Cumulative Review, you may need to be selective in making the review assignment. Certainly, not every student should be expected to complete all parts of every problem.

If you plan to administer a Unit Achievement Test following this review, you might find it advantageous to assign the exercises over a two-day period. Thus, the first day's assignment might include only the exercises related to Module 1 (Exercises 1–12) and Module 2 (Exercises 13–23) and the second day's assignment might include the exercises related to Module 3 (Exercises 24–29) and Module 4 (Exercises 30–39).

Encourage students to concentrate their efforts on problems related to those concepts they find most troublesome. Make answers available to the students so that they can check their own work and try to correct their own mistakes.

CUMULATIVE REVIEW

1. Set $A = \{0, 1, 3, 5, 7\}$
Set $B = \{0, 2, 5, 8, 11\}$

A List the elements in $A \cup B$.

B List the elements in $A \cap B$.

2. In the numeral **18 473 905**, which digit is in the

A hundreds' place?

B ten thousands' place?

C one millions' place?

3. Write the usual numeral for

$9 \cdot 1000 + 3 \cdot 100 + 5 \cdot 10 + 1$

4. 10^3 means 10 used as a factor ? times.

5. Give the usual numeral for 10^5.

6. Express 8 000 000 in scientific notation.

7. Round 6 746 283 to the nearest

A hundred. **B** ten thousand.

8. Give the correct symbol ($<$, $=$, or $>$) for each ▒.

A 2467 ▒ 2475

B 18 429 ▒ 18 249

9. Write a numeral telling how many dots there are using

A base-ten numerals.

B base-eight numerals.

C base-five numerals.

10. Find the sum and difference.

A $5_{(8)} + 6_{(8)} =$ ▒

B $14_{(8)} - 5_{(8)} =$ ▒

11. Write the Roman numeral for 126.

12. Write a decimal numeral for LXXIV.

13. Find the output number for the flow chart.

14. Write two subtraction equations related to the addition equation $14 + 8 = n$.

15. Write two division equations related to the multiplication equation $9 \cdot n = 27$.

16. **A** Complete the function table.

Function Rule: $5 \cdot n + 2$	
n	$f(n)$
2	12
1	▒
▒	22
0	▒
▒	52

B Give the function rule.

Function Rule: ▒	
n	$f(n)$
2	5
3	7
0	1
5	11
7	15

17. Match the equation with the basic principle.

A $17 \cdot 99 = 99 \cdot 17$

B $6 \cdot (20 + 3) = 6 \cdot 20 + 6 \cdot 3$

C $(59 \cdot 2) \cdot 5 = 59 \cdot (2 \cdot 5)$

D $106 \cdot 1 = 106$

E $38 = 38 + 0$

1 Commutative principle (+)

2 Commutative principle (×)

3 Associative principle (+)

4 Associative principle (×)

5 Distributive principle

6 Zero principle

7 One principle

18. Solve the equations.

A $(20 - 5) \div 3 = r$

B $3 \cdot 9 + 8 = s$

C $(10 \cdot 2 - 6) \div 2 = t$

19. Find the sum and difference.

A
$$\begin{array}{r} 475 \\ 963 \\ +\ 49 \end{array}$$

B
$$\begin{array}{r} 1075 \\ -\ 499 \end{array}$$

20. Find the products and quotients.

A $37 \cdot 10$ **C** $350 \div 7$

B $900 \cdot 80$ **D** $4800 \div 60$

21. What is the best estimate for 829×69?

A 48 000 **B** 55 000

C 60 000

22. Find the products.

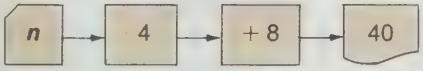
A
$$\begin{array}{r} 79 \\ \times 23 \end{array}$$

B
$$\begin{array}{r} 6438 \\ \times\ 305 \end{array}$$

23. Find the quotients and remainders.

A $32\overline{)729}$ **B** $495\overline{)19\,408}$

24. Write the equation suggested by this flow chart.



25. Make the inverse flow chart for Exercise 24.

26. What is n in Exercise 24?

27. Write and solve an equation for this statement.

When a certain number is multiplied by 8 and then 17 is subtracted from the product, the result is 39.

28. Estimate the answer to this problem.

A satellite travels at about 28000 km/h. About how far will it be in 48 hours?

29. Solve the problem. (A diagram may help you.)

The perimeter of a rectangle is 460 cm. The width is 100 cm. What is the length of the rectangle?

30. List the first five multiples of 4.

31. List all the factors of 24.

32. Is 17 a factor of **221** or **225**?

33. Which numbers in the set are prime?

$\{2, 9, 13, 27, 49, 59\}$

34. Complete the factor tree.

35. Write $2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 5$ using exponents.

36. Give the prime factorization of 45.

37. Find the product for $2^3 \cdot 3^2 \cdot 10^4$.

38. Find the GCF of these pairs of numbers.

A 15, 27 **B** 24, 40

39. Find the LCM of these groups of numbers.

A 9, 15 **B** 8, 13

C 3, 10, 12

MATHEMATICAL RECREATION

The repeating difference for any 4-digit number selected will be 6174.

Example: **5972**

$$\begin{array}{r} 9752 \\ -2579 \\ \hline 7173 \end{array}$$

$$\begin{array}{r} 7731 \\ -1377 \\ \hline 6354 \end{array}$$

$$\begin{array}{r} 6543 \\ -3456 \\ \hline 3087 \end{array}$$

$$\begin{array}{r} 8730 \\ -0378 \\ \hline 8352 \end{array}$$

$$\begin{array}{r} 8532 \\ -2358 \\ \hline 6174 \end{array}$$

$$\begin{array}{r} 7641 \\ -1467 \\ \hline 6174 \end{array}$$

These differences repeat

MATHEMATICAL RECREATIONS

THE REPEATING DIFFERENCE

Here is an interesting number investigation for you.

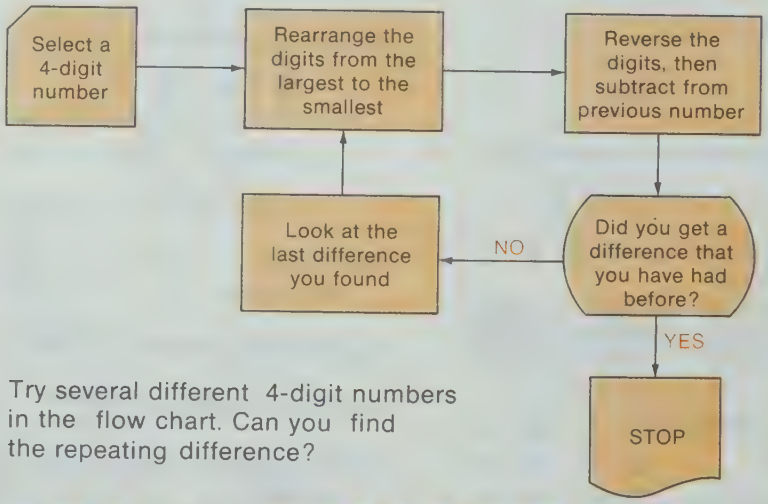
Choose any 4-digit number so that not all four digits are the same.

Rearrange the digits of the number from the largest to the smallest. Then reverse the order of the digits of the number and subtract from the previous number.

Rearrange the digits of the difference from the largest to the smallest. Then reverse the digits and subtract as before.

If you repeat this process several times, you will eventually get a difference which upon rearrangement and subtraction will give the same difference again and again.

The flow chart below may help you follow the steps for this investigation.


Select a 4-digit number
Rearrange the digits from the largest to the smallest
Reverse the digits, then subtract from previous number
Did you get a difference that you have had before?
NO
Look at the last difference you found
YES
STOP

Try several different 4-digit numbers in the flow chart. Can you find the repeating difference?

MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. 1, 2, 4, 5, 8, 10, 20, 40 **2.** 437 **3.** prime **4.** 5, 29, 47
5. $2 \cdot 5 \cdot 3 \cdot 7$ **6.** 110 **7.** $2 \cdot 3^3$ **8.** A 3^4; B $5^2 \cdot 7^2 \cdot 13^3$; C 10^6
9. 2^5 **10.** A 5; B 7 **11.** A 24; B 120

UNIT B: THE ARITHMETIC OF FRACTIONAL NUMBERS

Module 1: Fractions and Fractional Numbers

General Objectives

To review the fact that fractions are symbols that compare two sets or compare part of a set to a whole set.
To review pairs of equivalent fractions and sets of equivalent fractions.
To provide experience in finding lowest-terms fractions.
To extend the idea that a fractional number can be associated with a set of equivalent fractions and that there is a unique point on the number line for each fractional number.
To show that the set of fractional numbers can be ordered.

	Performance Objectives	Pupil Text	Reteach-Reinforce	Related Activities	
35	**RED** Given appropriate diagrams, the student can give fractions that compare sets and regions.	B-2, 3			**SWM 1** 218
36	**RED** Given a fraction, the student can produce a set of equivalent fractions.	B-4, 5	**WB** 33 **DM** 23	**ASC** F-1 **PD** c-1	**SWM 1** 222
37	**RED** Given a fraction, the student can write a lowest-terms fraction for it.	B-6, 7	**WB** 34 **DM** 24	**ASC** F-2 **SP** c-1	
38	**RED** Given two fractions, the student can determine whether or not they are equivalent.	B-10, 11	**WB** 35 **DM** 25	**SP** c-1 **PD** c-1	**SWM 1** 227
39	**RED** Given two fractions, the student can name the larger.	B-12, 13 B-14, 15	**WB** 36, 37 **DM** 26	**SP** c-1 **PD** c-1	
40	**YELLOW** Given an equation of the form $\frac{a}{b}=\frac{n}{c}$, the student can solve for n.	B-16, 17			
41	**YELLOW** Given simple word problems involving fractions, the student can solve them.	B-16, 17			
42	***GREEN*** *The student can use fractional numbers to compare graphical data and to solve problems.*	B-8, 9			
	Reviewing the Ideas	B-18	**WB** 38		

MATHEMATICS

The primary emphasis of this module is on the development of fractional number concepts in preparation for the later introduction of operations on the set of fractional numbers. The term *fractional numbers* is used to denote the set of nonnegative rational numbers.

Fractions are introduced as symbols, in the form of number pairs, that compare two sets. The work with sets of equivalent fractions is of particular importance to the development of fractional number concepts.

A formal definition of equivalent fractions follows:

Two fractions

$$\frac{a}{b} \text{ and } \frac{c}{d},\ b \neq 0,\ d \neq 0,$$

are equivalent to each other if and only if $a \cdot d = b \cdot c$.

Just as a cardinal number can be abstracted from a class of equivalent sets, so can we ab-

stract the fractional number one half from a set of equivalent fractions. Thus,

$$\left\{\tfrac{1}{2}, \tfrac{2}{4}, \tfrac{3}{6}, \tfrac{4}{8}, \tfrac{5}{10}, \tfrac{6}{12}, \tfrac{7}{14}, \ldots\right\}$$

can be associated with one fractional number, which in turn is associated with a unique point on the number line.

The fact that each fractional number has many different names (any fraction in the set) leads us to consider the concept of equality for fractional numbers. When we write $\frac{1}{2} = \frac{4}{8}$, we are indicating that $\frac{1}{2}$ and $\frac{4}{8}$ represent the same fractional number.

If we wish to compare two fractional numbers such as $\frac{3}{8}$ and $\frac{5}{8}$, we could easily arrive at $\frac{3}{8} < \frac{5}{8}$ by considering the physical interpretations from which the fractions $\frac{3}{8}$ and $\frac{5}{8}$ arise. If two fractions have different denominators, the most practical method of comparing them is by comparing fractions equivalent to the given fractions that have a common denominator. The following definition gives the shortest method of comparing two fractional numbers.

$\frac{a}{b} > \frac{c}{d}$ if and only if $a \cdot d > b \cdot c$.

Consider these examples:

Example 1

$\frac{3}{8} > \frac{5}{14}$ since $3 \cdot 14 > 8 \cdot 5$

Example 2

$\frac{9}{15} > \frac{4}{6}$ since $9 \cdot 6 > 15 \cdot 4$

TEACHING THE MODULE

Materials

Graph paper and metric measuring devices.

Vocabulary

centimetre
denominator
equivalent fractions
fraction
fractional number
gram
greatest common factor (GCF)
hectare
improper fraction
inequality
kilometre
litre
lowest-terms fraction
metre
metric system
number line
numerator
proper fraction
region
scale
set

Many of the ideas of this module may be review for your students, particularly if they have studied from earlier books in the *Investigating School Mathematics* series. If the basic ideas are clear to most students, you may find it helpful to spend extra time in building skills in finding lowest-terms fractions, solving fractional equations such as those in Exercise 4 on page B-11, or working with inequalities.

In this module, sequence is very important, for concepts of fractions and equivalent fractions must be developed prior to the introduction of the basic fractional number concepts. Once fractional numbers are introduced, it becomes feasible to proceed to ideas such as the number line, equality, and inequality.

There is relatively little computation in this module because the material is largely concept-oriented. Skills for computing with fractional numbers will be developed in subsequent modules.

Lesson Schedule

You should be able to move along at a relatively rapid pace through the material in the module. It is likely that 8 to 10 days will be sufficient time for all but the slower groups.

Evaluation

In evaluating student progress in this module, remember that the greater portion of the material is oriented toward concepts rather than computation. Since evaluation of concept understanding is often more difficult than evaluation of computational skills, you may want to give more attention to observing and interviewing students than to paper-pencil evaluation. The module review exercises and the student self-test can aid you in making your evaluation and can be used as models for teacher-made tests for the module. Also, Module and Unit Achievement Tests are available from the publisher.

RESOURCES FOR ACTIVE LEARNING

Activities

Aftermath 2, "Fraction Action," pp. 94–95, Creative Publications

Developmental Math Cards, "Bouncing Balls," K10, Addison-Wesley

Fraction Bars, Introductory Card Set, Scott Resources

Fractional Numbers, Geocards 6–14, Creative Publications

Geoboard Activity Card Kit, Cards 80–84, Cuisenaire Co.

Nuffield Project: *Computation and Structure 5* (Introduction in ordering of rationals), pp. 7–23, Wiley

Manipulative Devices

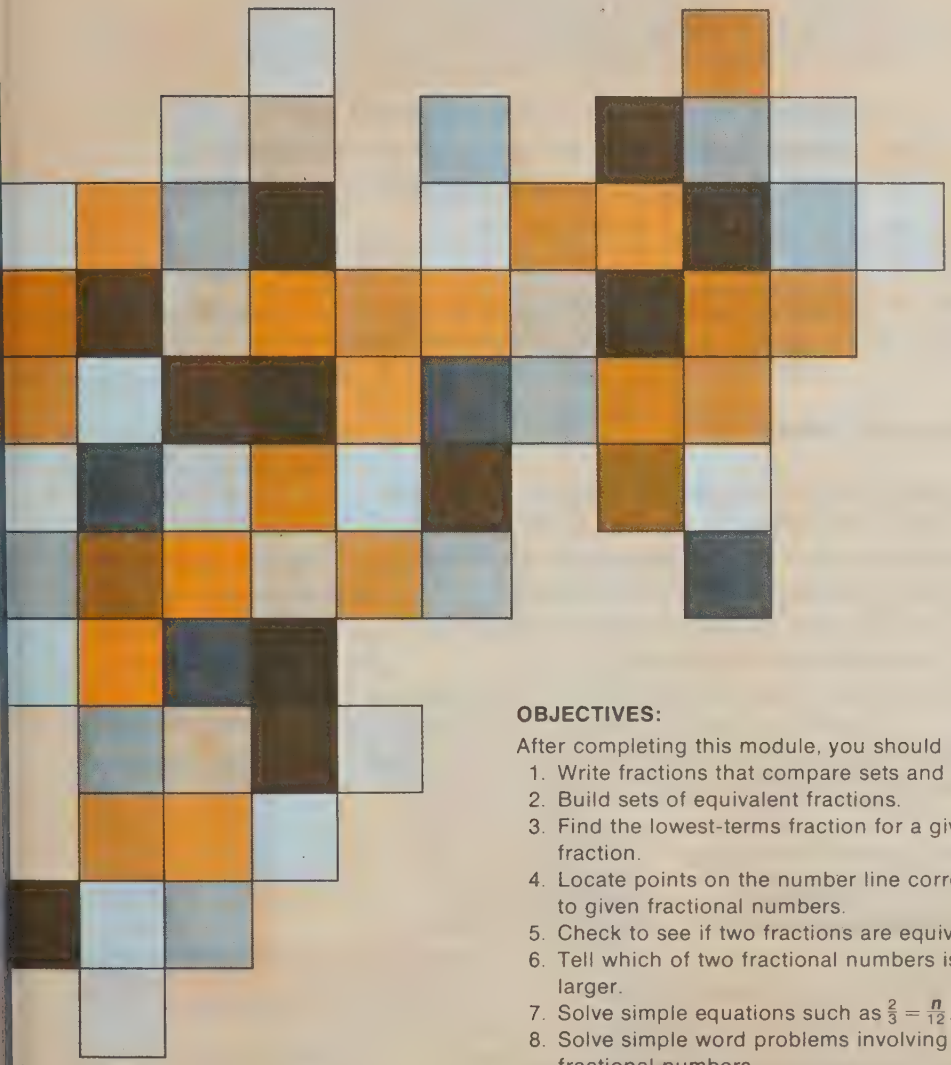
Fraction Bars manipulatives (Scott Resources)
Geoboards (Addison-Wesley)
Metric aids (Addison-Wesley; Balla; Creative Publications, Dick Blick; Educational Teaching Aids)

Games and Puzzles

Action Fractions (Lakeshore; Mind/Matter)
Fat Fraction (Educational Teaching Aids)
Fraction Dominoes (Creative Publications; Sigma)
In-order Game Kit (Midwest Publications)
Numo (Midwest Publications)
Spinner Number Games (Heath)
Tripletts (Midwest Publications)

UNIT B: *The Arithmetic of Fractional Numbers*

MODULE 1: *Fractions and Fractional Numbers*



OBJECTIVES:

After completing this module, you should be able to:

1. Write fractions that compare sets and regions.
2. Build sets of equivalent fractions.
3. Find the lowest-terms fraction for a given fraction.
4. Locate points on the number line corresponding to given fractional numbers.
5. Check to see if two fractions are equivalent.
6. Tell which of two fractional numbers is the larger.
7. Solve simple equations such as $\frac{2}{3} = \frac{n}{12}$.
8. Solve simple word problems involving fractional numbers.

Objective

The student can write a fraction that compares a part of a region to a whole region or a part of a set to a whole set of discrete elements.

PREPARATION

You may wish to review some elementary ideas about fractions before beginning the Investigation. Give simple examples such as:

The fraction $\frac{3}{5}$ tells what part of the rectangle is shaded. The denominator 5 tells how many parts of equal size. The numerator 3 tells how many of the parts are shaded.

INVESTIGATION

When you introduce the Investigation, suggest to the students that they make a table to record the information about the headline. For example:

Letter	Occurrences	Fraction
T	2	$\frac{2}{30}$
E	7	$\frac{7}{30}$
N	2	$\frac{2}{30}$

This will aid them in discussing Exercises 1 and 2.

DISCUSSION

Discussion Exercise 1 reviews the terms *numerator* and *denominator*. If students constructed a table showing the Investigation results, the answers to the questions are readily available.

When discussing Exercise 2, it is not necessary to discuss fractions in lowest terms, though students who know that $\frac{12}{30}$ in lowest terms is $\frac{2}{5}$ may give the simpler fraction. Lest the term *improper fraction* lead some students to think that such fractions are undesirable and must be written in mixed-numeral form, point out that often it is more convenient to work with fractions in that form than in mixed-numeral form.

Sets, Regions, and Fractions

Investigating the Ideas

Teenagers Enjoy Space Age Rocketry

In the newspaper headline, 2 of the 30 letters are T's.
The **fraction** $\frac{2}{30}$ compares the number of times the letter T appears to all the letters in the headline.
We say that $\frac{2}{30}$ of the letters are T's.

Can you write a fraction that compares the number of times each of the other letters appear to all of the letters in the headline?

Discussing the Ideas

1. The numeral above the bar of a fraction represents a number which is the **numerator** of the fraction. The numeral below the bar represents a number which is the **denominator** of the fraction.

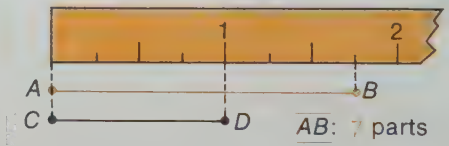
$\frac{2}{30}$ Numerator / Denominator

   A What is the denominator of each fraction that you found in the Investigation?
   B What are the numerators of the fractions that you found?

2. What fraction would compare the number of times a vowel (a, e, i, o, u) appears to all the letters in the headline?

3. A What part of the circular region is shaded?
   B What part is unshaded?

4. The fraction $\frac{7}{4}$ compares the length of $\overline{AB}$ to the unit $\overline{CD}$. The fraction $\frac{7}{4}$ is called an **improper fraction** because the numerator is greater than or equal to the denominator. The fraction $\frac{4}{7}$ is called a **proper fraction**.



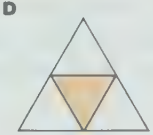
$\overline{AB}$: 7 parts
$\overline{CD}$: 4 parts

Tell whether each fraction below is a proper or an improper fraction.

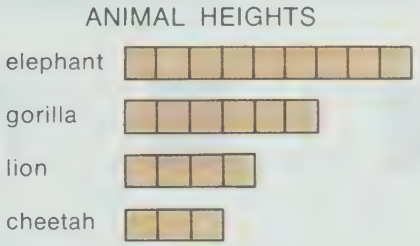
A $\frac{3}{8}$ B $\frac{9}{4}$ C $\frac{5}{5}$ D $\frac{1}{2}$ E $\frac{2}{3}$ F $\frac{3}{2}$ G $\frac{3}{3}$ H $\frac{5}{1}$

Using the Ideas

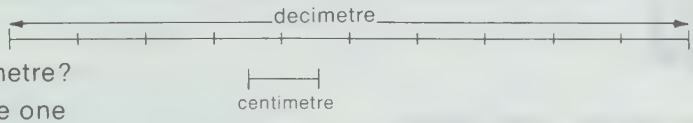
1. For each figure, write a fraction that indicates the part of the region that is shaded.

A B C D 

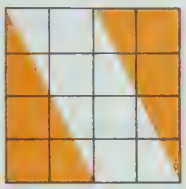
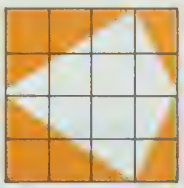
2. The bar graph compares the heights of four animals. Write a fraction that compares the heights in each part.
   - A lion to elephant
   - B cheetah to gorilla
   - C lion to cheetah
   - D elephant to gorilla
   - E cheetah to elephant



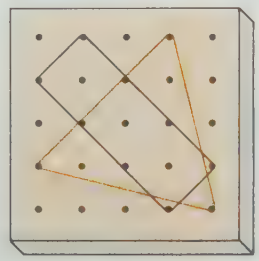
3. A A centimetre is what part of a decimetre?
   B Ten decimetres make one metre. What part of a metre is a centimetre?



4. Write a fraction that gives the part of each region that is shaded.

A  B C 

5. What part of the geoboard nails are
   - A inside the triangle?
   - B inside the rectangle?
   - C inside both figures?
   - D outside of the rectangle?
   - E outside of the triangle?
   - F outside both figures?



UTILIZATION

The exercises provide experiences with fractions as parts of regions or parts of sets.

In discussing Exercise 2, you may want to emphasize the language of comparison that is used with fractions. When comparing a number *x to a second number y*, we write $\frac{x}{y}$.

EXTENSION

Remedial: Ask students to write a fraction that indicates the part of the region that is shaded for each figure.

A. B.

C. D.

Enrichment: Have the students draw regions on graph paper and shade them to illustrate these fractions.

A. $\frac{5}{12}$ B. $\frac{7}{16}$ C. $\frac{19}{25}$ D. $\frac{23}{24}$ E. $\frac{61}{100}$

Assignments

Minimum 1–3. Average 1–4. Maximum 1–5.

Objective

The student can name several fractions equivalent to a given fraction.

PREPARATION

Materials: Graph paper, cm or $\frac{1}{2}$ cm (*Duplicator Masters,* pages 83 and 84).

The Investigation for this lesson will be easier for many students if they use graph paper to help them construct and shade the regions to represent the fractions equivalent to $\frac{3}{4}$.

INVESTIGATION

Have the students study examples A and B in the Investigation and give several more fractions for each. Then suggest that they draw several regions all the same size and shape on the graph paper and shade $\frac{3}{4}$ of each region. Now the student should further subdivide the region by doubling, tripling, and quadrupling the number of parts of equal size in the regions. Thus, a student might show figures like these:

Note that doubling, tripling, quadrupling, and so on, the number of subdivisions has the effect of multiplying the numerator and denominator of the fraction by the same whole number.

DISCUSSION

The main purpose in the Discussion is to give meaning to the method of finding fractions equivalent to a given fraction. Exercise 2 introduces the arithmetic needed:

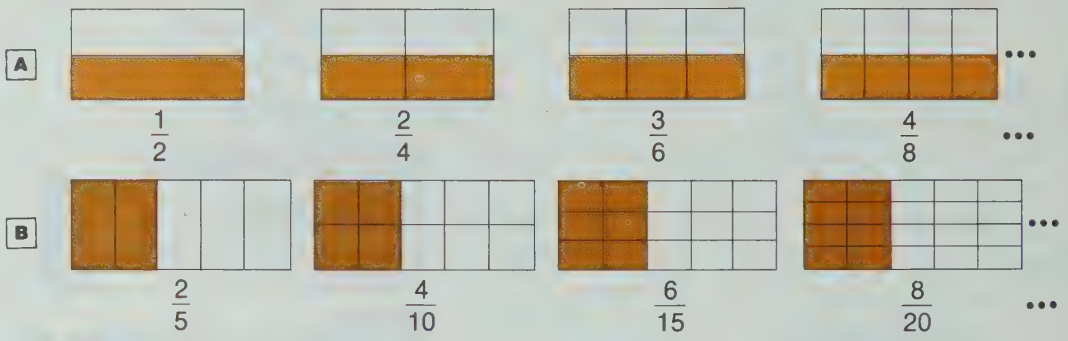
If $\frac{a}{b}$ is any fraction and n is any nonzero whole number, then $\frac{a}{b}$ is equivalent to $\frac{a \cdot n}{b \cdot n}$.

Exercise 3 states that n can be any whole number except zero. If a student should ask why n cannot be zero, you might note that this would give a product of zero in both numerator and denominator, and zero denominators are not permitted.

Equivalent Fractions

Investigating the Ideas

Two different fractions may be suggested by the same subset of a set or by the same part of a region. The fractions in **A** and **B** name the shaded part of each region. Give four more fractions for each sequence.



Can you draw and shade some regions for the fraction $\frac{3}{4}$?	Record the fractions for each of your regions.

Discussing the Ideas

1. Two different fractions which show the same part of a region or the same part of a set are called **equivalent fractions**. Use the figures in **A** and **B** above to answer these questions.

   A Is $\frac{1}{2}$ equivalent to $\frac{2}{4}$? **B** Is $\frac{2}{5}$ equivalent to $\frac{6}{15}$? **C** Is $\frac{1}{2}$ equivalent to $\frac{2}{5}$?

2. The diagram below may help you find other fractions for **B** above. What are the fractions for **A**, **B**, **C**, and **D**?

$$\frac{2}{5}\ \ \frac{2 \cdot 1}{5 \cdot 1}\ \ \frac{2 \cdot 2}{5 \cdot 2}\ \ \frac{2 \cdot 3}{5 \cdot 3}\ \ \frac{2 \cdot 4}{5 \cdot 4}\ \ \frac{2 \cdot 5}{5 \cdot 5}\ \ \frac{2 \cdot 6}{5 \cdot 6}$$

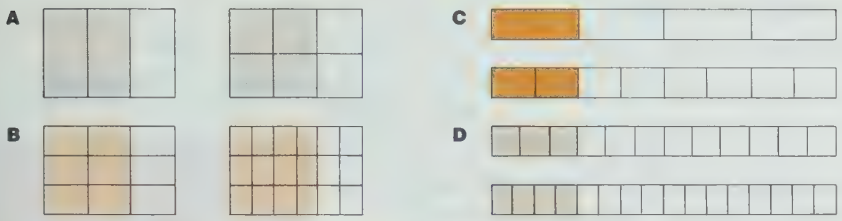
$$\downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow$$

$$\frac{2}{5}\ \ \frac{4}{10}\ \ \mathbf{A}\ \ \mathbf{B}\ \ \mathbf{C}\ \ \mathbf{D}$$

3. If n represents any whole number except zero, what can you conclude about the pair of fractions $\frac{7}{8}$ and $\frac{7 \cdot n}{8 \cdot n}$?

Using the Ideas

1. Each pair of figures suggests a pair of equivalent fractions. Write the pair of fractions.



2. Write the missing fractions for each row.

$\frac{5}{5}$ $\frac{5 \cdot 1}{6 \cdot 1}$ $\frac{5 \cdot 2}{6 \cdot 2}$ $\frac{5 \cdot 3}{6 \cdot 3}$ $\frac{5 \cdot 4}{6 \cdot 4}$ $\frac{5 \cdot 5}{6 \cdot 5}$ $\frac{5 \cdot 6}{6 \cdot 6}$

↓ ↓ ↓ ↓ ↓ ↓

$\frac{5}{6}$ A B C D E

$\frac{4}{3}$ $\frac{4 \cdot 1}{3 \cdot 1}$ $\frac{4 \cdot 2}{3 \cdot 2}$ $\frac{4 \cdot 3}{3 \cdot 3}$ $\frac{4 \cdot 4}{3 \cdot 4}$ $\frac{4 \cdot 5}{3 \cdot 5}$ $\frac{4 \cdot 6}{3 \cdot 6}$

↓ ↓ ↓ ↓ ↓ ↓

F G H I J K

3. Write the next three fractions in each **set of equivalent fractions**.

A $\left\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots\right\}$

B $\left\{\frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \frac{4}{16}, \ldots\right\}$

C $\left\{\frac{3}{7}, \frac{6}{14}, \frac{9}{21}, \frac{12}{28}, \ldots\right\}$

D $\left\{\frac{1}{6}, \frac{2}{12}, \frac{3}{18}, \ldots\right\}$

E $\left\{\frac{3}{8}, \frac{6}{16}, \frac{9}{24}, \ldots\right\}$

F $\left\{\frac{1}{9}, \frac{2}{18}, \frac{3}{27}, \ldots\right\}$

G $\left\{\frac{1}{5}, \frac{2}{10}, \ldots\right\}$

H $\left\{\frac{2}{7}, \frac{4}{14}, \ldots\right\}$

I $\left\{\frac{7}{10}, \ldots\right\}$

J $\left\{\frac{3}{2}, \ldots\right\}$

K $\left\{\frac{4}{9}, \ldots\right\}$

L $\left\{\frac{5}{5}, \ldots\right\}$

4. One of the fractions in each set is not equivalent to any of the other fractions in that set. Which fraction is it?

A $\left\{\frac{3}{4}, \frac{6}{8}, \frac{8}{12}, \frac{12}{16}, \frac{15}{20}\right\}$

B $\left\{\frac{1}{3}, \frac{2}{3}, \frac{3}{9}, \frac{4}{12}, \frac{5}{15}\right\}$

C $\left\{\frac{3}{8}, \frac{6}{16}, \frac{9}{24}, \frac{12}{36}, \frac{15}{40}\right\}$

D $\left\{\frac{2}{3}, \frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \frac{12}{16}\right\}$

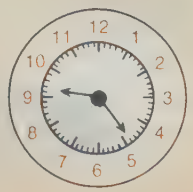
E $\left\{\frac{5}{4}, \frac{10}{6}, \frac{15}{12}, \frac{20}{16}, \frac{25}{20}\right\}$

F $\left\{\frac{6}{24}, \frac{5}{20}, \frac{2}{8}, \frac{4}{18}, \frac{3}{12}\right\}$

think

What part of the clock face is shaded?

Time: 9:24



More practice, page S-12, Set 19

UTILIZATION

Exercise 1 gives the students an opportunity to use regions to develop the general idea of equivalent fractions. The other exercises are designed to help your students understand the concept of equivalent fractions and the idea of building sets of equivalent fractions.

EXTENSION

To provide additional practice, make assignments from Supplementary Exercise Set 19 on page S-12, *Workbook* page 33, *Duplicator Masters* page 23, and *Arithmetic Skill Card* F-1.

Think Solution

At 9:24 the minute hand is exactly on the first minute mark before 5 and the hour hand is on the second minute mark after 9. There are 60 minute marks in all. Thus the shaded part covers $\frac{23}{60}$ of the clock face.

Assignments

Minimum 1, 2, 3A–C. Average 1–3. Maximum 1–4.

Objective

The student can decide whether a given fraction is in lowest terms and give the lowest-terms fraction for a given fraction.

PREPARATION

Review the idea of equivalent fractions by naming some fractions like $\frac{1}{3}$, $\frac{3}{4}$, $\frac{2}{5}$, $\frac{3}{5}$ orally and ask students to name or write several fractions equivalent to each one.

INVESTIGATION

This Investigation will require only a few minutes. Have the students record the "simpler" fraction for each part. Most students will probably name the lowest-terms fraction since those are the fractions most commonly used. To extend the Investigation, you might suggest that students make up some other statements that contain fractions which are not in lowest terms. For example:

I ate $\frac{30}{40}$ of a candy bar.

My notebook is about $\frac{25}{100}$ of a metre wide.

On a basketball team $\frac{6}{15}$ of the players are forwards.

Then have other students give simpler fractions for each.

DISCUSSION

The first exercise introduces the term "lowest-terms fraction," while the second exercise defines the term and suggests a method of finding the GCF of the numerator and denominator of the fraction.

Note that the flow chart in Exercise 3 gives a practical and commonly used method of finding the lowest-terms fraction for a given fraction. It is not necessary to divide numerator and denominator by the GCF, provided one repeats the division process until *all* common factors greater than 1 in the numerator and denominator have been divided out. Thus, for $\frac{12}{18}$ a student might proceed as follows:

$$\frac{12}{18} = \frac{12 \div 3}{18 \div 3} = \frac{4}{6}, \frac{4}{6} = \frac{4 \div 2}{6 \div 2} = \frac{2}{3}$$

Lowest-terms Fractions

Investigating the Ideas

Read each statement below.

1. Mary bought $\frac{50}{100}$ litre of milk.
2. $\frac{20}{30}$ of the students walk to school.
3. A dime is $\frac{10}{100}$ of a dollar.
4. The team won $\frac{14}{16}$ of its games.

Can you replace each fraction in the statements above with a "simpler" equivalent fraction?

Discussing the Ideas

1. For each set of equivalent fractions there is one fraction that is the "simplest" fraction in the set. This fraction is the **lowest-terms** fraction. What is the lowest-terms fraction for the set of equivalent fractions that contains $\frac{50}{100}$?

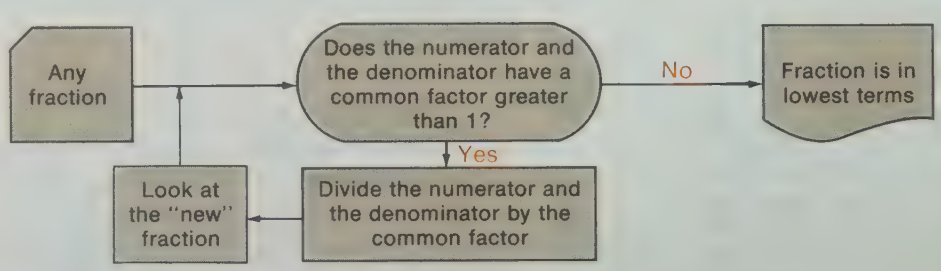
2. A fraction is in lowest terms if the greatest common factor (GCF) of both the numerator and the denominator is 1.

   A Which of the fractions $\frac{12}{15}$ or $\frac{9}{16}$ is in lowest terms?

   B What is the lowest-terms fraction for each fraction in the chart?

Fraction	Factors	GCF	Original fraction
$\frac{12}{15}$	{1, 2, 3, 4, 6, 12}	3	$\frac{4 \cdot 3}{5 \cdot 3}$
	{1, 3, 5, 15}		
$\frac{9}{16}$	{1, 3, 9}	1	$\frac{9 \cdot 1}{16 \cdot 1}$
	{1, 2, 4, 8, 16}		

3. Study the flow chart. Then explain how to use it in order to find the lowest-terms fraction for each fraction below.


Any fraction
Does the numerator and the denominator have a common factor greater than 1?
No
Fraction is in lowest terms
Yes
Divide the numerator and the denominator by the common factor
Look at the "new" fraction

A $\frac{12}{15}$ B $\frac{20}{24}$ C $\frac{10}{4}$ D $\frac{24}{36}$ E $\frac{7}{7}$ F $\frac{6}{29}$ G $\frac{18}{12}$

Using the Ideas

1. Give the lowest-terms fraction for each fraction.

A $\frac{8}{10}$	**C** $\frac{10}{16}$	**E** $\frac{18}{18}$	**G** $\frac{35}{50}$	**I** $\frac{18}{81}$	**K** $\frac{77}{98}$	**M** $\frac{45}{120}$
B $\frac{12}{14}$	**D** $\frac{9}{15}$	**F** $\frac{36}{40}$	**H** $\frac{12}{20}$	**J** $\frac{60}{45}$	**L** $\frac{105}{175}$	**N** $\frac{72}{180}$

2. Tell whether or not each fraction is in lowest terms. If not, give the lowest-terms fraction.

A $\frac{10}{12}$	**C** $\frac{8}{9}$	**E** $\frac{13}{39}$	**G** $\frac{11}{44}$	**I** $\frac{0}{30}$	**K** $\frac{70}{66}$	**M** $\frac{12}{121}$
B $\frac{15}{24}$	**D** $\frac{14}{15}$	**F** $\frac{25}{36}$	**H** $\frac{21}{28}$	**J** $\frac{4}{15}$	**L** $\frac{28}{331}$	**N** $\frac{38}{95}$

3. Find the lowest-terms fraction for each set.

A $\left\{\frac{4}{6}, \frac{6}{9}, \frac{10}{15}, \frac{20}{30}\right\}$

B $\left\{\frac{15}{60}, \frac{16}{64}, \frac{40}{160}, \frac{8}{32}\right\}$

C $\left\{\frac{90}{100}, \frac{900}{1000}, \frac{9000}{10\,000}, \frac{90\,000}{100\,000}\right\}$

D $\left\{\frac{65}{52}, \frac{25}{20}, \frac{100}{80}, \frac{15}{12}\right\}$

E $\left\{\frac{36}{27}, \frac{68}{51}, \frac{140}{105}\right\}$

F $\left\{\frac{10}{35}, \frac{24}{84}, \frac{42}{147}\right\}$

4. If the prime factorizations of the numerator and the denominator are known, then you can easily find the lowest-terms fraction. Study the example and then give the lowest-terms fraction for each part.

EXAMPLE: $\frac{30}{42} = \frac{2 \cdot 3 \cdot 5}{2 \cdot 3 \cdot 7} = \frac{5}{7}$

A $\frac{2 \cdot 3}{2 \cdot 5}$	**D** $\frac{19 \cdot 23}{19 \cdot 31}$	**G** $\frac{7 \cdot 11 \cdot 19}{11 \cdot 13 \cdot 19}$	**J** $\frac{3 \cdot 5 \cdot 5 \cdot 7}{3 \cdot 5 \cdot 7 \cdot 13}$
B $\frac{2 \cdot 3}{3 \cdot 5}$	**E** $\frac{2 \cdot 3 \cdot 5}{2 \cdot 3 \cdot 7}$	**H** $\frac{3 \cdot 3 \cdot 7}{3 \cdot 5 \cdot 7}$	**K** $\frac{2 \cdot 7 \cdot 11 \cdot 11}{7 \cdot 11 \cdot 13}$
C $\frac{7 \cdot 11}{11 \cdot 13}$	**F** $\frac{5 \cdot 7 \cdot 11}{2 \cdot 5 \cdot 7}$	**I** $\frac{11 \cdot 11 \cdot 13}{11 \cdot 13 \cdot 31}$	**L** $\frac{3 \cdot 3 \cdot 19 \cdot 31}{7 \cdot 3 \cdot 19}$

★5. Give the missing numerator or denominator in order to form fractions equivalent to the given lowest-terms fraction.

A $\frac{3}{4} \rightarrow \left\{\frac{\square}{20}, \frac{9}{\square}, \frac{\square}{40}, \frac{15}{\square}\right\}$

B $\frac{2}{7} \rightarrow \left\{\frac{\square}{21}, \frac{14}{\square}, \frac{6}{\square}, \frac{\square}{70}\right\}$

C $\frac{8}{9} \rightarrow \left\{\frac{24}{\square}, \frac{\square}{45}, \frac{72}{\square}, \frac{\square}{72}\right\}$

D $\frac{5}{6} \rightarrow \left\{\frac{\square}{18}, \frac{\square}{72}, \frac{40}{\square}, \frac{75}{\square}\right\}$

E $\frac{7}{11} \rightarrow \left\{\frac{\square}{22}, \frac{77}{\square}, \frac{\square}{77}, \frac{91}{\square}\right\}$

think

Discounting wind resistance and starting from rest, an object will fall approximately as indicated in the table below.

$490 \cdot 1^2$ cm in 1 sec	$490 \cdot 3^2$ cm in 3 sec
$490 \cdot 2^2$ cm in 2 sec	$490 \cdot n^2$ cm in n sec

If a skydiver jumps from 2960 metres, about how many seconds should he wait in order to pull his ripcord at about 1000 metres?

More practice, page S-12, Set 20

UTILIZATION

Gear your assignment to the needs and abilities of your students. If they are already proficient at expressing fractions in lowest terms, you may want to concentrate on exercises like Exercises 4 and 5, which provide a different approach to the ideas involved in finding the lowest-terms fraction.

EXTENSION

Supplementary Exercise Set 20 on page S-12 offers more practice. See also *Workbook* page 34, *Duplicator Masters* page 24, and *Arithmetic Skill Card* F-2.

Remedial: For students having difficulty finding lowest-terms fractions, give additional examples and have them use the flow-chart method in Discussion Exercise 3 to help them simplify the fractions. Avoid fractions with unwieldy numerators and denominators; real life presents few instances in which it is necessary to simplify such fractions.

Enrichment: The fractions below are partially factored. Ask students to find (a) the lowest-terms fraction and (b) the original fraction before factoring. (The lowest- terms fractions are given in parentheses.)

1. $\frac{6 \cdot 7}{15 \cdot 7}$ $\left(\frac{2}{5}\right)$
2. $\frac{9 \cdot 11}{3 \cdot 5 \cdot 11}$ $\left(\frac{3}{5}\right)$
3. $\frac{10 \cdot 7 \cdot 13}{15 \cdot 13}$ $\left(\frac{14}{3}\right)$
4. $\frac{39 \cdot 17}{15 \cdot 13}$ $\left(\frac{17}{5}\right)$
5. $\frac{21 \cdot 15}{9 \cdot 7}$ (5)
6. $\frac{33 \cdot 10 \cdot 6}{2 \cdot 30 \cdot 11}$ (3)

Think Solution

If the skydiver jumps from 2960 metres and must pull the ripcord at 1000 metres, he will free-fall a distance of $2960 - 1000 = 1960$ metres.

To arrive at the solution, find what number squared times 490 equals 196 000 (1960 metres = 196 000 cm).

$$490 \cdot n^2 = 196\,000$$
$$n^2 = 400$$
$$n = 20$$

Assignments

Minimum 1–3. Average 1–4. Maximum 1–5.

Objective

The student can associate one fractional number with a set of equivalent fractions and match the number with a unique point on the fractional number line.

PREPARATION

As preparation for this lesson, you might choose to review sets of equivalent fractions, to make sure that your students are able to build these sets efficiently. It might also be helpful to display on the chalkboard several sets of equivalent fractions each of which contains at least five or six fractions.

INVESTIGATION

This lesson is basic both to the development of the module and to the develolment of all the fractional number concepts in the text. This section makes the transition from the concept of fraction to that of number. Prior to this, the idea of number has not been associated with a given fraction. The concept of number will be abstracted from a set of equivalent fractions, just as, at some time, the number 2 was abstracted from a large class of equivalent sets.

The diagram in the Investigation points out the relationship between a set of equivalent fractions, a fractional number, and the correspondence of each fractional number to a unique point on the number line. If some students have some difficulty finding the number for the points on the number line, suggest that they count the number of divisions between 0 and 1 to find the denominator and then count over to the colored arrow to find the numerator.

DISCUSSION

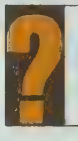
It is not necessary to be formal in making the transition from sets of equivalent fractions to fractional numbers. We use the term *fractional numbers* to denote the *non-negative rational numbers*. Emphasize that we usually find it convenient to use a lowest-terms fraction to name a fractional number, but this is not an absolute requirement.

Fractional Numbers

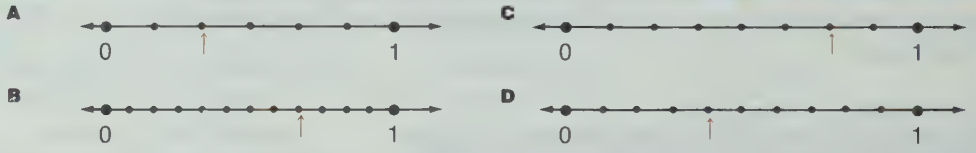
Investigating the Ideas

Fractions are symbols for **fractional numbers**. For each set of equivalent fractions, we think of exactly one fractional number. Each fractional number can be paired with exactly one point on the number line.

A set of equivalent fractions	One fractional number	One point on the number line
$\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \ldots\}$	two thirds	0, $\frac{2}{3}$, 1



Can you give a fraction that names the fractional number for the point above the arrow on each number line below? Record your answers.



Discussing the Ideas

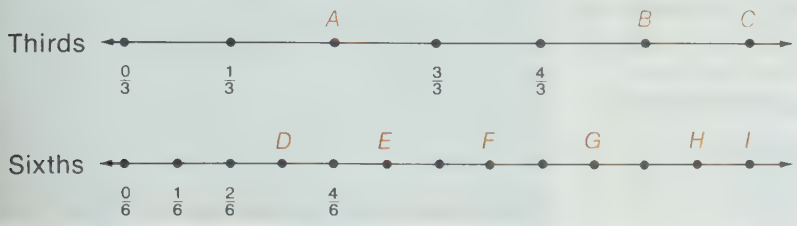
1. For the point on number line A, Steve wrote $\frac{2}{6}$. Margaret wrote $\frac{1}{3}$. Which student is correct?

2. Any fraction in a set of equivalent fractions can be used as a name for the fractional number associated with that set. Usually, the lowest-terms fraction is used as the name for the fractional number. Give the lowest-terms fraction for points *X*, *Y*, and *Z* on the number line.

3. The arrows designate points halfway between points shown on the number line. What number goes with each lettered point?

Using the Ideas

1. Use the denominator suggested at the left of each number line to give the fraction for the point indicated by each letter.



2. The number line is **scaled in fourths** because each segment between two whole-number points is divided into **four** congruent segments. Give the fractional number for each lettered point.

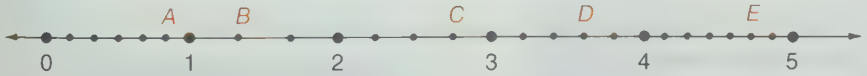
3. Give the fractional number for each lettered point.

4. One of the five numbers $\frac{7}{3}, \frac{4}{5}, \frac{9}{7}, \frac{7}{2}, \frac{11}{3}$, is matched with each point on the number line. Match the numbers with the lettered points.

5. Draw a number line and mark the points for 0, 1, 2, 3, 4, and 5. Then mark the approximate location of the point for each number below.

A $\frac{3}{2}$ **B** $\frac{12}{5}$ **C** $\frac{10}{3}$ **D** $\frac{25}{6}$ **E** $\frac{9}{2}$ **F** $\frac{1}{10}$ **G** $\frac{11}{10}$ **H** $\frac{21}{10}$

★6. Give the fractional numbers that go with the points *A*, *B*, *C*, *D*, and *E*.



UTILIZATION

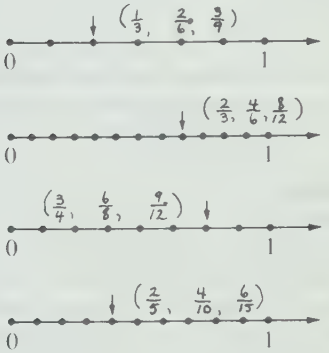
You may find it necessary to discuss with some students the meaning of a *scale* on the number line since Exercise 2 shows a number line scaled in fourths and the number line in Exercise 3 is scaled in fifths. The number line in Exercise 6 is scaled differently between each pair of whole numbers.

EXTENSION

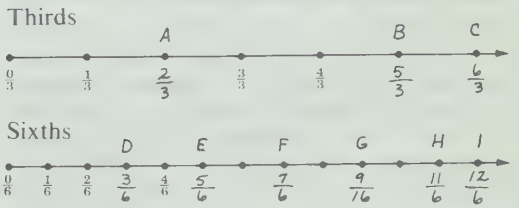
Remedial: Use examples such as the ones below for reteaching and reinforcement.

1 A. How is each number line scaled?

B. Give three fractions that name the rational number for the point indicated by the arrow on each number line.



2. Use the denominator suggested at the left of each number line to give the fraction for the point indicated by each letter.



Assignments

Minimum 1–3. Average 1–4. Maximum 1–6.

Objective

The student can use the "cross products" method to decide if two fractions are equivalent.

PREPARATION

Review some ways of building a set of equivalent fractions for a given fraction. Since each fractional number has many names, it is important for the student to have a short method of deciding whether two fractions name the same or different fractional numbers. This is the purpose of the lesson.

INVESTIGATION

Have the students study the method of checking equivalence of two fractions shown in the illustration. Then have them use this method to decide which pairs of fractions A through D are equivalent. If two fractions are equivalent, they name the same fractional number. If two fractions are not equivalent, they do not name the same fractional number.

DISCUSSION

The Investigation introduces the student to a simple test for determining whether or not two fractions are equivalent and, hence, name the same fractional number. The example in Exercise 1 shows why the two cross products for equivalent fractions will be the same. It is important that more able students understand this basic idea, while for slower students it is important only that they learn to use the test.

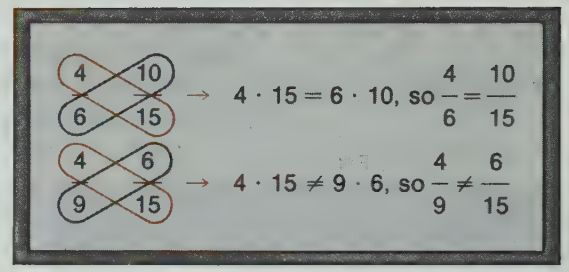
The example in Exercise 3 illustrates how the cross products method can be used to solve some simple fractional equations. Observe that, after finding the cross products to obtain $3 \cdot n = 24$, the student is expected to solve the equation by thinking of the missing factor or by using the definition of division to find that $n = 24 \div 3 = 8$. This method will be used later in solving simple proportions.

Equality of Fractional Numbers

Investigating the Ideas

Pete showed his classmates a method they could use to tell whether or not two fractions were equivalent.

Study his method in the examples shown on the chalkboard.



Can you use this method to decide which pairs of fractions are equivalent?

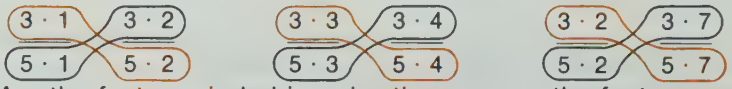
A $\frac{3}{8}, \frac{6}{16}$ **B** $\frac{4}{6}, \frac{6}{9}$ **C** $\frac{4}{7}, \frac{6}{10}$ **D** $\frac{5}{3}, \frac{15}{9}$

Discussing the Ideas

1. The class wondered whether Pete's method would always work. The teacher suggested that they look at some sets of equivalent fractions. She listed the set of fractions equivalent to $\frac{3}{5}$.

$$\left\{\frac{3}{5}, \frac{6}{10}, \frac{9}{15}, \frac{12}{20}, \frac{15}{25}, \frac{18}{30}, \frac{21}{35}, \cdots\right\}$$
$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$
$$\left\{\frac{3 \cdot 1}{5 \cdot 1}, \frac{3 \cdot 2}{5 \cdot 2}, \frac{3 \cdot 3}{5 \cdot 3}, \frac{3 \cdot 4}{5 \cdot 4}, \frac{3 \cdot 5}{5 \cdot 5}, \frac{3 \cdot 6}{5 \cdot 6}, \frac{3 \cdot 7}{5 \cdot 7}, \cdots\right\}$$

Then she chose several pairs of fractions from the set and circled the factors in the "cross products."



Are the factors circled in color the same as the factors circled in black for each pair of fractions?

2. How would you use the "cross products" method to decide which pairs of fractions are equivalent?

A $\frac{3}{4}, \frac{9}{12}$ **B** $\frac{6}{18}, \frac{4}{9}$ **C** $\frac{2}{2}, \frac{10}{10}$ **D** $\frac{3}{1}, \frac{12}{4}$ **E** $\frac{5}{6}, \frac{3}{4}$

3. The "cross products" method can be used to solve equations involving fractional numbers. Study the example. Then explain how you would solve these equations.

A $\frac{3}{4} = \frac{n}{20}$ **B** $\frac{n}{2} = \frac{9}{6}$ **C** $\frac{4}{n} = \frac{12}{15}$

EQUATION: $\frac{2}{3} = \frac{n}{12}$

SOLUTION: $3 \cdot n = 2 \cdot 12$
$3 \cdot n = 24$
$n = 8$

Using the Ideas

1. Which pairs of fractions are equivalent?

A $\frac{5}{6}, \frac{9}{10}$ **B** $\frac{8}{6}, \frac{12}{9}$ **C** $\frac{3}{10}, \frac{9}{100}$ **D** $\frac{4}{20}, \frac{5}{25}$ **E** $\frac{8}{12}, \frac{5}{25}$ **F** $\frac{1}{8}, \frac{3}{20}$ **G** $\frac{5}{3}, \frac{20}{12}$ **H** $\frac{0}{7}, \frac{0}{8}$ **I** $\frac{2}{1}, \frac{18}{9}$ **J** $\frac{1}{3}, \frac{33}{100}$

2. Copy each exercise and insert the correct symbol ($=$ or $\neq$) for each ▨.

A $\frac{6}{9} \square \frac{4}{6}$ **B** $\frac{4}{10} \square \frac{10}{25}$ **C** $\frac{20}{24} \square \frac{5}{6}$ **D** $\frac{8}{9} \square \frac{7}{8}$ **E** $\frac{10}{12} \square \frac{12}{14}$ **F** $\frac{8}{21} \square \frac{4}{10}$ **G** $\frac{7}{6} \square \frac{21}{18}$ **H** $\frac{5}{1} \square \frac{40}{8}$ **I** $\frac{3}{3} \square \frac{11}{11}$ **J** $\frac{90}{100} \square \frac{9}{10}$ **K** $\frac{11}{15} \square \frac{99}{135}$ **L** $\frac{23}{31} \square \frac{713}{961}$

3. Answer true (T) or false (F).

A $\frac{1}{3} = \frac{3}{9}$ **B** $\frac{2}{5} = \frac{5}{2}$ **C** $\frac{3}{7} = \frac{7}{13}$ **D** $\frac{7}{8} = \frac{56}{64}$ **E** $\frac{2}{15} = \frac{12}{90}$ **F** $\frac{5}{6} = \frac{45}{48}$ **G** $\frac{5}{9} = \frac{26}{45}$ **H** $\frac{11}{12} = \frac{55}{60}$ **I** $\frac{14}{16} = \frac{35}{80}$ **J** $\frac{63}{18} = \frac{49}{15}$ ★**K** $\frac{7}{13} = \frac{91}{169}$ ★**L** $\frac{11}{17} = \frac{187}{289}$

4. Solve the equations.

A $\frac{2}{3} = \frac{n}{15}$ **B** $\frac{n}{4} = \frac{7}{2}$ **C** $\frac{3}{n} = \frac{15}{40}$ **D** $\frac{16}{40} = \frac{n}{5}$ **E** $\frac{6}{5} = \frac{24}{n}$ **F** $\frac{8}{12} = \frac{6}{n}$ **G** $\frac{10}{12} = \frac{n}{18}$ **H** $\frac{5}{8} = \frac{n}{8}$ **I** $\frac{n}{10} = \frac{40}{100}$ ★**J** $\frac{4}{n} = \frac{n}{9}$

★**5.** Choose any four whole numbers greater than 0. Form two fractions, using these four numbers in the following way: Select two of your numbers. Their product is the numerator of the first fraction. The product of the remaining pair of numbers is the denominator of the second fraction. Now make another selection of two of your original numbers. Their product is the denominator of the first fraction. The product of the remaining pair is the numerator of the second fraction.

A Are the two fractions equivalent?

B Can you choose the numbers so that the two fractions will not be equivalent?

Think

Try to arrange the nine letters in the nine squares so that each letter is moved one square to the right or left or one square above or below its original position.

If this can be done, show an arrangement of the letters. If you feel it cannot be done, tell why it is not possible.

A	B	C
D	E	F
G	H	I

More practice, page S-13, Set 21

UTILIZATION

Exercises 1 through 3 provide ample practice in using the cross products method of checking for equivalence.

Solution, Exercise 5

Suppose the numbers chosen are 3, 5, 7, and 8. Then the steps for forming the two fractions might be as follows:

First fraction	Second fraction	First fraction	Second fraction
$3 \cdot 7$	$5 \cdot 8$	$3 \cdot 7$	$3 \cdot 5$
		$7 \cdot 8$	$5 \cdot 8$

Note that the cross products in each pair contain precisely the same factors; hence the fractions are equivalent.

EXTENSION

Additional practice is provided by Supplementary Exercise Set 21 on page S-13, *Workbook* page 35, and *Duplicator Masters* page 25.

Remedial: Write two different sets of equivalent fractions on the chalkboard, such as:

$\{\frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \frac{4}{16}, \frac{5}{20}\}$ $\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \frac{10}{15}\}$

Have the students check pairs in each set by the cross products method to show their equivalence. Then have the students choose a fraction from each set and show that they are *not* equivalent.

Enrichment: Discuss the following equations with the students and ask them to decide for each equation which statement is true.

(A) There is exactly one whole number for n.
(B) There is no whole number for n.
(C) There are many whole numbers for n.

1. $\frac{0}{6} = \frac{n}{7}$ (A)
2. $\frac{0}{n} = \frac{0}{8}$ (C)
3. $\frac{0}{6} = \frac{5}{n}$ (B)
4. $\frac{n}{7} = \frac{7}{7}$ (A)
5. $\frac{1}{1} = \frac{11}{n}$ (A)
6. $\frac{2}{2} = \frac{2 \cdot n}{3 \cdot n}$ (C)

Think Solution

It is impossible to make the specified arrangement. There will always be one of the shaded letters that cannot be moved as prescribed.

Assignments

Minimum 1, 2A–I, 3A–I. Average 1–4. Maximum 1–5.

Objective

The student can tell which of two fractional numbers is the greater by using regions, number lines, or fractions with common denominators.

PREPARATION

Before beginning the Investigation, conduct a brief review of equivalent fractions and locating points for fractional numbers on the number line.

INVESTIGATION

Discuss examples of A, B, and C in the Investigation and have the students give the correct word for the blank in each sentence. Ask them to explain their answers. Typical responses might be as follows:

A. Less of the region is shaded for $\frac{2}{3}$ than for $\frac{5}{6}$, so $\frac{2}{3}$ is *less* than $\frac{5}{6}$.

B. The point for $\frac{3}{4}$ is to the right of the point for $\frac{5}{8}$, so $\frac{3}{4}$ is *greater* than $\frac{5}{8}$.

C. Since $\frac{7}{8} = \frac{42}{48}$ and $\frac{5}{6} = \frac{40}{48}$, and $\frac{42}{48}$ is greater than $\frac{40}{48}$, then $\frac{7}{8}$ is *greater* than $\frac{5}{6}$.

Suggest that students choose fairly simple fractions to illustrate each method.

DISCUSSION

Work with inequalities takes on new significance when it involves the set of fractional numbers. Our efficient place-value notation makes it fairly easy to compare two whole numbers, but fractional numbers represented by non-unit fractions with unlike denominators are more difficult to compare (for example, $\frac{4}{5}$ and $\frac{15}{20}$). In such cases, some tests must be devised in order to find which of two fractions represents the larger number. The common denominator method shown in Exercise 5 is the most efficient, but some students may prefer the shortcut shown on Patty's paper. Observe that this method is based on using a common denominator which is the product of the two denominators, rather than the LCD of the fractions.

Comparing Fractional Numbers

Investigating the Ideas

You can use different methods to compare two fractional numbers.

A Compare parts of regions

$\frac{2}{3}$ $\frac{5}{6}$

$\frac{2}{3}$ is ? than $\frac{5}{6}$.

B Use points on the number line

$\frac{3}{4}$ is ? than $\frac{5}{8}$.

C Use common denominators

$\frac{7}{8} = \frac{7 \cdot 6}{8 \cdot 6} = \frac{42}{48}$

$\frac{5}{6} = \frac{5 \cdot 8}{6 \cdot 8} = \frac{40}{48}$

$\frac{7}{8}$ is ? than $\frac{5}{6}$.

Which is the correct word for each blank, **greater** or **less**?

Can you choose any two fractional numbers and compare them using each of the methods?

Discussing the Ideas

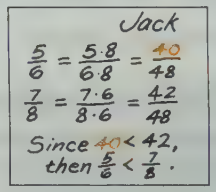
1. Which method above do you think is most practical to use? Why?

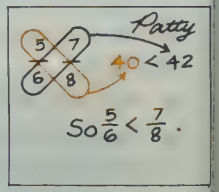
2. You can use the symbol $>$ for "greater than" and $<$ for "less than" for fractional numbers just as you did for the whole numbers.
   Tell which symbol goes in each ▨.
   A $\frac{1}{2}$ ▨ $\frac{3}{8}$ **B** $\frac{3}{5}$ ▨ $\frac{2}{3}$ **C** $\frac{1}{6}$ ▨ $\frac{5}{4}$ **D** $\frac{9}{10}$ ▨ $\frac{1}{8}$

3. **A** Give a fractional number for the shaded part of each region.
   B Which fractional number is greater?

4. **A** Which number is greater, the number for X or the number for Y?
   B Write an inequality statement about the numbers for X and Y.

5. Jack used the common denominator method to compare $\frac{5}{6}$ and $\frac{7}{8}$.
   Patty used a shortcut.
   Use either method to decide which fractional number in each pair is the larger.





A $\frac{4}{5}, \frac{7}{8}$ **B** $\frac{5}{9}, \frac{7}{12}$ **C** $\frac{4}{5}, \frac{7}{10}$ **D** $\frac{12}{10}, \frac{7}{6}$ **E** $\frac{12}{5}, \frac{5}{2}$ **F** $\frac{1}{4}, \frac{1}{6}$

Using the Ideas

1. Give the numbers for ***a*** and ***b***. Then give the correct symbol ($<$ or $>$) for each ▒.

A $\frac{1}{4} = \frac{a}{12}$, $\frac{1}{3} = \frac{b}{12}$ → $\frac{1}{4}$ ▒ $\frac{1}{3}$

B $\frac{5}{3} = \frac{a}{6}$, $\frac{5}{2} = \frac{b}{6}$ → $\frac{5}{3}$ ▒ $\frac{5}{2}$

C $\frac{3}{4} = \frac{a}{36}$, $\frac{7}{9} = \frac{b}{36}$ → $\frac{3}{4}$ ▒ $\frac{7}{9}$

D $\frac{2}{5} = \frac{a}{35}$, $\frac{3}{7} = \frac{b}{35}$ → $\frac{2}{5}$ ▒ $\frac{3}{7}$

E $\frac{5}{8} = \frac{a}{48}$, $\frac{7}{12} = \frac{b}{48}$ → $\frac{5}{8}$ ▒ $\frac{7}{12}$

F $\frac{7}{6} = \frac{a}{30}$, $\frac{6}{5} = \frac{b}{30}$ → $\frac{7}{6}$ ▒ $\frac{6}{5}$

G $\frac{3}{10} = \frac{a}{100}$, $\frac{33}{100} = \frac{b}{100}$ → $\frac{3}{10}$ ▒ $\frac{33}{100}$

H $\frac{1}{3} = \frac{a}{90}$, $\frac{7}{30} = \frac{b}{90}$ → $\frac{1}{3}$ ▒ $\frac{7}{30}$

2. Copy each problem and use any method you wish to find the correct symbol ($<$ or $>$) for each ▒.

A $\frac{1}{3}$ ▒ $\frac{1}{2}$ **B** $\frac{1}{4}$ ▒ $\frac{3}{16}$ **C** $\frac{1}{4}$ ▒ $\frac{5}{16}$ **D** $\frac{3}{8}$ ▒ $\frac{2}{5}$ **E** $\frac{1}{9}$ ▒ $\frac{1}{10}$ **F** $\frac{2}{3}$ ▒ $\frac{5}{7}$ **G** $\frac{7}{4}$ ▒ $\frac{8}{5}$ **H** $\frac{3}{7}$ ▒ $\frac{4}{9}$ **I** $\frac{9}{10}$ ▒ $\frac{10}{11}$ **J** $\frac{2}{5}$ ▒ $\frac{3}{8}$ **K** $\frac{5}{12}$ ▒ $\frac{3}{8}$ **L** $\frac{5}{16}$ ▒ $\frac{1}{3}$ **M** $\frac{7}{8}$ ▒ $\frac{11}{12}$ **N** $\frac{19}{25}$ ▒ $\frac{77}{100}$ **O** $\frac{9}{11}$ ▒ $\frac{35}{43}$

3. Which number in each pair is the larger?

A $\frac{1}{4}, \frac{1}{3}$ **B** $\frac{3}{5}, \frac{3}{8}$ **C** $\frac{3}{4}, \frac{4}{5}$ **D** $\frac{1}{2}, \frac{3}{5}$ **E** $\frac{1}{6}, \frac{6}{5}$ **F** $\frac{1}{3}, \frac{1}{2}$ **G** $\frac{3}{4}, \frac{5}{8}$ **H** $\frac{2}{3}, \frac{3}{5}$ **I** $\frac{3}{4}, \frac{7}{8}$ **J** $\frac{8}{7}, \frac{7}{6}$ **K** $\frac{1}{10}, \frac{1}{9}$ **L** $\frac{5}{12}, \frac{6}{11}$ **M** $\frac{4}{19}, \frac{3}{18}$ **N** $\frac{7}{50}, \frac{3}{25}$

think

A set of three consecutive even numbers → {24, 26, 28}

A set of four consecutive odd numbers → {7, 9, 11, 13}

1. Find a set of three consecutive **even** numbers whose sum is 216.
2. Find a set of four consecutive **odd** numbers whose sum is 216.
3. Find a set of three consecutive **whole** numbers whose sum is 216.

▒ + ▒ + ▒ = 216

More practice, page S-13, Set 22

UTILIZATION

Exercise 1 utilizes the common denominator method of comparing fractional numbers. Exercises 2 and 3 permit the students to use any method they choose.

EXTENSION

For further practice with the concepts of this lesson, see Supplementary Exercise Set 22 on page S-13, *Workbook* page 36, and *Duplicator Masters* page 26.

Enrichment: It is easy to compare fractional numbers when they are represented by fractions with the same denominator. The fraction with the largest numerator is the largest. Thus,

$$\frac{3}{8} < \frac{5}{8} < \frac{7}{8} < \frac{11}{8}.$$

Ask students to try to develop a way of telling which fractions represent the larger number when the numerators are the same but the denominators are different. For example, which fraction in the set below is the largest and which is the smallest?

$$\left\{\tfrac{5}{7}, \tfrac{5}{9}, \tfrac{5}{3}, \tfrac{5}{10}, \tfrac{5}{8}, \tfrac{5}{2}, \tfrac{5}{4}, \tfrac{5}{12}\right\}$$

Think Solution

1. $216 \div 3 = 72$. Thus, all three numbers cannot be larger than 72. Choosing the even numbers immediately preceding and following 72 gives the numbers {70, 72, 74}.
2. $216 \div 4 = 54$. Choosing the odd numbers preceding and following 54 gives {51, 53, 55, 57}.
3. $216 \div 3 = 72$. Choosing the preceding and following whole numbers gives {71, 72, 73}.

Assignments

Minimum 1, 2A–E. Average 1, 2A–I, 3A–J. Maximum 1–3.

Objective

Given three or more fractional numbers, the student can list them in order from smallest to largest.

PREPARATION

Review comparing two fractional numbers by using the common denominator method that was introduced in the last lesson.

INVESTIGATION

Have the students study the diagram to see if they can discover how the number halfway between $\frac{3}{4}$ and $\frac{4}{5}$ was found. By building sets of equivalent fractions for $\frac{3}{4}$ and $\frac{4}{5}$ the number $\frac{31}{40}$ is readily found. Of course, this number is the arithmetic mean of $\frac{3}{4}$ and $\frac{4}{5}$; i.e. $(\frac{3}{4} + \frac{4}{5}) \div 2$, but to find the mean would require skills in addition and division of fractional numbers which have not yet been developed in this book.

DISCUSSION

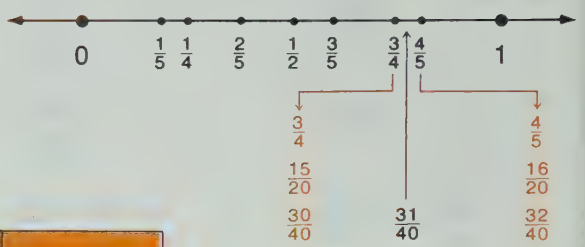
This lesson is an extension of the previous one, with emphasis shifted from inequalities to the ordering of sets of fractional numbers. Because of the fact that between any two fractional numbers there is always another fractional number, the fractional numbers have what is called the property of "density." Exercises 1, 2, and 3 emphasize the density property. No matter how close the points for two fractional numbers may be on the number line, we can always name another fractional number for a point between the two points.

Exercise 4 introduces the notation for continued inequalities. The relation "is less than" is a *transitive* relation. By this we mean that, if a, b, and c are fractional numbers such that $a < b$ and $b < c$, then $a < c$. Therefore we can introduce the notation $a < b < c$ to stand for the conjunction of $a < b$ and $b < c$. You might encourage students to read statements like $\frac{1}{2} < \frac{3}{4} < \frac{5}{6}$ as "$\frac{3}{4}$ is between $\frac{1}{2}$ and $\frac{5}{6}$."

Order of Fractional Numbers

Investigating the Ideas

Is there a fractional number for the point that is halfway between the points for $\frac{3}{4}$ and $\frac{4}{5}$?



Study the diagram to see why $\frac{31}{40}$ is halfway between $\frac{3}{4}$ and $\frac{4}{5}$.

Can you use this method to find a fractional number that is halfway between $\frac{1}{2}$ and $\frac{3}{5}$?

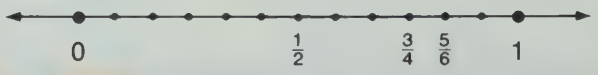
Discussing the Ideas

1. The number $\frac{31}{40}$ is halfway between $\frac{3}{4}$ and $\frac{4}{5}$. Is there a fractional number halfway between $\frac{3}{4}$ and $\frac{31}{40}$? Can you name a fraction for it?

2. Can you give the name of the fractional number that is halfway between $\frac{57}{100}$ and $\frac{58}{100}$ Explain how you found it.

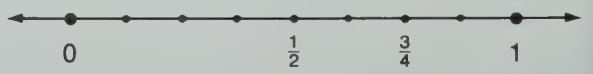
3. Do you think there is always another fractional number between any two different fractional numbers?

4. From the number line



you can see that $\frac{1}{2} < \frac{3}{4}$ and $\frac{3}{4} < \frac{5}{6}$. You can shorten this statement by writing $\frac{1}{2} < \frac{3}{4} < \frac{5}{6}$

5. A What is the name for the fractional number for the point halfway between the points for $\frac{1}{2}$ and $\frac{3}{4}$?



B Give a fraction for the $\frac{\text{IIII}}{\text{IIII}}$: $\frac{1}{2} < \frac{\text{IIII}}{\text{IIII}} < \frac{3}{4}$.

Using the Ideas

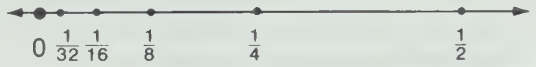
1. What fractional number is halfway between $\frac{2}{5}$ and $\frac{3}{5}$?

2. **A** Find the fractional number for the point on the number line that is just halfway between the points for $\frac{1}{3}$ and $\frac{1}{5}$.
   B Is $\frac{1}{4}$ larger or smaller than this number?
   C Is the point for $\frac{1}{4}$ closer to the point for $\frac{1}{3}$ or the point for $\frac{1}{5}$?
   D Find the fractional number for the point on the number line that is halfway between $\frac{1}{4}$ and $\frac{1}{5}$.

3. Write a shorter statement that has the same meaning as the given statement.
   EXAMPLE: For $\frac{1}{4} < \frac{1}{3}$ and $\frac{1}{3} < \frac{1}{2}$, write $\frac{1}{4} < \frac{1}{3} < \frac{1}{2}$
   A $\frac{1}{2} < \frac{3}{4}$ and $\frac{3}{4} < \frac{7}{8}$ **C** $\frac{5}{4} < \frac{5}{3}$ and $\frac{5}{3} < \frac{5}{2}$ **E** $\frac{1}{3} > \frac{1}{6}$ and $\frac{1}{6} > \frac{1}{12}$
   B $\frac{1}{5} < \frac{3}{10}$ and $\frac{3}{10} < \frac{3}{8}$ **D** $\frac{1}{7} < \frac{1}{6}$ and $\frac{1}{6} < \frac{1}{5}$ **F** $\frac{2}{5} < \frac{3}{4}$ and $\frac{7}{8} > \frac{3}{4}$

4. Give the missing numerator for each ||||.
   A $\frac{3}{8} < \frac{||||}{16} < \frac{4}{8}$ **C** $\frac{6}{10} < \frac{||||}{20} < \frac{7}{10}$ **E** $\frac{4}{9} < \frac{||||}{18} < \frac{5}{9}$ **G** $\frac{0}{4} < \frac{||||}{8} < \frac{1}{4}$
   B $\frac{1}{2} < \frac{||||}{8} < \frac{3}{4}$ **D** $\frac{5}{7} < \frac{||||}{14} < \frac{6}{7}$ **F** $\frac{2}{3} < \frac{||||}{24} < \frac{3}{4}$ **H** $\frac{7}{8} < \frac{||||}{48} < \frac{11}{12}$

5. **A** What is the smallest number in the set $\left\{\frac{1}{32}, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}\right\}$?
   B Give the names for two fractional numbers which are less than $\frac{1}{32}$.



6. What is the smallest number in the set $\left\{\frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}, \frac{5}{8}, \frac{6}{8}, \frac{7}{8}\right\}$?

7. What is the largest number in the set $\left\{\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}\right\}$?

8. **A** Which number is larger, $\frac{7}{8}$ or $\frac{6}{7}$?
   B Which number is smaller, $\frac{4}{5}$ or $\frac{5}{6}$?

9. If the numbers $\frac{3}{2}, \frac{3}{4}, \frac{7}{8}, \frac{7}{6}, \frac{15}{16}, \frac{5}{8}, \frac{2}{3}, \frac{19}{16}$ were listed in order of their closeness to 1, what would the order be?

10. Is there a "smallest" fractional number that is still larger than zero?

think

Give the lowest-terms fraction for each point *A* and *B* on the number line.

UTILIZATION

Exercises 1 and 2 provide students with further experiences in finding the fractional number halfway between two other fractional numbers.

In Exercise 3 the students are asked to write the equivalent shorter form for comparing three fractional numbers. It might be helpful to notice the different ways they can read a sentence that they have written. For example, $\frac{1}{2} < \frac{3}{4} < \frac{7}{8}$, when read from left to right, is "one half is less than three fourths is less than seven eighths." If you begin with $\frac{3}{4}$ and read first to the left and then to the right, it is "three fourths is greater than one half, and three fourths is less than seven eighths." You can even begin with $\frac{7}{8}$ and read from right to left, which would be "seven eighths is greater than three fourths, which is greater than one half." Each of these sentences means the same thing.

Exercise 10 offers an excellent opportunity to discuss the density of the set of fractional numbers. Ask the students to think of the smallest fractional number greater than zero that they know. After some students have given examples, ask others to find still smaller ones. For example, if a student says $\frac{1}{1000}$, a smaller fractional number would be $\frac{1}{10}$ of $\frac{1}{1000}$, or $\frac{1}{10\,000}$. Also, discuss the graphic implication of this. Although, $\frac{1}{1000}$ would be very close to zero on the number line, $\frac{1}{10\,000}$ would have to be between zero and $\frac{1}{1000}$, and so on.

EXTENSION

Further practice exercises are provided on page 37 of the *Workbook*.

Think Solution

The points for *A* and *B* divide the segment from $\frac{19}{2}$ to $\frac{21}{2}$ into thirds. Thus, writing equivalent fractions with a denominator of 6 we have $\frac{19}{2} = \frac{57}{6}$ and $\frac{21}{2} = \frac{63}{6}$. Hence the fraction for point *A* is $\frac{59}{6}$, and the fraction for point *B* is $\frac{61}{6}$.

Assignments

Minimum 1–4. Average 1–7. Maximum 1–10.

Objective

The student can use fractional numbers to work within the metric system.

PREPARATION

Materials: Metre sticks, centimetre, litre measures, centimetre cubes, standard metric masses.

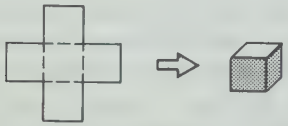
This lesson will be enhanced if you have available for use and discussion a variety of measurement devices. Some equipment might be obtainable from your science laboratory.

UTILIZATION

The exercises present no new mathematics; they deal with application of fractional numbers. The problems involve comparison of metric units with other units and no conversion of units is necessary.

EXTENSION

Enrichment: Encourage students to make a model of an "open-top cubic decimetre," by using cardboard and a pattern like the one below. Suggest that they wind tape several times around the sides of the box to make it more sturdy.

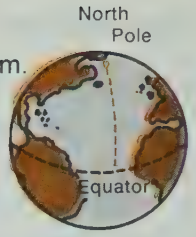


After they have completed the model, they can line the box with a plastic food bag and fill it with water. This will be a litre of water. Then have them fasten the top of the bag securely and weigh the bag of water on a metric scale, if one is available. The water should weigh about 1 kilogram (1000 grams).

Assignments

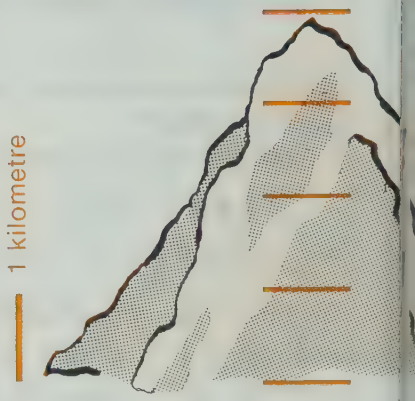
Minimum 1–5. Average 1–6. Maximum 1–7.

THE METRIC SYSTEM



1. The **metre** (m) is the standard unit of length in the metric system. Originally, the metre was a length such that the distance from the North Pole to the Equator measured along a certain meridian on Earth was 10 000 000 metres. What fraction compares one metre to the distance from the North Pole to the Equator?

2. A metre is divided into 100 parts of equal length called **centimetres** (cm). What fraction compares the length of a centimetre with the length of a metre.

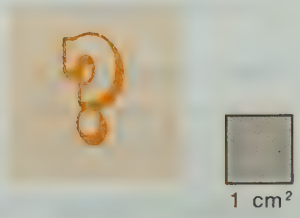


3. A **kilometre** (km) is 1000 metres. The drawing compares the height of Mt. Robson in British Columbia with a distance of one kilometre. What fraction compares the length of a kilometre with the height of Mt. Robson?

4 The **litre** (ℓ) is the metric unit of capacity. A litre is 1000 cubic centimetres (cm^3). What fraction compares a cubic centimetre to the litre?

5. The standard unit of weight in the metric system is a **kilogram** (kg). There are 1000 grams (g) in one kilogram. What fraction compares one gram with one kilogram?

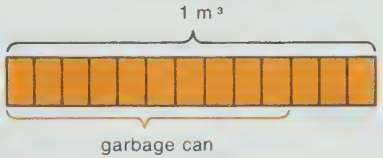
★ 6. The **square centimetre** (cm^2) is a metric unit of area. A **square metre** (m^2) is also a metric area. What fraction would you write to compare the size of the square centimetre to that of a square metre?



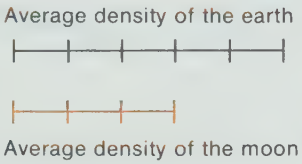
FRACTIONAL NUMBER PROBLEMS

1. A decimetre is $\frac{1}{10}$ of a metre. What lowest-terms fraction compares 4 decimetres with one metre?

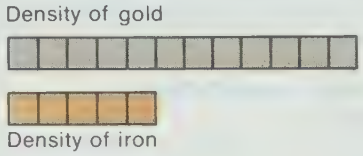
2. The shaded region shows what fractional part a large garbage can is of 1 cubic metre (m^3). What is this fraction?



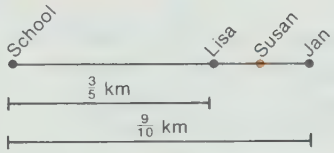
3. The average mass of the earth per cubic metre is greater than that of the moon. We say that the density of the moon is less than the density of the earth. The graph compares these densities. Express the average density of the moon as a fraction of the average density of the earth.



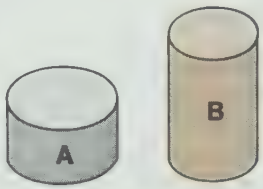
4. A cubic centimetre of gold is heavier than a cubic centimetre of iron. The graph compares the densities of these metals. Express the density of iron as a fraction of the density of gold.



5. Jan's house is $\frac{9}{10}$ km east of school. Lisa lives on the same street $\frac{3}{5}$ km east of school. Susan lives halfway between them. How far does Susan live from school?



6. Container A holds $\frac{3}{5}$ litre.
Container B holds $\frac{2}{3}$ litre.
Which container holds more?



7. A T-bone steak weighed $\frac{3}{10}$ kilogram. A piece of sirloin steak weighed $\frac{33}{100}$ kilogram. Which steak weighed less?

8. A **hectare** is a metric unit of area and is 10 000 square metres. The graph compares the size of a hectare with the size of a building lot. What fraction compares them?

Objective

The student can use fractional numbers to solve problems involving metric measures.

PREPARATION

Materials: Metre sticks.

UTILIZATION

The exercises require little factual knowledge of measurement since they involve application of fractional numbers for comparison purposes. However, you may want to discuss some of the exercises and extend them by exploring the relevant units of measure in greater detail.

EXTENSION

Enrichment: The *hectare*, mentioned in Exercise 8, may be less familiar to students than are some of the other metric units. If space is available on the school playground or athletic field, have students mark off a square that is 100 metres on each edge. Have one student stand at each corner of the square, which represents 1 hectare. This model of the hectare will help students estimate other areas in hectares.

Assignments

Minimum 1–5. Average 1–7. Maximum 1–8.

Objective

The student can demonstrate the ability to work with the ideas presented in this module.

UTILIZATION

In assigning the review exercises, notice that there are more than 50 responses required of the student if every part of each exercise is assigned. Therefore, you may wish to assign only one or two parts of some exercises. For example, if a student has demonstrated proficiency in finding lowest-terms fractions, then two or three parts of Exercise 4 may suffice. Other students, however, may need extra practice. With some students, you might consider giving the students themselves the responsibility of choosing how many parts of each problem they feel they should complete in order to adequately review the material covered in the module.

Additional review exercises are provided on page 38 of the *Workbook*.

REVIEWING THE IDEAS

1. Each figure suggests a pair of equivalent fractions. Give these fractions.

A B C

2. Give three fractions equivalent to the fraction given.

A $\frac{4}{5}$ B $\frac{1}{6}$ C $\frac{5}{8}$ D $\frac{3}{10}$ E $\frac{5}{16}$

3. Give the GCF of the numerator and denominator of each fraction.

A $\frac{12}{15}$ C $\frac{9}{27}$ E $\frac{26}{39}$ G $\frac{25}{30}$

B $\frac{8}{12}$ D $\frac{8}{9}$ F $\frac{14}{21}$ H $\frac{80}{90}$

4. Give the lowest-terms fraction for each fraction.

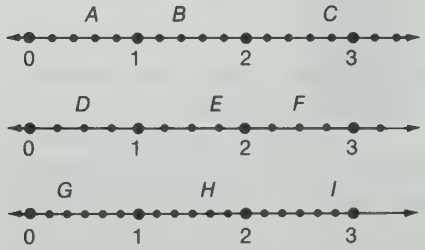
A $\frac{16}{20}$ E $\frac{20}{24}$ I $\frac{12}{9}$

B $\frac{10}{12}$ F $\frac{16}{28}$ J $\frac{48}{100}$

C $\frac{24}{36}$ G $\frac{2 \cdot 5}{7 \cdot 5}$ K $\frac{9}{30}$

D $\frac{16}{40}$ H $\frac{2 \cdot 2 \cdot 3}{2 \cdot 3 \cdot 5}$ L $\frac{60}{72}$

5. Give the lowest-terms fraction for each letter.



6. Solve the equations.

A $\frac{4}{5} = \frac{n}{20}$ D $\frac{3}{16} = \frac{n}{64}$

B $\frac{3}{4} = \frac{n}{16}$ E $\frac{9}{10} = \frac{n}{100}$

C $\frac{1}{2} = \frac{n}{30}$ F $\frac{2}{3} = \frac{n}{27}$

7. Copy and give the correct symbol ($<$, $=$, or $>$) for each

A $\frac{5}{8}$ $\frac{10}{12}$ D $\frac{3}{7}$ $\frac{4}{9}$

B $\frac{4}{3}$ $\frac{9}{6}$ E $\frac{5}{6}$ $\frac{4}{5}$

C $\frac{3}{2}$ $\frac{9}{6}$ F $\frac{5}{9}$ $\frac{4}{7}$

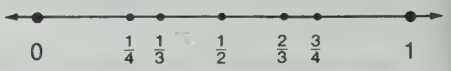
8. Copy and correctly fill in each screen.

A The statement, $\frac{5}{8} < \frac{2}{3} < \frac{5}{6}$, means $\frac{5}{8} <$ and $> \frac{5}{6}$.

B $\frac{3}{4} <$ $\frac{}{24} < \frac{5}{6}$

C $\frac{1}{2} <$ $\frac{}{16} < \frac{5}{8}$

9. Give the fractional number for the point that is halfway between the points for $\frac{2}{3}$ and $\frac{3}{4}$.



10. Give a whole number for n: $\frac{1}{6} < \frac{n}{60} < \frac{1}{5}$.

11. Mr. Knowlton sells a litre of cider for 48 cents. Mr. Shoup sells a litre of cider for 56 cents. What is the fraction that compares Mr. Knowlton's price to Mr. Shoup's price?

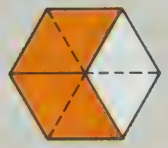
12. The graph compares the density of gasoline to the density of water.

What fraction compares the density of gasoline to the density of water?

TEST YOURSELF

1. What part of the set is composed of stars?

2. Name a pair of equivalent fractions suggested by the figure



3. Write three more fractions in this set of equivalent fractions.
$\left\{\frac{2}{5}, \frac{4}{10}, \frac{6}{15}, \ldots\right\}$

4. Give the lowest-terms fraction for each fraction.
   A $\frac{10}{12}$ B $\frac{15}{21}$ C $\frac{42}{36}$

5. Give the fractional numbers for *X*, *Y*, and *Z*.

6. A Is $\frac{9}{24}$ equivalent to $\frac{6}{16}$?
   B Is $\frac{8}{12}$ equivalent to $\frac{12}{16}$?

7. Find the number for *n*. $\frac{5}{3} = \frac{n}{12}$

8. Which number is larger, $\frac{5}{6}$ or $\frac{7}{10}$?

9. What number is halfway between $\frac{1}{4}$ and $\frac{1}{2}$?

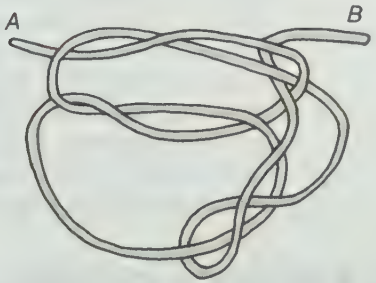
10. Give the missing numerator. $\frac{3}{5} < \frac{\text{IIII}}{80} < \frac{5}{8}$

RESEARCH PROJECTS

A Read about how the Egyptians used "unit fractions." Then explain their method to your class by giving example of your own. (See *Twenty-seventh Yearbook*, National Council of Teachers of Mathematics; Washington, D.C., 1963, pp. 221–226. *Science Awakening* by B. L. Van der Waerden; New York: John Wiley and Sons Canada, Ltd. 1963, pp. 19–23, 48–51.)

B Try writing some fractions using a base other than ten. For example, in base four the fraction $\frac{5}{8}$ would be written as $\frac{11}{13_{(4)}}$. In base two, the fraction $\frac{7}{10}$ would be written $\frac{111}{1010_{(2)}}$. List some sets of equivalent fractions using other bases. Make a chart that compares fractions in other bases with fractions in base ten.

C How are knots and mathematics related? Is the figure below a knot or a false knot? Find out about problems like this one.



(*See Topology: The Rubber-Sheet Geometry* by Donovan Johnson and William H. Glenn; St. Louis: Webster Publishing Co., 1960 available from McGraw-Hill Ryerson, Ltd.)

TEST YOURSELF

After completing the self-evaluation test, students should check their answers against the answer key and rework any problems that gave them difficulty. Encourage them to refer back to the appropriate lessons of the module as an aid in correcting their own mistakes.

RESEARCH PROJECTS

Students who undertake Research Project A might find it interesting to read about the Rhind Papyrus in James R. Newman's *The World of Mathematics;* New York: Simon and Schuster, 1956, Vol. I, pp. 170-178. (Available from Musson Book Company, Don Mills, Ontario.)

Students who are stimulated by the challenge of Research Project B might be encouraged to create a set of exercises using fractions with a chosen base other than ten. Such exercises might even be duplicated and given to other students to solve.

If students are intrigued by the sort of topological concepts suggested by Research Project C, refer them to William Simon's *Mathematical Magic;* New York: Scribner's, 1964, pp. 37-59. (Available from John Wiley and Sons Canada, Ltd., Rexdale, Ontario.) There they will find some interesting topological tricks that they will enjoy trying.

MATHEMATICAL RECREATION

This mathematical recreation was invented by the famous English mathematician, William Rowen Hamilton (1805–1865).

The playing board network has the same connections as a dodecahedron and, in fact, the puzzle could be tried using a dodecahedron except for the inconvenience of handling the solid.

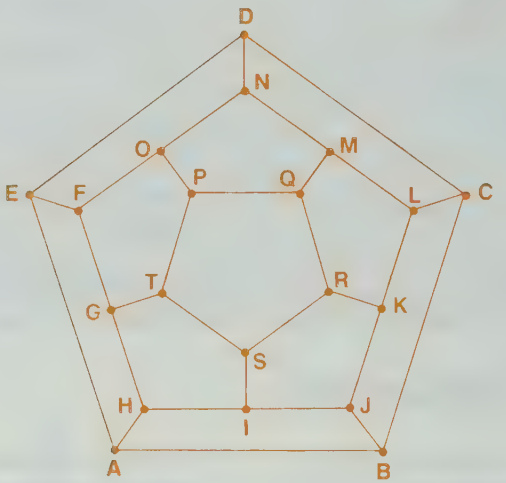
Given the condition that no more than five consecutive towns are first connected, the puzzle can always be solved. That is, a path passing through each of the remaining towns only once and ending at the starting point can always be found.

For variation, suggest that students try to find paths that start and end at specified points. For example, find a path that starts at point A and ends at point R, passing through each of the other lettered points once and only once. Of course, not every "road" connecting the points must be used.

MATHEMATICAL RECREATIONS

the pentagonal circuit

The 20 points on the diagram below represent 20 towns. The segments connecting the points represent roads.



One player selects five towns that can be reached in consecutive order such as I, S, T, P, O and draws a path connecting them. The second player must start at the last town (in this case, O) and draw a path which passes through each of the remaining towns exactly once and then return to the starting point. If the second player cannot find a path, he loses the game. Tracing paper can be placed over the diagram and the paths drawn on the tracing paper. Or, counters can be placed on the diagram to mark the routes taken.

MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. $\frac{4}{9}$ 2. $\frac{2}{3}, \frac{4}{6}$ 3. $\frac{8}{20}, \frac{10}{25}, \frac{12}{30}$ 4. A $\frac{5}{6}$; B $\frac{5}{7}$; C $\frac{7}{6}$ 5. $X: \frac{1}{6}$; $Y: \frac{1}{3}$; $Z: \frac{5}{6}$
6. A yes; B no 7. $n = 20$ 8. $\frac{5}{6}$ 9. $\frac{3}{8}$ 10. 49

UNIT B: THE ARITHMETIC OF FRACTIONAL NUMBERS

Module 2: Adding and Subtracting Fractional Numbers

General Objectives

To provide experience in working with the addition and subtraction algorithms for fractional numbers.

To provide practice in addition and subtraction of fractional numbers using mixed numerals.

To introduce functions and graphs of functions using fractional numbers.

To round fractional numbers to the nearest whole number.

To extend the students' problem-solving skills in word problems involving fractional numbers.

	Performance Objectives	Pupil Text	Reteach-Reinforce	Related Activities	
43	**RED** Given two fractional numbers with like denominators, the student can find their sum or difference.	B-22, 23	**WB** 39		
44	**RED** Given two fractional numbers with unlike denominators, the student can find the least common denominator and then find the sum or difference of the numbers.	B-24, 25	**WB** 40 **DM** 27	**ASC** F-5, 6 **SP** c-2, 3 **PD** c-1, 2	**SWM 3** 110
45	**RED** The student can write a mixed numeral for an improper fraction or an improper fraction for a mixed numeral.	B-26, 27	**WB** 41 **DM** 28	**ASC** F-3, 4 **SP** c-1 **PD** c-1	
46	**RED** The student can find sums and differences of fractional numbers using mixed numerals or improper fractions.	B-28, 29	**WB** 42 **DM** 29	**ASC** F-7, 8 **SP** c-2, 3 **PD** c-1, 2 **PS** g-1	**SWM 2** 86
47	**YELLOW** The student can solve appropriate word problems involving addition and subtraction of fractional numbers.	B-30, 31			
48	**YELLOW** The student can complete function tables and draw graphs using fractional number pairs of the function.	B-32, 33	**WB** 43		
49	***GREEN*** *The student can round fractional numbers to the nearest whole number.*	B-34, 35			
	Reviewing the Ideas	B-36	**WB** 44		

MATHEMATICS

In teaching addition of fractional numbers, it is important to introduce the student first to sums of fractions with like denominators. Thus, if $\frac{a}{b}$ and $\frac{c}{b}$ are two fractional numbers, then $\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$. The truth of this statement can be demonstrated by means of the number line or other semiconcrete objects. Once this concept has been mastered, addition with fractions having unlike denominators can be considered. If $\frac{a}{b}+\frac{c}{d}$ are two fractional numbers and $b \neq d$, then we can choose fractions having the same denominator by choosing fractions equivalent to $\frac{a}{b}$ and $\frac{c}{d}$. Formally, we can show that

$$\frac{a}{b}+\frac{c}{d}=\frac{a\cdot d}{b\cdot d}+\frac{b\cdot c}{b\cdot d}=\frac{a\cdot d+b\cdot c}{b\cdot d}.$$

Subtraction of fractional numbers can be defined in terms of addition. Thus, if $\frac{x}{y}$ is a fractional number such that $\frac{x}{y}+\frac{c}{d}=\frac{a}{b}$, then we say that $\frac{x}{y}$ is the difference of $\frac{a}{b}$ and $\frac{c}{d}$ and we write $\frac{a}{b}-\frac{c}{d}=\frac{x}{y}$.

Proceeding much as in addition, we can then show that

$$\frac{a}{b}-\frac{c}{d}=\frac{a\cdot d-b\cdot c}{b\cdot d}.$$

Although any common denominator may be chosen for the fractions used in addition and subtraction of fractional numbers, it is helpful to encourage students to use the least common denominator (LCD) which is the least common multiple (LCM) of the denominators.

TEACHING THE MODULE

Materials

graph paper, sheets of standard typing paper, centimetre rulers, scissors, co-ordinate grid paper.

Vocabulary

associative principle
commutative principle
denominator
equivalent fractions
function
graph
improper fraction
least common denominator (LCD)
mixed numeral
number pairs
numerator
prime factorization
slide rule
zero principle

This module develops the basic algorithms for addition and subtraction of fractional numbers. Along with this development, the module introduces some special topics, such as the fractional number functions and graphing, rounding fractional numbers, as well as a variety of word problems involving computation with fractional numbers.

The general purpose of this module is to develop students' skills in fractional number computation to an acceptable degree of proficiency. Students should also develop an understanding of the special topics as well as the basic skills for solving word problems.

Keep in mind that when, as in this module, there is an emphasis on routine computing and problem solving, it is especially important to keep the material interesting, even if this means covering some topics in less detail than you would like. You can return to such topics at another time or stress them in later reviews.

Lesson Schedule

Assigning a time limit for this module is difficult since much of the work concentrates upon the development of skills which may or may not have been attained in the earlier grades. Your time schedule should be flexible enough to fit the needs and abilities of the students. It would be reasonable for a very weak class to spend as long as 15 days in this module. A group of students with a strong elementary mathematics background and average ability could be expected to cover the material in 10 days or less. With a very good class, in addition to covering the basic topics, you should encourage solution of the Think problems, the Research Projects, and the Mathematical Recreation as means of stimulating interest.

Evaluation

Since much of this module is devoted to computation, it is easy to determine whether your students can add or subtract fractional numbers with some facility. It is somewhat more difficult, however, to discover the source of any problems students may be having. Close attention to the students' day-by-day progress, their responses in the Discussion stages, and analyses of their written work are often revealing and will aid you in spotting sources of difficulty.

Encourage students who do poorly on the self-evaluation test to seek help in overcoming their difficulties before proceeding to new material.

RESOURCES FOR ACTIVE LEARNING

Activities

Aftermath 2, "Sum Fractions," p. 59; "Add-Sub Slide Rule," pp. 29–31, Creative Publications
Fraction Bars, Card Set 1, Scott Resources
Fractional Numbers, Geocards 15–18, Creative Publications
Nuffield Project: *Computation and Structure 5*, pp. 24–29, Wiley

Manipulative Devices

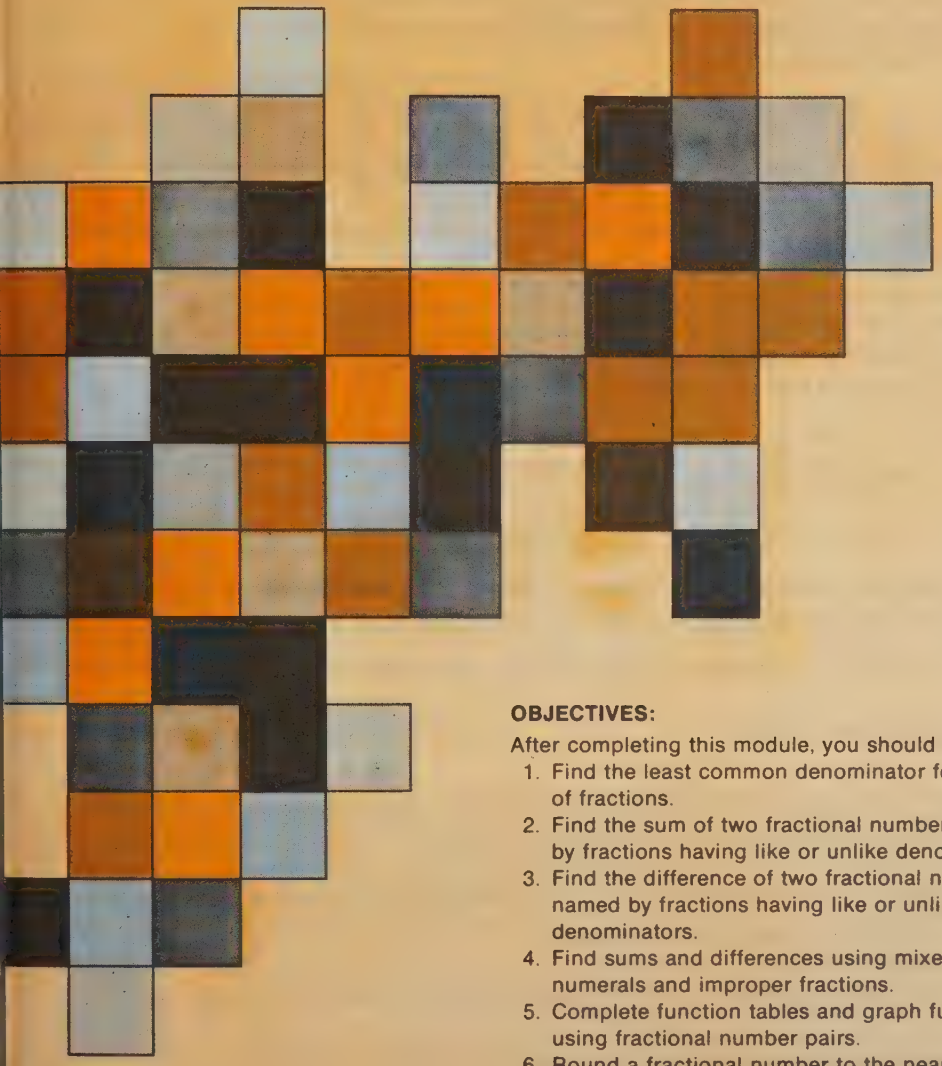
Fraction Bars manipulatives (Scott Resources)
Geoboards (Addison-Wesley)

Games and Puzzles

Action Fractions (Lakeshore; Mind/Matter)
Come Out Even (Holt)
Here-to-There (Scott Foresman)
Spinner Number Games (Heath)
TUF (Cuisenaire Co.; TUF)

UNIT B: *The Arithmetic of Fractional Numbers*

MODULE 2: *Adding and Subtracting Fractional Numbers*



OBJECTIVES:

After completing this module, you should be able to:

1. Find the least common denominator for a pair of fractions.
2. Find the sum of two fractional numbers named by fractions having like or unlike denominators.
3. Find the difference of two fractional numbers named by fractions having like or unlike denominators.
4. Find sums and differences using mixed numerals and improper fractions.
5. Complete function tables and graph functions using fractional number pairs.
6. Round a fractional number to the nearest whole number.
7. Solve word problems involving addition or subtraction of fractional numbers.

Objective

The student can find the sum or difference of two fractional numbers by using fractions with the same denominator.

PREPARATION

Materials: Graph paper strips (*Duplicator Masters*, page 83).

Provide graph paper for the students to construct the addition slide rule for the Investigation. If the strips are mounted to a like-sized strip of posterboard, the slide rule will be easier to manipulate. You might wish to save some time by having duplicated copies of the slide rule available for students to cut out and use.

INVESTIGATION

After students have constructed their slide rules, have them study the example to see how the slide rule is used. You may want to point out that the slide rule is used much like a number line in showing addition and subtraction.

Have students record their sums they find and ask some of them to display the sums they have found on the chalkboard.

DISCUSSION

Most students will probably recall the method of adding fractional numbers using fractions with like denominators. Formally, this can be stated:

If $\frac{a}{b}$ and $\frac{c}{b}$ are fractional numbers,

then $\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$.

Exercise 2 introduces addition involving fractions with unlike denominators, but since the fractions are $\frac{1}{2}$ and $\frac{3}{4}$, the student can readily use the slide rule and choose $\frac{4}{8}$ and $\frac{6}{8}$ in order to determine the sum.

To find the difference $\frac{5}{8}-\frac{3}{8}$ (Exercise 3), the student can use the slide rule in the same position as for the addition problem shown in the Investigation ($\frac{2}{8}+\frac{3}{8}$). However, the jump is to the left from $\frac{5}{8}$ to $\frac{2}{8}$.

Exercise 4 will help the student think about choosing fractions with like denominators to perform addition and subtraction.

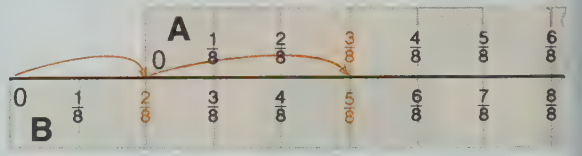
Adding and Subtracting Fractional Numbers

Investigating the Ideas

Make a number line scaled in eighths on both sides of the line on a strip of graph paper. Cut along the line so as to divide the strip into parts **A** and **B**.

You can use the two parts, **A** and **B**, as an addition slide rule. For example, with the strips in the position shown at the right, you can see that

$$\frac{2}{8}+\frac{3}{8}=\frac{5}{8}.$$



Can you use your slide rule to find at least 10 more sums?	Write an equation for each sum that you find.	

Discussing the Ideas

1. **A** How can you use your slide rule to find $\frac{3}{8}+\frac{7}{8}$?
   B What is the sum as a lowest-terms fraction?

2. **A** What fractions would you use on the slide rule to find $\frac{1}{2}+\frac{3}{4}$?
   B What is the lowest-terms fraction for this sum?

3. How can you use the slide rule to find the difference $\frac{5}{8}-\frac{3}{8}$?

4. Study the examples. Then give the missing sum or difference.
   A To find $\frac{1}{2}+\frac{3}{8}$, we think $\left\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10} \ldots\right\}$
   and write $\frac{4}{8}+\frac{3}{8}=\frac{\text{IIII}}{\text{IIII}}$.
   B To find $\frac{11}{8}-\frac{3}{4}$, we think $\left\{\frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \frac{12}{16}, \ldots\right\}$
   and write $\frac{11}{8}-\frac{6}{8}=\frac{\text{IIII}}{\text{IIII}}$.

Using the Ideas

1. Give an addition and a subtraction equation for the slide rule. Use the fractions shown in color.

2. Find each sum and difference.

A $\frac{5}{8}+\frac{1}{8}$ **C** $\frac{8}{8}+\frac{8}{8}$ **E** $\frac{3}{4}+\frac{7}{4}$ **G** $\frac{3}{4}+\frac{5}{4}$ **I** $\frac{5}{16}+\frac{8}{16}$ **K** $\frac{16}{15}-\frac{4}{15}$

B $\frac{9}{8}-\frac{3}{8}$ **D** $\frac{9}{8}-\frac{8}{8}$ **F** $\frac{5}{2}-\frac{3}{2}$ **H** $\frac{9}{4}+\frac{3}{4}$ **J** $\frac{4}{12}+\frac{10}{12}$ **L** $\frac{7}{24}+\frac{9}{24}$

3. Give the numerators ***x*** and ***y***.

A $\frac{3}{4}+\frac{1}{2}=\frac{3}{4}+\frac{x}{4}=\frac{y}{4}$ $\left\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \ldots\right\}$

B $\frac{13}{8}-\frac{5}{4}=\frac{13}{8}-\frac{x}{8}=\frac{y}{8}$ $\left\{\frac{5}{4}, \frac{10}{8}, \frac{15}{12}, \ldots\right\}$

C $\frac{2}{3}+\frac{5}{12}=\frac{x}{12}+\frac{5}{12}=\frac{y}{12}$ $\left\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots\right\}$

4. Give the numerators ***r*** and ***s***.

A $\frac{7}{8}+\frac{1}{2}=\frac{7}{8}+\frac{r}{8}=\frac{s}{8}$

B $\frac{11}{4}-\frac{3}{2}=\frac{11}{4}-\frac{r}{4}=\frac{s}{4}$

C $\frac{1}{8}+\frac{9}{4}=\frac{1}{8}+\frac{r}{8}=\frac{s}{8}$

D $\frac{9}{2}-\frac{7}{8}=\frac{r}{8}-\frac{7}{8}=\frac{s}{8}$

E $2+\frac{5}{4}=\frac{r}{4}+\frac{5}{4}=\frac{s}{4}$

F $\frac{3}{4}+\frac{9}{16}=\frac{r}{16}+\frac{9}{16}=\frac{s}{16}$

G $\frac{3}{5}-\frac{3}{10}=\frac{r}{10}-\frac{3}{10}=\frac{s}{10}$

H $3-\frac{5}{6}=\frac{r}{6}-\frac{5}{6}=\frac{s}{6}$

I $\frac{11}{3}+\frac{5}{6}=\frac{r}{6}+\frac{5}{6}=\frac{s}{6}$

5. Find the sums and differences.

A $\frac{3}{8}+\frac{3}{2}$ **C** $\frac{1}{2}+\frac{1}{8}$ **E** $\frac{9}{4}+\frac{3}{8}$ **G** $\frac{13}{2}-\frac{9}{8}$ **I** $\frac{5}{8}-\frac{3}{16}$ **K** $5+\frac{3}{2}$

B $\frac{11}{8}-\frac{1}{4}$ **D** $\frac{3}{2}-\frac{3}{4}$ **F** $\frac{5}{8}-\frac{1}{4}$ **H** $\frac{3}{4}+\frac{5}{16}$ **J** $4-\frac{7}{4}$ **L** $\frac{15}{2}+4$

★**6.** If ***a***, ***b***, and ***c*** are whole numbers and $b \neq 0$, then $\frac{a}{b}+\frac{c}{b}=$ ▨.

think

The ancient Greek mathematicians gave the name **triangular numbers** to the following set: {1, 3, 6, 10, . . .}

The **square numbers** are the numbers in this set: {1, 4, 9, 16, . . .}

1. Give the next five triangular numbers.
2. Give the next five square numbers.
3. Pick any triangular number. Add it to the next one. Do this several times. What do you notice?

UTILIZATION

The main thrust of the exercises is to find sums and differences by using fractions having the same denominator. Although Exercises 3, 4, and 5 contain fractions that have unlike denominators, all of them involve denominators with halves, fourths, eighths, and sixteenths, so students should not find them difficult. Fractions having denominators that pose greater problems will be discussed in the following lesson.

Exercise 6 deals with the generalization for adding with fractions having the same denominator:

$$\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b},\ b \neq 0.$$

A similar generalization for subtraction would be:

$$\frac{a}{b}-\frac{c}{b}=\frac{a-c}{b},\ b \neq 0.$$

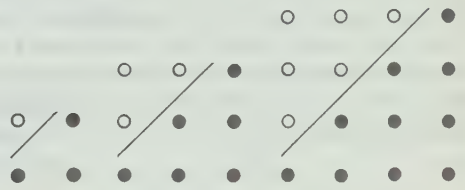
EXTENSION

Workbook page 39 offers exercises that would be appropriate as an assignment for additional practice.

Think Solution

This is a problem that students of all levels of ability can do. You will note in the problem that the number 1 is considered to be both a triangular number and a square number, simply for convenience. Although it is difficult to think literally of a dot as being either a square or a triangle, this convention plays a vital role in helping us construct the patterns for triangular and square numbers.

The pictures below may help some students see why the sum of any two consecutive triangular numbers is a square number.



Assignments

Minimum 1–3. Average 1–4. Maximum 1–6.

Objective

The student can add or subtract two fractional numbers represented by fractions with unlike denominators.

PREPARATION

As an introductory activity, display a pair of numbers, such as 6 and 8, 4 and 6, or 8 and 10, and ask the students to list 6 to 8 multiples of each. Then ask them to find the multiples that are common to both lists.

INVESTIGATION

The purpose of the Investigation is to enable students to understand that to find sums and differences of fractional numbers they must choose fractions for the fractional numbers that have the same denominator. Encourage students to observe that the various fractions which they write for the sum of $\frac{1}{2}$ and $\frac{2}{3}$, $\{\frac{7}{6}, \frac{14}{12}, \frac{21}{18}, \frac{28}{24}, \ldots\}$, form a set of equivalent fractions. This shows that the correct sum is found regardless of which common denominator is selected, though the computation is simpler when the LCD is chosen.

DISCUSSION

If they consider the LCM of the two denominators, most students will not find it difficult to select the LCD for two fractions, especially if the denominators are not very large. However, if the denominators are large, then the LCD is often easier to find by considering prime factors of each denominator. (See Exercise 3.) Consider 72, 45, and 60.

$$72 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$$
$$45 = 3 \cdot 3 \cdot 5$$
$$60 = 2 \cdot 2 \cdot 3 \cdot 5$$

The LCM for these numbers must include every prime factor found in the different numbers, and these factors must be included the maximum number of times that they occur in the prime factorization of any one of the numbers. Thus:

Factors of 60 · Factors of 45 / Factors of 72

$2 \cdot \overbrace{2 \cdot 2}^{\text{Factors of 60}} \cdot \overbrace{3 \cdot 3 \cdot 5}^{\text{Factors of 45}} = 360$ (LCM)

Choosing the Least Common Denominator (LCD)

Investigating the Ideas

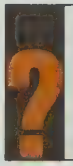
You can find the sum of two fractional numbers by choosing fractions that have the same denominator.

Consider the sum $\frac{1}{2} + \frac{2}{3}$.

$\frac{1}{2} \rightarrow \{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \frac{6}{12}, \frac{7}{14}, \frac{8}{16}, \frac{9}{18}, \frac{10}{20}, \frac{11}{22}, \frac{12}{24}, \ldots\}$

$\frac{2}{3} \rightarrow \{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \frac{10}{15}, \frac{12}{18}, \frac{14}{21}, \frac{16}{24}, \frac{18}{27}, \ldots\}$

$\frac{3}{6} + \frac{4}{6} =$ ▮ $\quad \frac{6}{12} + \frac{8}{12} =$ ▮ $\quad \frac{9}{18} + \frac{12}{18} =$ ▮ $\quad \frac{12}{24} + \frac{16}{24} =$ ▮



Can you give each of the missing fractional numbers for the screens above and show that they are equivalent to each other?

Write at least three more addition equations for the sum of $\frac{1}{2}$ and $\frac{2}{3}$.

Discussing the Ideas

1. Although you may use many different pairs of fractions to find a sum, your work will be easier if you choose pairs with the smallest possible common denominator. This denominator is called the **least common denominator** (LCD) for the two fractions given. In the example above, 6 was the LCD for $\frac{1}{2}$ and $\frac{2}{3}$. Notice that 6 is the least common multiple (LCM) of the denominators 2 and 3. What is the LCD for $\frac{5}{8}$ and $\frac{1}{6}$?

2. **A** List the multiples of 8 up to 80.
   B List the multiples of 14 up to 140.
   C What is the LCM of 8 and 14.
   D What is the LCD for $\frac{3}{8}$ and $\frac{5}{14}$?

3. You can find the LCD for two fractions by thinking about the prime factorization of their denominators.

   Consider the sum $\frac{9}{30} + \frac{13}{42}$.

   Prime factorization of $30 = 2 \cdot 3 \cdot 5$
   Prime factorization of $42 = 2 \cdot 3 \cdot 7$
   LCM of 30 and 42: $2 \cdot 3 \cdot 5 \cdot 7 = 210$; LCD for $\frac{9}{30}$ and $\frac{13}{42}$ is 210

   A Find the sum $\frac{9}{30} + \frac{13}{42}$.
   B Explain how to use this method to find the LCD for $\frac{5}{18} + \frac{7}{24}$.

Using the Ideas

1. **A** Give two common multiples of 16 and 40.

B Give the LCM of these two numbers.

C What is the LCD for $\frac{1}{16}$ and $\frac{1}{40}$?

D Find the sum $\frac{5}{16} + \frac{7}{40}$.

2. Find the LCD for the pair of fractions by finding the LCM of the denominators.

A $\frac{2}{3}, \frac{7}{9}$ **B** $\frac{1}{6}, \frac{1}{8}$ **C** $\frac{2}{15}, \frac{3}{10}$ **D** $\frac{5}{8}, \frac{1}{12}$ **E** $\frac{1}{4}, \frac{1}{9}$ **F** $\frac{2}{7}, \frac{5}{8}$ **G** $\frac{1}{3}, \frac{4}{5}$ **H** $\frac{3}{10}, \frac{7}{100}$ **I** $\frac{7}{12}, \frac{9}{16}$ **J** $\frac{1}{2}, \frac{3}{11}$ **K** $\frac{5}{9}, \frac{1}{12}$. **L** $\frac{11}{16}, \frac{5}{6}$

3. **A** Find the LCM of 27 and 45.

B What is the LCD for $\frac{7}{27}$ and $\frac{11}{45}$?

C Find $\frac{7}{27} + \frac{11}{45}$.

D Find $\frac{7}{27} - \frac{11}{45}$.

4. In simple addition problems, practice finding the LCD mentally.

PROBLEM:	THINK:	WRITE:
$\frac{3}{4} + \frac{1}{6}$	LCD is 12	$\frac{3}{4} = \frac{9}{12}$ $+\frac{1}{6} = \frac{2}{12}$ $\frac{11}{12}$

Find the sum or difference. Give your answer in lowest terms.

A $\frac{2}{3} + \frac{3}{4}$ **B** $\frac{3}{4} + \frac{3}{8}$ **C** $\frac{11}{12} + \frac{1}{4}$ **D** $\frac{7}{8} - \frac{5}{6}$ **E** $\frac{11}{20} - \frac{3}{8}$

F $\frac{15}{16} - \frac{3}{4}$ **G** $\frac{5}{6} - \frac{1}{3}$ **H** $\frac{4}{5} + \frac{1}{6}$ **I** $\frac{3}{2} - \frac{3}{4}$ **J** $\frac{9}{10} - \frac{1}{3}$

K $\frac{77}{100} - \frac{7}{10}$ **L** $\frac{3}{5} + \frac{1}{2}$ ★**M** $\frac{12}{20} + \frac{14}{16}$ ★**N** $\frac{18}{27} - \frac{32}{64}$ ★**O** $\frac{18}{28} + \frac{4}{56}$

5. For each problem, rename two of the three fractional numbers so that the fractions have the same denominators. Find the sum of the three numbers.

A $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ **B** $\frac{1}{4}, \frac{3}{8}, \frac{5}{16}$ **C** $\frac{1}{2}, \frac{3}{4}, \frac{5}{16}$ **D** $\frac{3}{2}, \frac{2}{3}, \frac{5}{6}$ **E** $\frac{7}{2}, \frac{6}{5}, \frac{3}{10}$ **F** $\frac{1}{3}, \frac{4}{5}, \frac{7}{15}$ **G** $\frac{3}{5}, \frac{1}{6}, \frac{7}{30}$ **H** $\frac{5}{8}, \frac{6}{7}, \frac{3}{56}$ **I** $\frac{4}{7}, \frac{7}{10}, \frac{11}{70}$ **J** $\frac{5}{4}, \frac{11}{25}, \frac{71}{100}$

6. Find the sums.

A $\frac{1}{35} + \frac{2}{55}$ **B** $\frac{2}{15} + \frac{2}{21}$ **C** $\frac{5}{14} + \frac{3}{22}$ **D** $\frac{7}{24} + \frac{4}{9}$ **E** $\frac{3}{22} + \frac{5}{26}$ **F** $\frac{8}{105} + \frac{5}{63}$ **G** $\frac{1}{3} + \frac{1}{4} + \frac{1}{6}$ **H** $\frac{2}{5} + \frac{1}{6} + \frac{7}{30}$ ★**I** $\frac{5}{30} + \frac{7}{42} + \frac{9}{54}$ ★**J** $\frac{7}{35} + \frac{3}{18} + \frac{7}{105}$ ★**K** $\frac{7}{9} + \frac{5}{24} + \frac{11}{36}$ ★**L** $\frac{3}{8} + \frac{5}{12} + \frac{7}{10}$

More practice, page S-14, Set 23

UTILIZATION

Adjust the amount of work assigned in this section to the abilities of your students. When your students have finished the assignment, the additional set of Supplementary Exercises on page S-23 may be used for those who need extra practice.

EXTENSION

To provide further practice, make assignments from *Workbook* page 40, *Duplicator Masters page 27,* and *Arithmetic Skill Cards* F-5 and F-6.

Remedial: Have students build sets of equivalent fractions for $\frac{1}{2}$, $\frac{2}{3}$, $\frac{4}{5}$, and $\frac{3}{4}$.

$\frac{1}{2}: \{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \ldots\}$

$\frac{2}{3}: \{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots\}$

$\frac{4}{5}: \{\frac{4}{5}, \frac{8}{10}, \frac{12}{15}, \frac{16}{20}, \ldots\}$

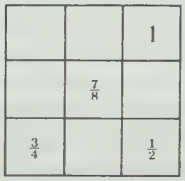
$\frac{3}{4}: \{\frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \frac{12}{16}, \ldots\}$

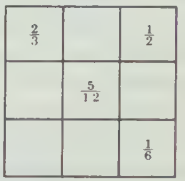
Then write several addition and subtraction problems on the chalkboard, using the lowest-terms fractions. The students can then find the sums and differences by selecting from the sets of equivalent fractions those that have common denominators.

Suggested Problems:

A. $\frac{1}{2} + \frac{2}{3}$ B. $\frac{4}{5} - \frac{2}{3}$ C. $\frac{3}{4} + \frac{2}{3}$ D. $\frac{4}{5} - \frac{1}{2}$ E. $\frac{3}{4} - \frac{2}{3}$ F. $\frac{1}{2} + \frac{3}{4}$ G. $\frac{3}{4} - \frac{1}{2}$ H. $\frac{2}{3} - \frac{1}{2}$ I. $\frac{4}{5} + \frac{2}{3}$

Enrichment: Ask students to copy and complete these magic squares.





Assignments

Minimum 1, 2A–H, 3, 4A–E. Average 1–4. Maximum 1–6.

Objective

The student can write an improper fraction for a mixed numeral or write a mixed numeral for an improper fraction.

PREPARATION

Since you will want to allow sufficient time for most students to discover the pattern and complete the Investigation, you should keep any preparatory activities to a minimum.

INVESTIGATION

The Investigation defines an improper fraction and a mixed numeral and gives the student an example of the way a mixed numeral can be found for an improper fraction. Note that this is a *power skill* method here. (The flow chart in Discussion Exercise 3 will help to develop the *speed skill* aspect of the method.)

DISCUSSION

Students may find more than one way of extending the pattern of fractions. Using the sequence of improper fractions, they may see that the sum of the numerator and denominator of a given fraction is the denominator of the next fraction. This sum added to the denominator gives the numerator of the next fraction. Thus, considering $\frac{17}{12}$, we have $17 + 12 = 29$, and $29 + 12 = 41$, so the next fraction is $\frac{41}{29}$. Then $41 + 29 = 70$, and $70 + 29 = 99$, giving $\frac{99}{70}$. Another pattern students may discover is that doubling the denominator of a given fraction and adding the denominator of the previous fraction gives the denominator of the next fraction.

In Exercises 2 and 3 the student is exposed to the power and speed skill methods of finding a mixed numeral for an improper fraction. The flow chart in Exercise 4 gives the speed skill method for finding an improper fraction for a mixed numeral. However, you may wish to give a few examples showing the *power* development before discussing the flow chart. Example: Express $2\frac{3}{4}$ as an improper fraction.

$$2\tfrac{3}{4} = 2 + \tfrac{3}{4}$$
$$\downarrow \quad \downarrow$$
$$\tfrac{8}{4} + \tfrac{3}{4} = \tfrac{11}{4}$$

Improper Fractions and Mixed Numerals

Investigating the Ideas

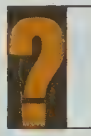
A fraction whose numerator is greater than or equal to its denominator is called an **improper fraction.** Sometimes a **mixed numeral** is written for an improper fraction.

Improper fraction $\rightarrow \frac{9}{5} = \frac{5}{5} + \frac{4}{5} = 1 + \frac{4}{5} = 1\frac{4}{5} \leftarrow$ **Mixed numeral**

Study the sequences of improper fractions and mixed numerals below.

Improper fractions: $\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \ldots$

Mixed numerals: $1, 1\frac{1}{2}, 1\frac{2}{5}, 1\frac{5}{12}, 1\frac{12}{29}, \ldots$



Can you find the next three improper fractions and mixed numerals for each sequence?

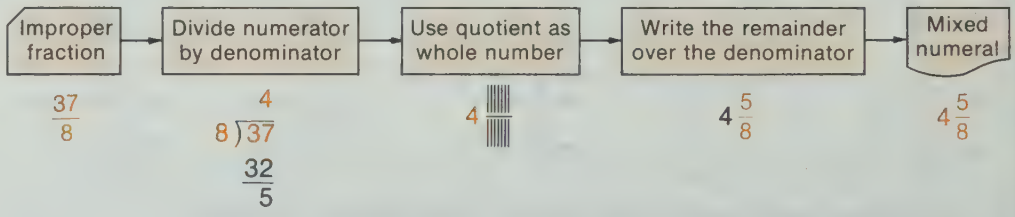
Discussing the Ideas

1. Explain the pattern for the sequences in the Investigation.

2. Give whole numbers for ***a*** and ***b*** and a mixed numeral for **c**.

   A $\frac{17}{12} = \frac{12}{12} + \frac{a}{12} = 1 + \frac{b}{12} = c$ **B** $\frac{22}{3} = \frac{21}{3} + \frac{a}{3} = 7 + \frac{b}{3} = c$

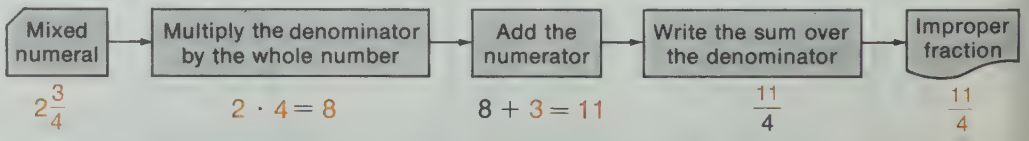
3. The flow chart shows a shortcut for finding mixed numerals for improper fractions.



   Use the flow chart to help you find mixed numerals for these improper fractions.

   A $\frac{12}{5}$ **B** $\frac{11}{2}$ **C** $\frac{45}{8}$ **D** $\frac{100}{3}$ **E** $\frac{38}{9}$ **F** $\frac{17}{17}$ **G** $\frac{63}{15}$

4. This flow chart shows a shortcut for finding an improper fraction for a mixed numeral.



   Use the flow chart to find improper fractions for these mixed numerals.

   A $1\frac{2}{3}$ **B** $3\frac{1}{5}$ **C** $3\frac{4}{5}$ **D** $7\frac{3}{8}$ **E** $9\frac{9}{10}$ **F** $5\frac{3}{7}$ **G** $10\frac{7}{8}$

Using the Ideas

1. Write a mixed numeral for each improper fraction.

A $\frac{7}{3}$	**D** $\frac{25}{4}$	**G** $\frac{39}{8}$	**J** $\frac{31}{16}$	**M** $\frac{49}{10}$	**P** $\frac{610}{80}$
B $\frac{13}{5}$	**E** $\frac{47}{12}$	**H** $\frac{66}{7}$	**K** $\frac{111}{20}$	**N** $\frac{223}{100}$	**Q** $\frac{75}{24}$
C $\frac{27}{10}$	**F** $\frac{27}{2}$	**I** $\frac{74}{9}$	**L** $\frac{51}{13}$	**O** $\frac{5287}{1000}$	**R** $\frac{90}{60}$

2. Write an improper fraction for each mixed numeral.

A $7\frac{1}{8}$	**D** $8\frac{2}{3}$	**G** $6\frac{3}{4}$	**J** $37\frac{1}{2}$	**M** $16\frac{2}{3}$	**P** $30\frac{1}{4}$
B $2\frac{5}{6}$	**E** $33\frac{1}{3}$	**H** $10\frac{9}{10}$	**K** $18\frac{3}{7}$	**N** $11\frac{3}{7}$	**Q** $62\frac{1}{2}$
C $3\frac{4}{5}$	**F** $10\frac{5}{6}$	**I** $9\frac{7}{8}$	**L** $3\frac{1}{3}$	**O** $19\frac{3}{4}$	**R** $83\frac{1}{3}$

3. Give the missing numerators of the fractions.

A $3 = \frac{\square}{1} = \frac{\square}{2} = \frac{\square}{3}$ **C** $10 = \frac{\square}{4} = \frac{\square}{8} = \frac{\square}{10}$ **E** $9 = \frac{\square}{7} = \frac{\square}{8} = \frac{\square}{9}$

B $7 = \frac{\square}{3} = \frac{\square}{5} = \frac{\square}{7}$ **D** $13 = \frac{\square}{1} = \frac{\square}{2} = \frac{\square}{3}$ **F** $0 = \frac{\square}{5} = \frac{\square}{11} = \frac{\square}{20}$

4. Give the numerators ***a*** and ***b***.

A $4\frac{1}{8} = 3 + \left(\frac{a}{8} + \frac{1}{8}\right) = 3\frac{b}{8}$ **C** $3\frac{2}{3} = 2 + \left(\frac{a}{3} + \frac{2}{3}\right) = 2\frac{b}{3}$

B $6\frac{1}{4} = 5 + \left(\frac{a}{4} + \frac{1}{4}\right) = 5\frac{b}{4}$ **D** $7\frac{4}{5} = 6 + \left(\frac{a}{5} + \frac{4}{5}\right) = 6\frac{b}{5}$

5. Solve the equations. Give your answers in lowest terms.

A $4 + n = 3\frac{5}{4}$ **C** $12\frac{12}{9} = 13 + n$ **E** $30\frac{7}{4} = 31 + n$

B $10 + n = 9\frac{11}{8}$ **D** $23 + n = 22\frac{18}{16}$ **F** $72\frac{19}{16} = 73 + n$

6. Write each mixed numeral in such a way that the fractional part is not an improper fraction and is also in lowest terms.

A $13\frac{10}{8}$	**C** $26\frac{9}{6}$	**E** $10\frac{18}{15}$	**G** $32\frac{15}{12}$
B $9\frac{10}{4}$	**D** $1\frac{4}{3}$	**F** $3\frac{24}{10}$	**H** $11\frac{28}{24}$

Think

The number ***n*** in Problem A is the same as in Problem B. The quotient in Problem A is the same as in Problem B. Find ***n*** and the missing quotient.

A $15\overline{)n}$, R13

B $18\overline{)n}$, R1

More practice, page S-14, Set 24

UTILIZATION

These exercises will help to develop skills needed in adding and subtracting fractional numbers represented by mixed numerals or improper fractions; they also provide useful background for later work in multiplication and division involving such fractions.

Unfortunately, the word "improper" may suggest to some students that there is something "wrong" with these fractions, so be sure to stress that improper fractions represent fractional numbers just as legitimately as do proper fractions. The term "improper fraction" need not be stressed, but it should be introduced because students are likely to encounter it in future work.

EXTENSION

Supplementary Exercise Set 24 on page S-14, *Workbook* page 41, and *Duplicator Masters* page 28 are recommended for students who would benefit from additional practice with the ideas presented in this lesson. Also see *Arithmetic Skill Cards* F-3 and F-4.

Think Solution

The reasoning for this problem might be: The number n must be equal to both

[15 · (the number) + 13] and

[18 · (the same number) + 1].

15 times any number ends in 5 or 0, so [15 · (the number) + 13] ends in 8 or 3. The preceding fact tell us that [18 · (the number) + 1] must end in 8 or 3, which means that [18 · (the number)] must end in 7 or 2. Since no number times 18 ends in 7 (because 18 times any number is even), we only need a product of 18 and a number that ends in 2. [18 · (a number ending in 4)] ends in 2, and [18 · (a number ending in 9)] ends in 2. Using this information, the student might try [(18 · 4) + 1] to see if it equals [(15 · 4) + 13]. Since 72 + 1 = 60 + 13] and since 72 + 1 = 60 + 13, the solution is:

$15\overline{)73}$ 4, R13 $18\overline{)73}$ 4, R1

Assignments

Minimum 1A–I, 2A–I, 3. Average 1–5. Maximum 1–6.

Objectives

The student can identify and use the basic principles of commutativity, associativity, and zero for addition of fractional numbers.

The student can find sums and differences of fractional numbers represented by mixed numerals.

PREPARATION

Since there is no Investigation for this lesson, you might review the basic principles for addition of whole numbers, so that students will be ready to discuss these principles for fractional numbers.

DISCUSSION

Students are likely already to be aware of the fact that the basic principles that hold for addition of whole numbers also hold for addition of fractional numbers. The principles for fractional numbers are formalized in Exercise 1. In Exercise 2, it is brought out that in most applications, we use both commutative and associative principles freely and simultaneously.

Plan to spend some time discussing the examples in Exercises 5 and 6. Slower students may need extra help in the subtraction example since regrouping is required. If Exercises 4, 5, and 6 on page B-27 were not assigned or discussed previously, it would be helpful to include them in this discussion. Supply extra examples like the ones below as needed.

A. $\frac{5}{8} = \frac{15}{24}$
$+\frac{5}{6} = \frac{20}{24}$
$\frac{35}{24} = 1\frac{11}{24}$

B. $7\frac{1}{4} = 7\frac{2}{8}$
$+3\frac{7}{8} = 3\frac{7}{8}$
$10\frac{9}{8} = 11\frac{1}{8}$

C. $27\frac{1}{3} = 27\frac{2}{6} = 26\frac{8}{6}$
$-18\frac{5}{6} = -18\frac{5}{6} = -18\frac{5}{6}$
$8\frac{3}{6} = 8\frac{1}{2}$

Basic Principles for Addition of Fractional Numbers

Discussing the Ideas

1. If $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ represent any fractional numbers, the basic principles in the table below hold for addition of fractional numbers.

Commutative principle	Associative principle	Zero principle
$\frac{a}{b}+\frac{c}{d}=\frac{c}{d}+\frac{a}{b}$	$\left(\frac{a}{b}+\frac{c}{d}\right)+\frac{e}{f}=\frac{a}{b}+\left(\frac{c}{d}+\frac{e}{f}\right)$	$\frac{a}{b}+0=\frac{a}{b}$

A Which principle states that the sum will be the same even though the **order** of the addends is changed?

B Which principle enables you to change the **grouping** of the addends without changing the sum?

2. Since addition is both commutative and associative, we may omit parentheses and arrange the addends in any way that we choose.

$$\frac{1}{3}+\frac{3}{4}+\frac{5}{6}=\frac{5}{6}+\frac{1}{3}+\frac{3}{4}$$

What are some other ways that you could arrange the addends $\frac{1}{3}$, $\frac{3}{4}$, and $\frac{5}{6}$?

3. **A** Which principle enables you to find the sum of $\frac{23}{42}+0$ mentally?

B What is the sum?

4. Explain how you can use the basic principles to help you find these sums.

A $\frac{1}{8}+\left(\frac{7}{8}+\frac{1}{4}\right)$ **B** $\left(\frac{1}{3}+0\right)+\frac{1}{3}$ **C** $\left(2+\frac{1}{4}\right)+\left(6+\frac{3}{4}\right)$

5. Explain the steps used to find this sum.

$$12\tfrac{1}{3} = 12\tfrac{4}{12}$$
$$+19\tfrac{3}{4} = 19\tfrac{9}{12}$$
$$31\tfrac{13}{12} = 32\tfrac{1}{12}$$

6. Explain the steps used to find this difference.

$$23\tfrac{1}{2} = 23\tfrac{3}{6} = 22\tfrac{9}{6}$$
$$-12\tfrac{2}{3} = 12\tfrac{4}{6} = 12\tfrac{4}{6}$$
$$10\tfrac{5}{6}$$

Using the Ideas

1. Use the commutative, associative, and zero principles to find the sums.

A $\frac{1}{6}+\left(\frac{5}{6}+\frac{1}{4}\right)$ **C** $\frac{3}{10}+\left(\frac{7}{10}+\frac{1}{3}\right)$ **E** $\frac{5}{6}+\frac{9}{10}+\frac{1}{6}$ **G** $\frac{6}{13}+\frac{0}{11}+\frac{2}{13}$

B $\left(\frac{3}{8}+\frac{3}{4}\right)+\frac{1}{4}$ **D** $\left(\frac{1}{2}+0\right)+\frac{1}{2}$ **F** $\frac{7}{11}+\frac{5}{9}+\frac{4}{11}$ **H** $\frac{3}{7}+\frac{2}{7}+\frac{0}{3}$

2. Find the sums.

A $\frac{3}{4}+\frac{2}{3}$ **B** $\frac{9}{10}+\frac{3}{5}$ **C** $\frac{5}{12}+\frac{5}{8}$ **D** $\frac{1}{8}+\frac{8}{9}$ **E** $\frac{7}{15}+\frac{3}{5}$ **F** $\frac{8}{9}+\frac{5}{12}$

G $23\frac{9}{10}+32\frac{1}{5}$ **H** $11\frac{7}{10}+16\frac{3}{4}$ **I** $83\frac{1}{3}+16\frac{2}{3}$ **J** $52\frac{1}{2}+29\frac{7}{10}$ **K** $41\frac{1}{8}+33\frac{5}{6}$ **L** $19\frac{5}{9}+20\frac{5}{6}$

3. Find the differences.

A $7\frac{7}{8}-3\frac{3}{4}$ **B** $16\frac{1}{4}-12\frac{3}{8}$ **C** $35\frac{1}{6}-28\frac{1}{4}$ **D** $8\frac{1}{5}-7\frac{9}{10}$ **E** $24\frac{2}{3}-11\frac{1}{2}$ **F** $40-27\frac{5}{8}$

G $11\frac{1}{10}-5\frac{7}{10}$ **H** $14\frac{1}{2}-8\frac{2}{3}$ **I** $19\frac{3}{8}-13\frac{3}{4}$ **J** $66\frac{2}{3}-33\frac{3}{4}$ **K** $53\frac{3}{16}-42\frac{1}{2}$ **L** $31\frac{1}{8}-17\frac{1}{2}$

4. The price of a certain stock was listed at $68\frac{1}{4}$. Three days later, it was listed at $65\frac{1}{2}$. How much did the price drop?

5. Roberta lives $2\frac{1}{2}$ blocks from Joan. She ran $1\frac{3}{4}$ blocks and walked the rest of the way to Joan's house. How far did she walk?

think

Find the sums for Exercises 1 through 4. Use these results to find the sum for Exercise 5.

1. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}$

2. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}$

3. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}$

4. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}$

5. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{99\cdot 100}$

More practice, page S-15, Set 25

UTILIZATION

Exercise 1 is designed to encourage the student to use the associative, commutative, and zero principles freely. Many of your students will be able to do the parts of this problem mentally.

It may be helpful to have parts of Exercises 2 through 5 put on the board and explained to the class. Some of your students will be quite efficient at this type of addition-subtraction exercise, but others will need more practice.

EXTENSION

To provide further practice for the students, make selective assignments from Supplementary Exercise Set 25 on page S-15, *Workbook* page 42, *Duplicator Masters* page 29, and *Arithmetic Skill Cards* F-5 through F-8.

Enrichment: Ask students who solve the Think problem to create three more problems that follow pattern 4 and show that the pattern holds for the problems they write.

Think Solution

Notice that the sums for Exercises 1 through 4 follow a pattern. The factors of the denominator of the last fraction written as a proper fraction are equivalent, in lowest terms, to the sum. For example, the sum

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}$$

equals $\frac{4}{6}$ or $\frac{2}{3}$, which is a proper fraction using the factors of the last denominator.

Assignments

Minimum 1, 2A–C, 2G–I, 3A–F.
Average 1, oral; 2G–L, 3A–F, 4.
Maximum 1, oral; 2–5.

Objective

The student can use addition and subtraction to solve fractional number problems involving length and perimeter.

PREPARATION

Materials: Standard typing paper (two sheets per student), centimetre rulers, scissors.

Before assigning the problems, you might review the idea that the perimeter of a rectangle is the sum of the lengths of the sides.

UTILIZATION

Since these problems involve measurement, you should allow for some variation in answers. However, the following comments and the given answers are based on a sheet of paper that measures $21\frac{6}{10}$ by $27\frac{9}{10}$ centimetres.

If feasible, allow pairs of students to work together on these problems. The students can alternate measuring and recording the required information. Also, they can share rulers and scissors. While it is possible to complete the problems without actually cutting the paper, most students will find it helpful to do so.

In Exercise 4 if the strips are cut as indicated by the drawing, there will be two strips about $27\frac{9}{10}$ cm long and two strips about $17\frac{6}{10}$ cm long, so the total length will be about $27\frac{9}{10} + 27\frac{9}{10} + 17\frac{6}{10} + 17\frac{6}{10}$, or 91 cm. Other addends are possible, depending on how the strips are cut.

For Exercise 5, the perimeter of the remaining $17\frac{6}{10}$ by $23\frac{9}{10}$ cm rectangle is 83 cm. Similarly, for Exercise 6, using the smaller rectangle ($13\frac{6}{10}$ by $19\frac{9}{10}$ cm), the perimeter is 67 cm.

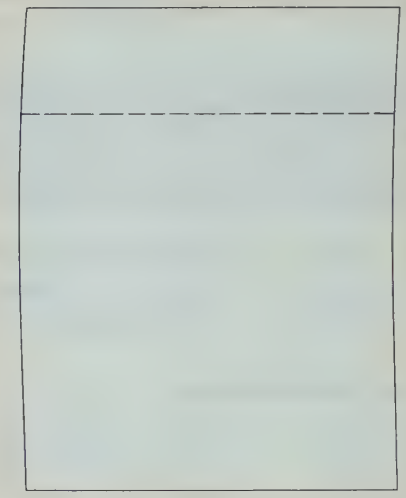
Exercise 7 can be solved by using the formula for the area of a rectangle. However, this involves multiplication of fractional numbers. Students can avoid this difficulty by placing all the strips they have removed in Exercises 4 and 6 on the remaining rectangle to see whether or not the strips will completely cover it. They will find that the strips will more than cover it, so it must be less than half the size of the original sheet.

Assignments

Minimum 1–3. Average 1–6. Maximum 1–7.

PAPER PROBLEMS

The problems below refer to the size of a sheet of paper that is ordinarily used for writing and typing.

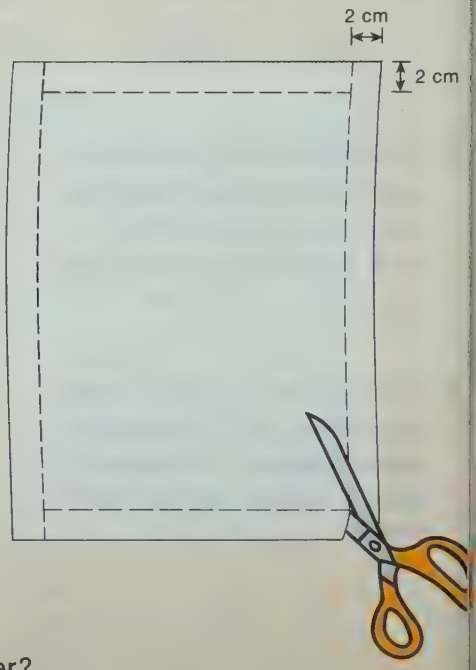


1. What is the perimeter (distance around) of a sheet of typing paper?

2. Measure one of the diagonals of the paper to the nearest $\frac{1}{10}$ centimetre. What is its length?

3. If you cut a square from the paper whose sides are the same length as the shorter side of the sheet of paper, how long and how wide is the rectangular piece of paper that is left?

4. Suppose that you cut a strip of paper two centimetres wide from each of the four sides of a piece of typing paper and then placed the four strips end-to-end lengthwise. What would be the total length of the four strips?



5. What would be the perimeter of the rectangle that is left after the 2-centimetre strips were cut from the sheet?

6. Repeat Exercise 4 using the rectangle that was left after the first set of 2-centimetre strips have been removed? What is the perimeter of the new smaller rectangle that is left after the second cutting?

☆ 7. Is the last rectangle more or less than half the size of the original sheet of paper?

1500-METRE OLYMPIC RECORDS

For years it seemed nearly impossible that man would be able to run the Olympic distance of 1500 metres in less than four minutes. But in 12 Olympic meets many runners have run the distance in less than four minutes. The table gives the men who have won this event over the past seventy six years.

NAME	YEAR	TIME	
		min	sec
Edwin Flack (Br.)	1896	4	$33\frac{1}{5}$
Charles Bennett (Br.)	1900	4	6
James Lightbody (U.S.)	1904	4	$5\frac{2}{5}$
James Lightbody (U.S.)	1906	4	12
Mel Sheppard (U.S.)	1908	4	$3\frac{2}{5}$
Arnold Jackson (Br.)	1912	3	$56\frac{8}{10}$
Albert Hill (Br.)	1920	4	$1\frac{4}{5}$
Paavo Nurmi (Fin.)	1924	3	$53\frac{6}{10}$
Harry Larva (Fin.)	1928	3	$53\frac{1}{5}$
Luigi Beccali (Italy)	1932	3	$51\frac{2}{10}$
Jack Lovelock (N.Z.)	1936	3	$47\frac{8}{10}$
Henri Eriksson (Swed.)	1948	3	$49\frac{8}{10}$
Joseph Barthel (Lux.)	1952	3	$45\frac{2}{10}$
Ron Delany (Ire.)	1956	3	$41\frac{2}{10}$
Herb Elliot (Australia)	1960	3	$35\frac{6}{10}$
Peter Snell (N.Z.)	1964	3	$38\frac{1}{10}$
Kipchoge Keino (Kenya)	1968	3	$34\frac{9}{10}$
Pekka Vasala (Fin.)	1972	3	$36\frac{3}{10}$

1. According to the table, who was the first to run the 1500 in less than 4 minutes? How much less than 4 minutes was his time?

2. How many seconds less did Ron Delany take to run the event than did Jack Lovelock?

3. Nurmi and Larva were two famous Finnish runners. How much difference was there in their Olympic times?

4. How many seconds difference is there in James Lightbody's record time and Pekka Vasala's time?

5. Which person in the table was the first to run the 1500 in almost a minute less than Edwin Flack?

6. In 1967, Jim Ryun set a record time for the 1500 metre race. His time for this race was 3 minutes $33\frac{1}{10}$ seconds. How much better is this time than the record Olympic time for the event?

Objective

The student can use addition and subtraction of fractional numbers to solve word problems.

PREPARATION

You may find it convenient to assign the exercises on both this page and the preceding page at the same time.

UTILIZATION

These problems may have particular appeal to those students who are interested in track and field events. You may capitalize on this interest by having some interested students provide records in other track and field events or in other sports.

Assignments

Minimum 1–4. Average 1–6. Maximum 1–7.

Objective

Given the input numbers and a function rule, the student can compute the output numbers and graph the points for the number pairs of the function.

PREPARATION

Materials: Graph paper or duplicated co-ordinated grids (*Duplicator Masters*, pages 83 and 85).

Review the use of the function machine using whole numbers as inputs and single addition or subtraction rules such as $n + 3$ or $10 - n$. You may also want to review locating points in the co-ordinate plane given the co-ordinates of the points.

INVESTIGATION

Hand out duplicated co-ordinate grids, or graph paper so that students can make their own co-ordinate grids. The two axes of the grid in the student text extend only to 3 units. You may prefer that the students try input numbers other than those in the table. If so, they may need to use a larger grid.

DISCUSSION

For Exercises 2 and 3, students can use the same co-ordinate grid that they used for the Investigation. In Exercise 3 students can estimate the co-ordinates in eighths fairly easily as halfway between the co-ordinates for the fourths, which are indicated by the grid lines.

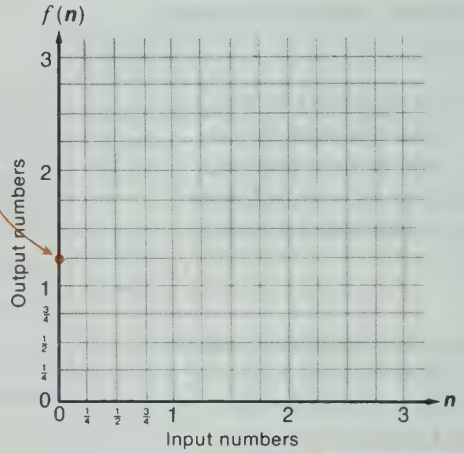
Graphing Functions with Fractional Numbers

Investigating the Ideas

Study the function machine rule to see how the number pairs for the function can be used as co-ordinates on the graph.

Function Rule: $n + 1\frac{1}{4}$

n	$f(n)$	Number pairs
0	$1\frac{1}{4}$	$(0, 1\frac{1}{4})$
$\frac{1}{4}$	$1\frac{1}{2}$	$(\frac{1}{4}, 1\frac{1}{2})$
$\frac{1}{2}$	$1\frac{3}{4}$	$(\frac{1}{2}, 1\frac{3}{4})$
$\frac{3}{4}$	2	$(\frac{3}{4}, 2)$
1	$2\frac{1}{4}$	$(1, 2\frac{1}{4})$
$1\frac{1}{4}$	$2\frac{1}{2}$	$(1\frac{1}{4}, 2\frac{1}{2})$
$1\frac{1}{2}$	$2\frac{3}{4}$	$(1\frac{1}{2}, 2\frac{3}{4})$



Can you mark the points on the grid for the co-ordinates?

Discussing the Ideas

1. Do the points for the co-ordinates for the function above follow a pattern? Describe the pattern.

2. **A** Complete the function table.
   B Use the number pairs of this function as co-ordinates and mark the points for them on a grid.
   C Do the points for this function follow a pattern?

Function Rule: $3 - n$

n	$f(n)$
0	3
$\frac{1}{4}$	$2\frac{3}{4}$
$\frac{1}{2}$	
$\frac{3}{4}$	
1	
$1\frac{1}{2}$	
2	

3. **A** Try the numbers $\frac{3}{8}$, $\frac{5}{8}$, and $\frac{7}{8}$ as input numbers for the function rule in Exercise 2.
   B Estimate the position of the points for the co-ordinates in part **A** and mark them on your graph. Do these points fall in line with your other points?

Using the Ideas

In Exercises 1, 2, and 3, graph the number pairs for each function. Use a grid similar to the one in the Investigation.

1.

Function Rule: $n + \frac{1}{2}$	
n	$f(n)$
0	$\frac{1}{2}$
$\frac{1}{2}$	1
1	$1\frac{1}{2}$
$1\frac{1}{2}$	2
2	$2\frac{1}{2}$
$2\frac{1}{2}$	3

2.

Function Rule: $n - \frac{1}{4}$	
n	$f(n)$
2	$1\frac{3}{4}$
$1\frac{3}{4}$	$1\frac{1}{2}$
$1\frac{1}{2}$	$1\frac{1}{4}$
$1\frac{1}{4}$	1
1	$\frac{3}{4}$
$\frac{3}{4}$	$\frac{1}{2}$

3.

Function Rule: $2 + n$	
n	$f(n)$
0	2
$\frac{1}{2}$	$2\frac{1}{2}$
1	3
$1\frac{1}{2}$	$3\frac{1}{2}$
$1\frac{3}{4}$	$3\frac{3}{4}$
2	4

In Exercises 4, 5, and 6, complete each function table. Then graph the number pairs of the function.

4.

Function Rule: $n + \frac{3}{4}$		
	n	$f(n)$
A	$\frac{1}{4}$	
B	$\frac{1}{2}$	
C	$\frac{3}{4}$	
D	1	
E	$1\frac{1}{4}$	
F	$1\frac{1}{2}$	
G	$1\frac{3}{4}$	

5.

Function Rule: $n - 1\frac{1}{2}$		
	n	$f(n)$
A	3	
B	$2\frac{3}{4}$	
C	$2\frac{1}{2}$	
D	$2\frac{1}{4}$	
E	2	
F	$1\frac{3}{4}$	
G	$1\frac{1}{2}$	

6.

Function Rule: $n + n$		
	n	$f(n)$
A	0	
B	$\frac{1}{4}$	
C	$\frac{1}{2}$	
D	$\frac{3}{4}$	
E	1	
F	$1\frac{1}{4}$	
G	$1\frac{1}{2}$	

7. Invent a function rule. Make a table for your function and then graph the number pairs of the function.

Think

A microbe colony with plenty of food doubles its population every three hours. At noon the colony had a billion microbes. When was the population 500 million?

UTILIZATION

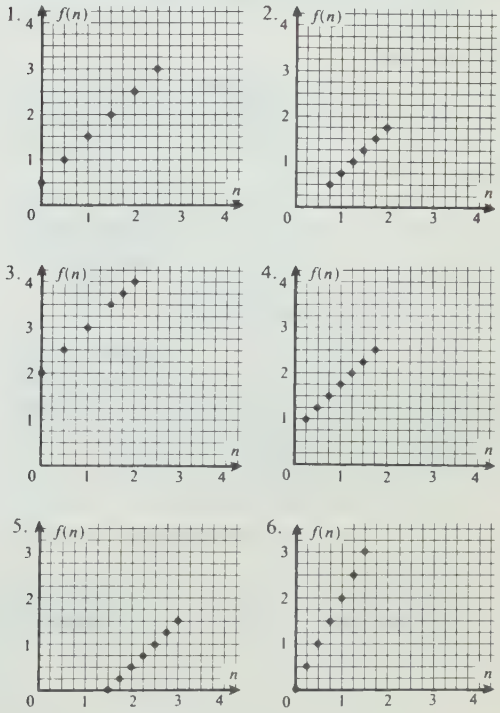
Provide co-ordinate grid or graph paper for the exercises. Suggest that students scale their grids in fourths. Each axis should be at least 4 units long.

After students have completed the exercises, it would be helpful to have selected students display the graphs for each of the functions on the overhead projector or the chalkboard. Other students can compare their graphs with those displayed.

EXTENSION

Suitable further practice exercises are provided on page 43 of the *Workbook*.

Answers, Exercises 1–6



Assignments

Minimum 1–3. Average 1–5. Maximum 1–7.

Objective

The student will be able to round a fractional number to the nearest whole number and find sums and differences using rounded fractional numbers.

PREPARATION

The Investigation involves a discovery problem for which no special preparation is needed.

INVESTIGATION

The symbol $\overrightarrow{\underleftarrow{n}}$ means to round to the nearest whole number, with halves being rounded to the next larger whole number. Most students should be able to discover this special function by studying function table A. Then they can complete function table B to demonstrate their understanding of the function rule.

DISCUSSION

In the discussion of Exercise 3, bring out the fact the function rule $\overrightarrow{\underleftarrow{n}}$ rounds fractional numbers like $\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$, $3\frac{1}{2}$, . . . , *up* to the next larger whole number.

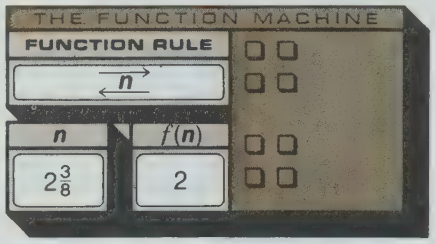
The flow charts in Discussion Exercise 5 show that if we add two fractional numbers and round the sum, we may get an answer different from the one we get if we round each addend and then find the sum.

This interesting function opens the door for invention of a variety of other kinds of rounding functions. As a follow-up activity, you might have some of your more able students invent some special functions of their own.

A Special Function

Investigating the Ideas

The input and output numbers in tables A and B describe a special function rule. The symbol $\overrightarrow{\underleftarrow{n}}$ is used for this function rule.



A

Function Rule $\overrightarrow{\underleftarrow{n}}$	
n	$f(n)$
2	2
$2\frac{3}{8}$	2
$2\frac{1}{2}$	3
$2\frac{3}{4}$	3
$3\frac{1}{8}$	3
$3\frac{5}{8}$	4

B

Function Rule $\overrightarrow{\underleftarrow{n}}$	
n	$f(n)$
$\frac{5}{6}$	
$\frac{7}{4}$	
$\frac{1}{8}$	
$\frac{15}{4}$	
$8\frac{7}{12}$	
$16\frac{1}{2}$	

Can you discover the rule for this special function?	Complete function table B to show that you know the rule.

Discussing the Ideas

1. Do you think that the output numbers for the function rule $\overrightarrow{\underleftarrow{n}}$ will always be a whole number?

2. Mike said, "The function rule rounds each input number to the nearest whole number." Do you agree with him?

3. What does the function rule do about numbers halfway between two whole numbers such as $2\frac{1}{2}$ and $3\frac{1}{2}$?

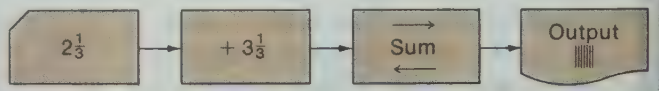
4. Give the missing whole numbers.

   A $\overrightarrow{\underleftarrow{\frac{5}{2}}} = ?$ B $\overrightarrow{\underleftarrow{6\frac{1}{2}}} = ?$ C $\overrightarrow{\underleftarrow{\frac{14}{4}}} = ?$ D $\overrightarrow{\underleftarrow{\frac{33}{2}}} = ?$ E $\overrightarrow{\underleftarrow{\frac{1}{2}}} = ?$

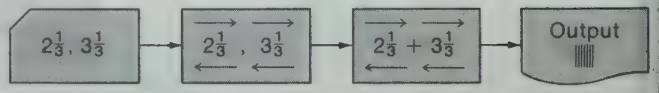
5. A Find the output numbers for flow charts A and B.

   B Explain why the output numbers for the two flow charts are different.

A



B



Using the Ideas

For Exercises 1, 2, and 3, give the output number $f(n)$ for each input number n.

1.

Function Rule $\overrightarrow{\ n\ }$

	n	$f(n)$
A	1	
B	$\frac{4}{3}$	
C	$\frac{5}{3}$	
D	2	
E	$\frac{7}{3}$	
F	$\frac{11}{3}$	

2.

Function Rule $\overrightarrow{\ n\ }$

	n	$f(n)$
A	$\frac{9}{10}$	
B	$\frac{17}{16}$	
C	$\frac{20}{6}$	
D	$\frac{44}{9}$	
E	$\frac{100}{3}$	
F	$\frac{1}{6}$	

3.

Function Rule $\overrightarrow{\ n\ }$

	n	$f(n)$
A	$18\frac{5}{8}$	
B	$9\frac{4}{11}$	
C	$\frac{127}{241}$	
D	$\frac{289}{144}$	
E	$1099\frac{9}{10}$	
F	$\frac{11}{24}$	

In Exercises 4 through 7, solve the equations.

4. $\overrightarrow{1\frac{3}{8}} + \overrightarrow{2\frac{1}{4}} = n$ **5.** $\overrightarrow{1\frac{3}{8} + 2\frac{1}{4}} = n$ **6.** $\overrightarrow{8\frac{3}{4}} - \overrightarrow{3\frac{3}{8}} = n$ **7.** $\overrightarrow{8\frac{3}{4} - 3\frac{3}{8}} = n$

8. Tell which symbol (= or <) is covered by each ▒.

A $3 \text{ ▒ } \overrightarrow{3\frac{1}{4}} \text{ ▒ } 4$ B $4 \text{ ▒ } \overrightarrow{\frac{23}{5}} \text{ ▒ } 5$ C $6 \text{ ▒ } \overrightarrow{\frac{41}{6}} \text{ ▒ } 7$

9. You can find many numbers for N in the equation.

$$\overrightarrow{2\tfrac{2}{3}} \cdot \overrightarrow{N} = 6$$

A What is the smallest number for N?

B The number for N must be less than ▒.

C Is there a largest number for N?

10. Draw a graph of the function $\overrightarrow{n}$. Use fractional numbers from 0 through 5 as input numbers.

Think

Discover the rule for the function shown in the table.

Function Rule ▒

n	$f(n)$
$\frac{3}{4}$	1
$2\frac{1}{7}$	10
$2\frac{6}{7}$	11
$5\frac{7}{8}$	110
$\frac{46}{3}$	1111
$\frac{31}{2}$	10000
$13\frac{2}{5}$	1101

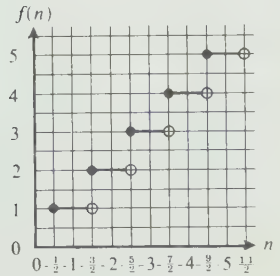
UTILIZATION

Exercises 1, 2, 3 give further practice using the rounding function. The remaining exercises involve some computation.

Solutions, Exercises 9 and 10

9. Since the given factor $2\frac{2}{3}$ rounds to 3, the number chosen for N must round to 2 in order to equal the given product 6. Therefore, the smallest number that rounds to 2, according to this special function, is $1\frac{1}{2}$. Also, N must be less than $2\frac{1}{2}$ to round to 2.

Because of the density of fractional numbers, for any choice of fractional number N less than $2\frac{1}{2}$, there is always a larger fractional number between the chosen number and $2\frac{1}{2}$. Thus N could approach $2\frac{1}{2}$ but never reach it. Therefore, there is no largest fractional number for N.

10. 

Think Solution

The solution to this problem will be difficult for students who have not studied bases other than 10. However, it will be easy for other students to discover the rule: Round each input number to the nearest whole number and express the output as a base-two numeral.

Assignments

Minimum 1–3. Average 1–7. Maximum 1–10.

Objective

The student can demonstrate the ability to work with the concepts presented in this module.

UTILIZATION

The review exercises can be utilized in a variety of ways. They can, of course, be assigned as individual written work. Another way to utilize the problems would be to assign specific problems to various students and then have them explain the solutions to the class. Still another way would be to let pairs of students work and discuss the problems together to help reinforce each others' ideas and to correct each others' errors.

Further module review exercises are provided on page 44 of the *Workbook*.

REVIEWING THE IDEAS

1. Tell what addition fact is shown by the number line, given that it is scaled in

A units C ninths

B halves D twenty-fourths

2. Give the number for n.

A $\frac{3}{5} + n = 1$ E $4\frac{1}{3} - n = 3\frac{2}{3}$

B $\frac{5}{7} + n = 1\frac{3}{7}$ F $n - \frac{3}{4} = \frac{5}{4}$

C $1\frac{1}{3} + n = 2$ G $n - \frac{11}{7} = \frac{12}{7}$

D $\frac{7}{9} - n = \frac{4}{9}$ H $\frac{4}{3} \cdot n = \frac{28}{3}$

3. A What is the least common multiple (LCM) of 4 and 6?

B What is the least common denominator (LCD) for $\frac{3}{4}$ and $\frac{5}{6}$?

4. Find the sums. Give your answers in lowest-terms fractions.

A $\frac{1}{6} + \frac{1}{10}$ C $\frac{2}{15} + \frac{2}{21}$

B $\frac{9}{16} + \frac{1}{4}$ D $\frac{3}{8} + \frac{1}{6}$

5. Give an improper fraction for each mixed numeral.

A $10\frac{1}{3}$ C $2\frac{7}{8}$ E $6\frac{3}{4}$

B $7\frac{3}{10}$ D $3\frac{2}{3}$ F $5\frac{5}{6}$

6. Give a mixed numeral for each improper fraction.

A $\frac{12}{5}$ C $\frac{33}{7}$ E $\frac{127}{14}$

B $\frac{17}{3}$ D $\frac{37}{10}$ F $\frac{441}{100}$

7. Use the associative and commutative principles to obtain the sums easily.

A $2\frac{1}{2} + \frac{3}{5} + 1\frac{1}{2}$ C $\frac{7}{13} + \frac{5}{11} + \frac{6}{13}$

B $\frac{7}{9} + 3\frac{2}{5} + \frac{2}{9}$ D $2\frac{3}{5} + 3\frac{4}{7} + 5\frac{2}{5}$

8. Find the sums.

A $7\frac{1}{6} + 4\frac{2}{3}$ B $28\frac{3}{4} + 29\frac{7}{8}$

9. Find the differences.

A $16\frac{2}{3} - 9\frac{1}{4}$ B $42\frac{1}{4} - 27\frac{3}{8}$

10. Complete the function rule table.

Function Rule: $n + \frac{3}{4}$

	n	$f(n)$
	$\frac{1}{4}$	1
A	$\frac{1}{2}$	
B	$\frac{3}{4}$	
C	1	
D	$\frac{2}{3}$	

11. If $\overrightarrow{\underleftarrow{n}}$ means "round n to the nearest whole number," what are each of the following?

A $\overrightarrow{\underleftarrow{1\frac{2}{3}}}$ B $\overrightarrow{\underleftarrow{7\frac{2}{5}}}$ C $\overrightarrow{\underleftarrow{16\frac{1}{2}}}$

12. A strip of paper is $21\frac{1}{10}$ cm long. If you cut off a piece $9\frac{7}{10}$ cm long how long is the remaining piece?

TEST YOURSELF

1. Give the sum as a lowest-terms fraction.
   $\frac{3}{8} + \frac{7}{8}$

2. Give the difference as a lowest-terms fraction.
   $\frac{13}{4} - \frac{7}{4}$

3. What is the least common denominator (LCD) for $\frac{1}{4}$ and $\frac{5}{6}$?

4. Find the sum. $\frac{1}{4} + \frac{5}{6}$

5. Find the difference. $\frac{4}{3} - \frac{7}{8}$

6. Give a mixed numeral for each improper fraction.
   A $\frac{15}{4}$ B $\frac{25}{10}$

7. Give an improper fraction for each mixed numeral.
   A $3\frac{1}{5}$ B $8\frac{5}{6}$

8. Find the sums.

   A $12\frac{1}{3} + 29\frac{3}{4}$ B $53\frac{7}{8} + 19\frac{3}{4}$

9. Find the differences.

   A $31\frac{1}{2} - 12\frac{1}{5}$ B $100 - 76\frac{9}{10}$

10. Complete the function table.

	Function Rule $n + \frac{2}{3}$	
	n	$n + \frac{2}{3}$
	$\frac{1}{3}$	1
A	$\frac{1}{6}$	
B	$\frac{1}{2}$	

11. Round each fractional number to the nearest whole number.
    A $1\frac{7}{8}$ B $5\frac{3}{16}$ C $3\frac{1}{2}$

RESEARCH PROJECTS

A Find the average yearly rainfall for your area. Use this information to compute the number of kilolitres of water that will fall on each square kilometre. Find the mass of the water that falls on each square kilometre.

B Do you know the speed of light? Can you find how long it takes light from the sun to reach the earth? Look up the speed of light and the distance from the earth to the sun and find the amount of time.

C Do you think there are more fractional numbers than there are whole numbers? You know that between any two whole numbers there are infinitely many fractional numbers. Surprisingly, it can be shown that there are just as many whole numbers as there are fractional numbers. Find out how this can be shown. (See *The Number Story* by Herta T. Freitag and Arthur H. Freitag; Washington, D.C.: National Council of Teachers of Mathematics, 1960.)

TEST YOURSELF

The self-evaluation test covers all the major ideas presented in the module. If the lesson on rounding fractional numbers (pages B-34 and B-35) was omitted, students should not be expected to do Exercise 11. Of course, the same is true for any other exercises that embody ideas, concepts, or skills that may have been omitted from your coverage of the module.

RESEARCH PROJECTS

Research Project A requires the student to refer to tables of measures and to make several conversions in units of area and volume.

For Research Project B, the student will need to use a reference book to find the velocity of light. A commonly used approximation of the speed of light is 300,000 kilometres per second; a more precise value is 299,728 kilometres per second. Thus, students' findings for this project may vary somewhat. The mean distance from the earth to the sun is 148,000,000 kilometres.

Research Project C deals with the topic of infinite sets. The proof that there is a one-to-one correspondence between the set of whole numbers and the set of fractional numbers, in other words, that they have the same (transfinite) cardinal number, was first given by Georg Cantor (1845–1918). Another reference that you might suggest for students who undertake this project is Carl B. Boyer's *A History of Mathematics;* New York: Wiley, 1968, pp. 612–613. (Available from John Wiley and Sons Canada Ltd., Rexdale, Ontario.)

MATHEMATICAL RECREATION

"Peasant multiplication" can be explained in various ways, but perhaps the simplest way to understand why the method works is to see that the principle involved is that multiplication is repeated addition.

First Factor	Second Factor
35	$1 \cdot 27 = 27$
17	$2 \cdot 27 = 54$
~~8~~	~~$4 \cdot 27 = 108$~~
~~4~~	~~$8 \cdot 27 = 216$~~
~~2~~	~~$16 \cdot 27 = 432$~~
1	$32 \cdot 27 = 864$
	$35 \cdot 27 \quad 945$

By striking out the lines under the first factor that contain even numbers, we are eliminating power of 2 multiples of 27 which correspond to the zeros in the binary representation of 35, as the table below shows.

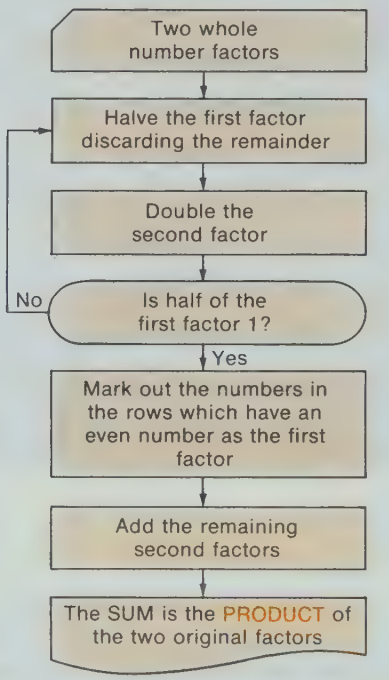
	2^5	2^4	2^3	2^2	2^1	2^0
$35_{(ten)} =$	1	0	0	0	1	$1_{(2)}$

Since every numeral in base ten has a unique base-two representation, this method can be used to find any product of whole numbers.

MATHEMATICAL RECREATIONS

PEASANT MULTIPLICATION

An old method of multiplying whole numbers is shown below. It is sometimes called "peasant multiplication." Study the flow chart which explains the steps and the example that accompanies it to show how to find the product $35 \cdot 27$.


Two whole number factors
Halve the first factor discarding the remainder
Double the second factor
No
Is half of the first factor 1?
Yes
Mark out the numbers in the rows which have an even number as the first factor
Add the remaining second factors
The SUM is the PRODUCT of the two original factors

First factor	Second factor
35	27
17	54
~~8~~	~~108~~
~~4~~	~~216~~
~~2~~	~~432~~
1	864

Total $945 = 35 \cdot 27$

Can you use peasant multiplication to find these products?

1. $25 \cdot 51$ **2.** $19 \cdot 43$ **3.** $39 \cdot 59$ **4.** $24 \cdot 101$

MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. $\frac{5}{4}$ or $1\frac{1}{4}$ **2.** $\frac{3}{2}$ or $1\frac{1}{2}$ **3.** 12 **4.** $\frac{13}{12}$ or $1\frac{1}{12}$ **5.** $\frac{11}{24}$

6. A $3\frac{3}{4}$; B $2\frac{1}{2}$; **7.** A $\frac{16}{5}$; B $\frac{53}{6}$ **8.** A $42\frac{1}{12}$; B $73\frac{5}{8}$

9. A $19\frac{3}{10}$; B $23\frac{1}{10}$ **10.** A $\frac{5}{6}$; B $1\frac{1}{6}$ **11.** A 2; B 5; C 4

UNIT B: THE ARITHMETIC OF FRACTIONAL NUMBERS

Module 3: Multiplying and Dividing Fractional Numbers

General Objectives

To review multiplication of fractional numbers.
To increase understanding and skill in division of fractional numbers.
To introduce complex fractions.
To introduce methods of simplifying complex fractions.
To improve problem solving skills with fractional numbers.

	Performance Objectives	Pupil Text	Reteach-Reinforce	Related Activities	
50	**RED** The student can find the product of factors which are unit fractions and/or whole numbers.	B-40, 41 B-42, 43	**WB** 45, 46		
51	**RED** Given two fractional numbers, the student can find their product.	B-44, 45 B-46, 47	**WB** 47, 48 **DM** 30	**ASC** F-9, 10, 11 **SP** d-1, 2 **PD** c-3	
52	**RED** Given a nonzero fractional number, the student can give the reciprocal of the number.	B-48, 49	**WB** 49	**SP** d-2 **PD** c-4	
53	**RED** Given two fractional numbers, the student can find their quotient.	B-50, 51 B-52, 53	**WB** 50, 51 **DM** 31, 32	**ASC** F-12 **SP** d-3, 4 **PD** c-4	**SWM 2** 94
54	**YELLOW** The student can solve equations by using the basic principles for multiplication of fractional numbers.	B-42, 43 B-46, 47	**WB** 46, 48		
55	**YELLOW** The student can simplify a complex fraction.	B-54, 55	**WB** 52		
56	**YELLOW** The student can solve word problems involving multiplication or division of fractional numbers.	B-47, 58, 59	**DM** 33		
57	***GREEN*** *The student can solve functions or word problems involving fractional numbers.*	B-56, 57 B-60, 61	**WB** 53		
	Reviewing the Ideas	B-62	**WB** 54		

MATHEMATICS

To develop understanding of the multiplication algorithm for fractional numbers, the steps outlined below are developed in the student text.

A. Establish the following two unit-fraction principles:

$$\frac{1}{a}\cdot\frac{1}{b}=\frac{1}{a\cdot b} \text{ and } a\cdot\frac{1}{b}=\frac{a}{b}$$

B. Assume the commutative and associative principles hold for multiplication of fractional numbers.

C. Apply the basic principles to derive the product of any two fractional numbers.

$$\begin{aligned}\frac{a}{b}\cdot\frac{c}{d}&=a\cdot\frac{1}{b}\cdot c\cdot\frac{1}{d}\\&=a\cdot c\cdot\frac{1}{b}\cdot\frac{1}{d}\\&=a\cdot c\cdot\frac{1}{b\cdot d}\\&=\frac{a\cdot c}{b\cdot d}\end{aligned}$$

One of the features that distinguishes the set of fractional numbers from the set of whole numbers is that each nonzero fractional number has a unique reciprocal, or multiplicative inverse. Because of this feature, division, except by zero, is always possible and is closed on the set of fractional numbers.

While there are various ways that could be used to develop the traditional rule, or algorithm, for division of fractional numbers, the method used in this book is one which relies

strongly on the inverse relation between multiplication and division of fractional numbers.

To find the quotient $\frac{a}{b} \div \frac{c}{d}$, we seek a number $\frac{x}{y}$ such that $\frac{x}{y} \cdot \frac{c}{d} = \frac{a}{b}$. If c is a factor of a and d is a factor of b, then x and y and hence $\frac{x}{y}$ are easily found. In the more general case we choose a fraction equivalent to $\frac{a}{b}$ and the most convenient choice is $\frac{a}{b} = \frac{a \cdot (c \cdot d)}{b \cdot (c \cdot d)}$. Then it is easily seen that $\frac{a \cdot (c \cdot d)}{b \cdot (c \cdot d)} \div \frac{c}{d} = \frac{a \cdot d}{b \cdot c} = \frac{a}{b} \cdot \frac{d}{c}$. Thus, we have proved the division algorithm:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

TEACHING THE MODULE

Vocabulary

associative principle
commutative principle
complex fraction
denominator
distributive principle
equilibrium
factor
function
multiplicative inverse
number line
numerator
one principle
product
quotient
reciprocal
region
unit fraction
unit fraction principles

Although the basic purpose of the module is to review and extend work with multiplication and division of fractional numbers, there is ample material dealing with special functions and applications of fractional numbers to various thematic problem sets to meet a wide variety of abilities and interests.

Since many students have had previous experience with the multiplication and division algorithms, you may want to emphasize understanding of the algorithms and increased facility in their use. Additional practice sets as well as accompanying workbook and duplicator masters material may aid you in this effort.

If some of your students have good computational ability with fractional numbers, you should encourage them to devote more of their efforts to the special word problem sets and the special research projects suggested at the end of the module.

Lesson Schedule

If one day is devoted to each lesson and additional time is alloted to review and testing, you will need 10 to 12 days to cover the module. However, with groups of students for whom most of the material will be review, you may be able to cover the material adequately in somewhat less time.

Evaluation

This module, like the previous one, is largely devoted to the development of computational algorithms. It is not hard to determine whether a given student can correctly use such an algorithm in a straightforward computational situation. Applying the algorithms properly in word problems is usually more difficult for most students, and it is sometimes hard to determine whether errors are the result of computational mistakes or faulty reasoning. The module review exercises, the self-evaluation test at the end of the module, and your observations and contact with the students should help you make a satisfactory assessment of their progress. Also, you might want to administer the Module Achievement Test that is included in the test package available from the publisher.

RESOURCES FOR ACTIVE LEARNING

Activities

Fraction Bars, Card Set II, Scott Resources
Fractional Numbers, Geocards 19–23, Creative Publications
Let's Play Games in Mathematics, Vol. 7, "Number line Football," pp. 15–16, National Textbook Co.
Nuffield Project: *Computation and Structure 5,* pp. 30–44, Wiley

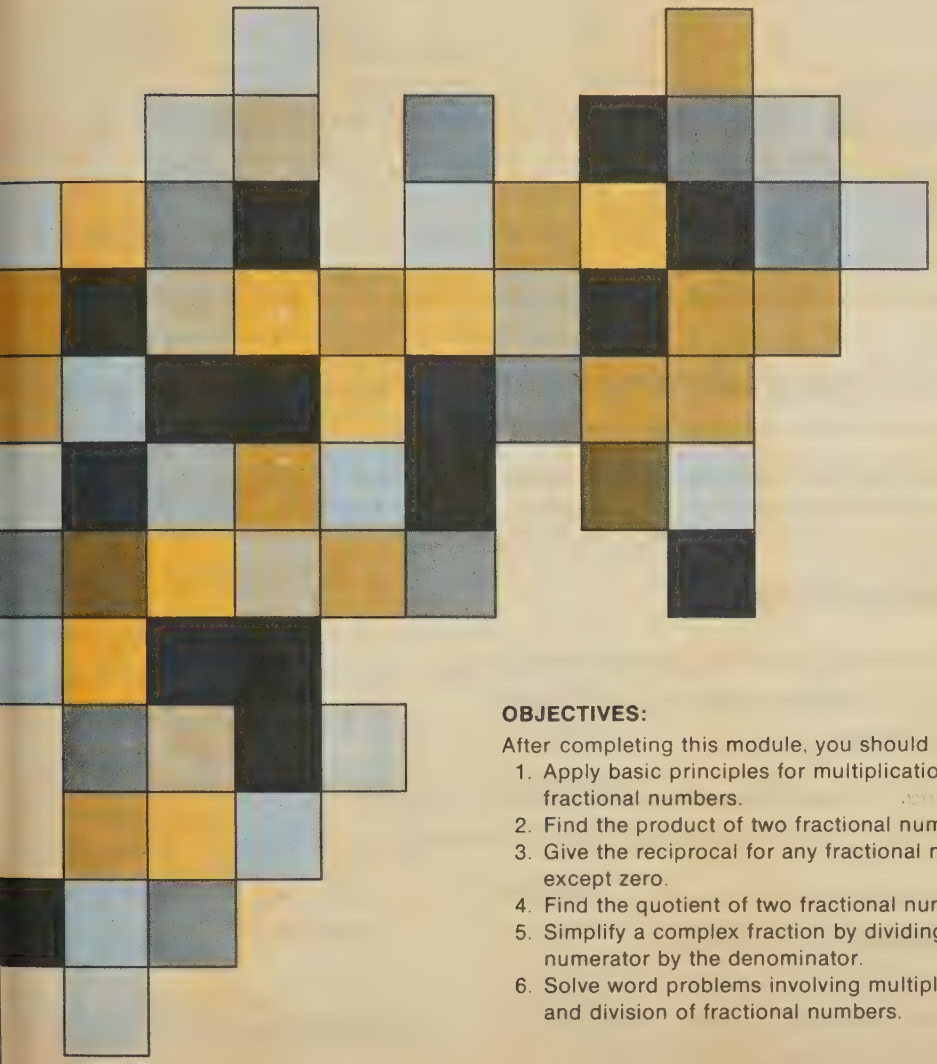
Manipulative Devices

Fraction Bars manipulatives (Scott Resources)
Geoboards (Addison-Wesley)

Games and Puzzles

Spinner Number Games (Heath)

MODULE 3: *Multiplying and Dividing Fractional Numbers*



OBJECTIVES:

After completing this module, you should be able to:

1. Apply basic principles for multiplication of fractional numbers.
2. Find the product of two fractional numbers.
3. Give the reciprocal for any fractional number except zero.
4. Find the quotient of two fractional numbers.
5. Simplify a complex fraction by dividing the numerator by the denominator.
6. Solve word problems involving multiplication and division of fractional numbers.

Objective

The student can find the product of two unit fractions or the product of a whole number and a unit fraction.

PREPARATION

If possible, prepare some transparencies of shaded regions for use with an overhead projector to illustrate multiplication of unit fractions during the Investigation and Discussion. For example, shade $\frac{1}{3}$ of a rectangular region as in figure A. Then make figure B by drawing the horizontal line and shading "$\frac{1}{2}$ of $\frac{1}{3}$" or $\frac{1}{6}$ of the region.

Figure A

Figure B

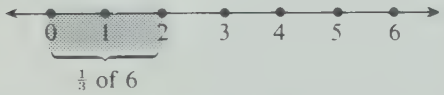
INVESTIGATION

After students have studied the Investigation, you can show additional examples like the one suggested above. Other examples of jumps on the fractional number line could also be illustrated. Encourage students to keep their examples very simple, and use only fractions that have numerators of 1. You might suggest that they choose one from each list below.

Unit fraction times unit fraction	Whole number times unit fraction
$\frac{1}{2} \cdot \frac{1}{4}$	$2 \cdot \frac{1}{3}$
$\frac{1}{3} \cdot \frac{1}{2}$	$3 \cdot \frac{1}{4}$
$\frac{1}{3} \cdot \frac{1}{4}$	$5 \cdot \frac{1}{8}$
$\frac{1}{2} \cdot \frac{1}{2}$	$6 \cdot \frac{1}{5}$

DISCUSSION

In discussing question 3, note that the number line could also be used to find $\frac{1}{3}$ of 6. The student simply thinks, "$\frac{1}{3}$ of the way to 6 is 2, so $\frac{1}{3} \cdot 6 = 2$."



The statements about *unit fractions* in Exercise 4 are important as background for the later development of the multiplication algorithm.

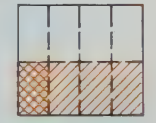
Multiplying with Unit Fractions

Investigating the Ideas

Study examples **A** and **B** below.

A

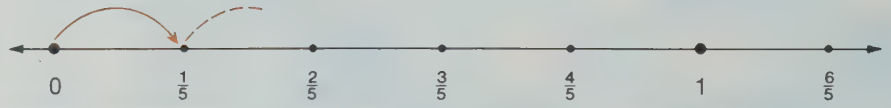
Shade $\frac{1}{2}$ of a region.



Shade $\frac{1}{4}$ of $\frac{1}{2}$ of a region.

What part of the region has been shaded twice?

B



Make 3 jumps of $\frac{1}{5}$ on the number line. Where do you end?

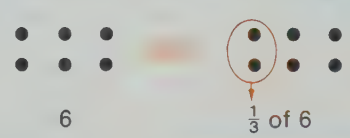
Can you find the answers to questions for A and B?	Make up two more examples using different fractions.

Discussing the Ideas

1. Example A suggests a multiplication equation.
   Since $\frac{1}{4}$ **of** $\frac{1}{2}$ **of the region is** $\frac{1}{8}$ **of a region**, you can write $\frac{1}{4} \cdot \frac{1}{2} = \frac{|}{|||}$.

2. Example B suggests another multiplication equation.
   Since 3 jumps of $\frac{1}{5}$ takes you to $\frac{3}{5}$, you can write $3 \cdot \frac{1}{5} = \frac{|||}{|||}$.

3. **A** What is $\frac{1}{3}$ of 6?
   B Complete: $\frac{1}{3} \cdot 6 = \frac{6}{3} = |||$.



4. Fractions such as $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ are often called **unit fractions** because their numerators are 1. The Investigation and the Discussion above suggest the statements shown at the right. Give a few more examples of multiplication using unit fractions.

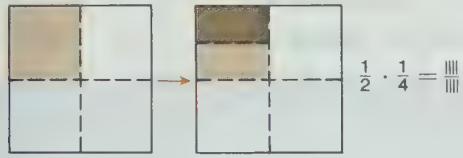
If a and b are any whole numbers, except zero, $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{a \cdot b}$

If n is any whole number, $n \cdot \frac{1}{a} = \frac{1}{a} \cdot n = \frac{n}{a}$

Using the Ideas

1. Copy and complete the multiplication equation suggested by each pair of figures.

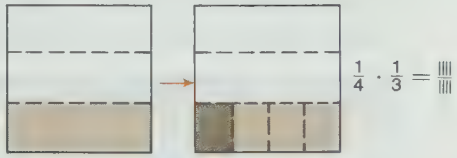
A



$\frac{1}{2} \cdot \frac{1}{4} =$ ||||/||||

$\frac{1}{4}$ of a region $\frac{1}{2}$ of $\frac{1}{4}$

B

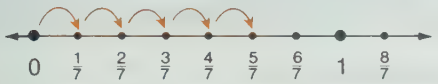


$\frac{1}{4} \cdot \frac{1}{3} =$ ||||/||||

$\frac{1}{3}$ of a region $\frac{1}{4}$ of $\frac{1}{3}$

2. Give the multiplication fact suggested by each number line.

A



B

C

D

3. Solve the equations.

A $\frac{1}{2} \cdot \frac{1}{3} = n$	C $\frac{1}{5} \cdot \frac{1}{5} = n$	E $\frac{1}{3} \cdot \frac{1}{6} = n$	G $\frac{1}{10} \cdot \frac{1}{10} = n$	I $\frac{1}{8} \cdot \frac{1}{4} = n$
B $\frac{1}{4} \cdot \frac{1}{5} = n$	D $\frac{1}{2} \cdot \frac{1}{4} = n$	F $\frac{1}{5} \cdot \frac{1}{4} = n$	H $\frac{1}{10} \cdot \frac{1}{100} = n$	J $\frac{1}{100} \cdot \frac{1}{100} = n$

4. Find the products.

A $3 \cdot \frac{1}{4}$	C $10 \cdot \frac{1}{10}$	E $10 \cdot \frac{1}{100}$	G $0 \cdot \frac{1}{3}$	I $30 \cdot \frac{1}{5}$
B $5 \cdot \frac{1}{8}$	D $4 \cdot \frac{1}{2}$	F $7 \cdot \frac{1}{1000}$	H $1 \cdot \frac{1}{8}$	J $200 \cdot \frac{1}{4}$

5. Write a multiplication equation for each part and then solve it.

EXAMPLE: $\frac{1}{4}$ of 24 ANSWER: $\frac{1}{4} \cdot 24 = \frac{24}{4} = 6$

A $\frac{1}{3}$ of 30	C $\frac{1}{2}$ of 14	E $\frac{1}{8}$ of 32	G $\frac{1}{9}$ of 54	I $\frac{1}{10}$ of 20
B $\frac{1}{10}$ of 100	D $\frac{1}{6}$ of 54	F $\frac{1}{5}$ of 5	H $\frac{1}{4}$ of 20	J $\frac{1}{25}$ of 100

6. In a carton of eggs, $\frac{1}{4}$ of a dozen remain. If you use $\frac{1}{3}$ of the $\frac{1}{4}$ dozen of eggs, what part of a dozen did you use?

think

A bug tries to climb an 8-metre pole. Each day he climbs up 2 metres. Each night while he sleeps, he slips down 1 metre. When will he reach the top of the pole?

More practice, page S-15, Set 26

UTILIZATION

If your students seem to have grasped the use of unit fractions in multiplication, you may proceed quickly through the exercises. For many students, oral discussions of Exercises 1 through 4 would suffice. Adjust your assignments to the level of your students.

EXTENSION

For additional practice, see Supplementary Exercise Set 26 on page S-15 and *Workbook* page 45.

Enrichment: Give students oral practice with multiplication problems involving a whole number and unit fractions in which the product can easily be computed mentally. Examples:

A. $\frac{1}{2} \cdot 60$	G. $300 \cdot \frac{1}{3}$
B. $\frac{1}{3} \cdot 300$	H. $21 \cdot \frac{1}{7}$
C. $\frac{1}{10} \cdot 50$	I. $28 \cdot \frac{1}{4}$
D. $\frac{1}{4} \cdot 32$	J. $100 \cdot \frac{1}{10}$
E. $\frac{1}{8} \cdot 16$	K. $1000 \cdot \frac{1}{100}$
F. $25 \cdot \frac{1}{5}$	L. $\frac{1}{6} \cdot 42$

Think Solution

1. The net gain for the bug is 1 metre per day. However, consider that there is a high point which the bug reaches at the end of each day:

Day	1	2	. . .	5	6	7
High point (m)	2	3	. . .	6	7	8
Net point (m)	1	2	. . .	5	6	

2. As an additional question for this Think, ask the following: If the young nephew of the bug brings him his lunch every day at noon and if the bug climbs steadily so that each day he has climbed 1 m by noon, how far will the nephew climb up and down the pole?

Answer: Total = 2 + 4 + 6 + 8 + 10 + 12 + 14 = 56 m.

Assignments

Minimum 1–4. Average 1–2, oral; 3; 4. Maximum 1–4, oral; 5; 6.

Objective

The student can identify and apply the basic principles for multiplication of fractional numbers.

PREPARATION

Review the basic principles for addition of fractional numbers. (See page B-28.) It may be helpful to show these principles on the chalkboard or overhead projector to compare them with the principles for multiplication.

DISCUSSION

Have the students study the basic principles at the top of the page. Compare them with basic principles for addition. Point out that just as 0 is the *identity* element for addition because $\frac{a}{b}+0=\frac{a}{b}$, 1 is the multiplicative identity element since $\frac{a}{b}\cdot 1=\frac{a}{b}$.

In discussing Exercise 1, bring out the fact that since $\frac{1}{2}\cdot\frac{1}{5}$ and $\frac{1}{5}\cdot\frac{1}{2}$, both have $\frac{1}{10}$ as their product, we can write $\frac{1}{2}\cdot\frac{1}{5}=\frac{1}{5}\cdot\frac{1}{2}$, which illustrates a particular instance of commutativity. In the same way, parts D, E, and F illustrate associativity.

As in addition, associativity and commutativity can be used freely to arrange factors in any order or to group in any convenient manner. This should be stressed in connection with Exercise 3.

In Discussion Exercise 4, some students may not realize why zero denominators are not permitted. One argument against zero denominators is that we do not permit division by zero. A better argument, at this stage, is that if a symbol like $\frac{0}{0}$ were used for a fraction, then all fractions would have to be equivalent to each other. Thus, $\frac{0}{0}$ would be equivalent to $\frac{2}{5}$ because, using the "cross product" check, we have $0\cdot 5=0\cdot 2$. $\frac{0}{0}$ would be equivalent to $\frac{3}{4}$ because $0\cdot 4=0\cdot 3$. Therefore we would be led to think that $\frac{2}{5}$ and $\frac{3}{4}$ are also equivalent fractions, which is false.

Basic Principles for Multiplication of Fractional Numbers

Discussing the Ideas

In the table of basic principles $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ represent any fractional number and n represents a whole number.

Commutative principle	Associative principle	Unit fraction principles
$\frac{a}{b}\cdot\frac{c}{d}=\frac{c}{d}\cdot\frac{a}{b}$	$\left(\frac{a}{b}\cdot\frac{c}{d}\right)\cdot\frac{e}{f}=\frac{a}{b}\cdot\left(\frac{c}{d}\cdot\frac{e}{f}\right)$	$\frac{1}{a}\cdot\frac{1}{b}=\frac{1}{a\cdot b}$
One principle	**Distributive principle**	
$\frac{a}{b}\cdot 1=\frac{a}{b}$	$\frac{a}{b}\cdot\left(\frac{c}{d}+\frac{e}{f}\right)=\left(\frac{a}{b}\cdot\frac{c}{d}\right)+\left(\frac{a}{b}\cdot\frac{e}{f}\right)$	$\frac{1}{a}\cdot n=n\cdot\frac{1}{a}=\frac{n}{a}$

1. Find the number for x. Then find the number for y.

A $\frac{1}{2}\cdot\frac{1}{5}=x$ $\quad\frac{1}{5}\cdot\frac{1}{2}=y$

B $\frac{1}{3}\cdot\frac{1}{7}=x$ $\quad\frac{1}{7}\cdot\frac{1}{3}=y$

C $\frac{1}{8}\cdot\frac{1}{9}=x$ $\quad\frac{1}{9}\cdot\frac{1}{8}=y$

D $\frac{1}{2}\cdot\left(\frac{1}{3}\cdot\frac{1}{4}\right)=x\rightarrow\left(\frac{1}{2}\cdot\frac{1}{3}\right)\cdot\frac{1}{4}=y$

E $\frac{1}{5}\cdot\left(\frac{1}{2}\cdot\frac{1}{3}\right)=x\rightarrow\left(\frac{1}{5}\cdot\frac{1}{2}\right)\cdot\frac{1}{3}=y$

F $\frac{1}{2}\cdot\left(\frac{1}{4}\cdot\frac{1}{5}\right)=x\rightarrow\left(\frac{1}{2}\cdot\frac{1}{4}\right)\cdot\frac{1}{5}=y$

2. Give the principle that justifies each step of the multiplication problem below.

A $\left(4\cdot\frac{1}{3}\right)\cdot\frac{1}{5}=4\cdot\left(\frac{1}{3}\cdot\frac{1}{5}\right)$

B $4\cdot\left(\frac{1}{3}\cdot\frac{1}{5}\right)=4\cdot\left(\frac{1}{15}\right)$

C $4\cdot\left(\frac{1}{15}\right)=\frac{4}{15}$

3. Which two principles used together allow you to change the order and grouping of the factors as shown below?

Instead of: $\left(\frac{2}{3}\cdot\frac{7}{8}\right)\cdot\frac{3}{2}$ — You can write: $\frac{2}{3}\cdot\left(\frac{3}{2}\cdot\frac{7}{8}\right)$

4. **A** If $a\neq 0$, what whole number is $\frac{a}{a}$? **B** If $a\neq 0$, what whole number is $\frac{0}{a}$?

5. Find the products.

A $\frac{5}{7}\cdot\frac{9}{9}$ **B** $\frac{13}{25}\cdot\frac{47}{47}$ **C** $\frac{73}{42}\cdot\frac{0}{61}$ **D** $\left(\frac{7}{11}\cdot\frac{5}{9}\right)\cdot\frac{0}{23}$ **E** $\frac{31}{31}\cdot\frac{0}{25}$ **F** $\frac{67}{67}\cdot\frac{99}{99}$

6. Complete the equations. Use the distributive principle.

A $\left(\frac{5}{6}\cdot\frac{1}{5}\right)+\left(\frac{5}{6}\cdot\frac{4}{5}\right)=\frac{5}{6}\cdot\left(\frac{1}{5}+x\right)$ **B** $\frac{3}{4}\cdot\left(\frac{1}{3}+\frac{1}{2}\right)=\left(\frac{3}{4}\cdot\frac{1}{3}\right)+x$

Using the Ideas

1. Use the commutative principle to find n.

A Since $\frac{3}{5} \cdot \frac{2}{3} = \frac{2}{5}$, then $\frac{2}{3} \cdot \frac{3}{5} = n$.

B Since $\frac{1}{2} \cdot \frac{4}{7} = \frac{2}{7}$, then $\frac{4}{7} \cdot \frac{1}{2} = n$.

C Since $\frac{3}{4} \cdot \frac{6}{5} = \frac{9}{10}$, then $\frac{6}{5} \cdot \frac{3}{4} = n$.

D Since $\frac{2}{5} \cdot \frac{7}{10} = \frac{7}{25}$, then $\frac{7}{10} \cdot \frac{2}{5} = n$.

E Since $\frac{8}{3} \cdot \frac{5}{8} = \frac{5}{3}$, then $\frac{5}{8} \cdot \frac{8}{3} = n$.

F Since $5 \cdot \frac{2}{3} = \frac{10}{3}$, then $\frac{2}{3} \cdot 5 = n$.

2. Solve the equations.

A $\frac{1}{4} \cdot 1 = n$ **B** $1 \cdot \frac{1}{8} = n$ **C** $n \cdot 1 = \frac{5}{6}$ **D** $\frac{3}{5} \cdot \frac{3}{3} = n$ **E** $\frac{5}{2} \cdot \frac{4}{4} = n$ **F** $\frac{9}{10} \cdot \frac{10}{10} = n$ **G** $\frac{1}{9} \cdot n = \frac{1}{9}$ **H** $\frac{0}{3} \cdot \frac{1}{4} = n$ **I** $\frac{5}{12} \cdot n = 0$ **J** $n \cdot \frac{5}{5} = \frac{7}{8}$

3. Solve the equations.

A $3 \cdot \left(\frac{1}{2} \cdot \frac{1}{4}\right) = n$ **B** $\left(3 \cdot \frac{1}{2}\right) \cdot \frac{1}{4} = n$ **C** $\left(3 \cdot \frac{1}{4}\right) \cdot \frac{1}{2} = n$ **D** $5 \cdot \left(\frac{1}{3} \cdot \frac{1}{4}\right) = n$ **E** $\left(5 \cdot \frac{1}{4}\right) \cdot \frac{1}{3} = n$ **F** $\left(5 \cdot \frac{1}{3}\right) \cdot \frac{1}{4} = n$ **G** $7 \cdot \left(\frac{1}{2} \cdot \frac{1}{5}\right) = n$ **H** $\left(7 \cdot \frac{1}{5}\right) \cdot \frac{1}{2} = n$

4. Solve the equations.

A $\left(6 \cdot \frac{1}{4}\right) \cdot \frac{1}{3} = 6 \cdot \left(\frac{1}{4} \cdot n\right)$ **B** $\left(7 \cdot \frac{1}{5}\right) \cdot n = 7 \cdot \left(\frac{1}{5} \cdot \frac{1}{8}\right)$ **C** $\left(n \cdot \frac{1}{2}\right) \cdot \frac{1}{5} = 7 \cdot \left(\frac{1}{2} \cdot \frac{1}{5}\right)$

5. Find each product by first finding the product of the unit fractions.

A $3 \cdot \frac{1}{2} \cdot \frac{1}{3}$ **B** $7 \cdot \frac{1}{5} \cdot \frac{1}{6}$ **C** $\frac{1}{4} \cdot 5 \cdot \frac{1}{2}$ **D** $\frac{1}{10} \cdot \frac{1}{10} \cdot 100$ **E** $7 \cdot \frac{1}{3} \cdot \frac{1}{5}$ **F** $3 \cdot \frac{1}{10} \cdot \frac{1}{10}$ **G** $23 \cdot \frac{1}{10} \cdot \frac{1}{100}$ **H** $\frac{1}{4} \cdot 5 \cdot \frac{1}{4}$ **I** $\frac{1}{3} \cdot 4 \cdot \frac{1}{7}$ **J** $\frac{1}{8} \cdot 24 \cdot \frac{1}{3}$

6. Find the answers in the easiest way possible.

A $\frac{3}{5} \cdot \left(\frac{3}{4} + \frac{1}{4}\right)$ **B** $\frac{6}{7} \cdot \left(\frac{5}{8} + \frac{3}{8}\right)$ **C** $\left(\frac{15}{19} + \frac{4}{19}\right) \cdot \frac{17}{23}$ **D** $\left(\frac{23}{39} + \frac{16}{39}\right) \cdot \frac{37}{45}$ **E** $\left(\frac{5}{7} + \frac{2}{7}\right) \cdot \frac{2}{5}$ **F** $\frac{8}{9} \cdot \left(\frac{2}{3} + \frac{1}{3}\right)$ **G** $\left(\frac{3}{10} + \frac{7}{10}\right) \cdot \frac{10}{11}$ **H** $\left(\frac{4}{15} + \frac{11}{15}\right) \cdot \frac{18}{18}$

7. Solve the equations.

A $4 \cdot \frac{1}{8} \cdot 2 \cdot \frac{1}{3} = n \cdot \left(\frac{1}{8} \cdot \frac{1}{3}\right)$

B $3 \cdot \frac{1}{5} \cdot 2 \cdot \frac{1}{7} = n \cdot \left(\frac{1}{5} \cdot \frac{1}{7}\right)$

C $10 \cdot \frac{1}{3} \cdot 10 \cdot \frac{1}{3} = n \cdot \frac{1}{9}$

D $6 \cdot \frac{1}{5} \cdot 3 \cdot \frac{1}{4} = (6 \cdot 3) \cdot n$

E $5 \cdot \frac{1}{2} \cdot 7 \cdot \frac{1}{3} = 35 \cdot n$

F $11 \cdot \frac{1}{4} \cdot 5 \cdot \frac{1}{2} = n \cdot \frac{1}{8}$

G $12 \cdot \frac{1}{5} \cdot 3 \cdot \frac{1}{4} = 36 \cdot n$

think

If the number of oranges in a bin in a fruit market are counted by twos, threes, fives, and sevens, there is one orange left over each time. What is the smallest number of oranges that could be in the bin?

UTILIZATION

The exercises emphasize use of the basic principles as an aid in finding products using whole numbers and unit fractions. These exercises form a necessary prelude to the development of the multiplication algorithm in the following lesson.

EXTENSION

Workbook page 46 offers more practice exercises suitable for use with this lesson.

This might be a good time to call students' attention to Research Project A at the end of this module (page B-63). Considerable interest and excitement in such a project can be created if daily changes in the stock prices are shown by graphs posted on the classroom bulletin board. Such a project would be suitable for students of all ability levels.

Think Solution

In order to solve this problem, the LCM of 2, 3, 5, and 7 must be determined. Since 2, 3, 5, and 7 are all prime numbers, the LCM is $2 \cdot 3 \cdot 5 \cdot 7$, or 210. Thus, the smallest integer that would leave a remainder of 1 when divided by all four numbers would be $210 + 1$, or 211.

Assignments

Minimum 1–2, oral; 3; 4.
Average 1–6. Maximum 1–7.

Objective

Given two fractional numbers, the student can find the product by using the multiplication algorithm for fractional numbers.

PREPARATION

Since this lesson extends the concepts of the previous lesson, you can briefly review the previous lesson or begin immediately on the Investigation.

INVESTIGATION

The students should have sufficient experience with multiplication involving unit fractions and whole numbers to compute the products for any selection from the set of factors.

Disregarding variations due to associativity and commutativity, all 11 different products can be found.

$5 \cdot \frac{1}{2} = \frac{5}{2}$ $\quad 5 \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{5}{6}$

$5 \cdot \frac{1}{3} = \frac{5}{3}$ $\quad 7 \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{7}{6}$

$7 \cdot \frac{1}{2} = \frac{7}{2}$ $\quad 5 \cdot 7 \cdot \frac{1}{2} = \frac{35}{2}$

$7 \cdot \frac{1}{3} = \frac{7}{3}$ $\quad 5 \cdot 7 \cdot \frac{1}{3} = \frac{35}{3}$

$7 \cdot 5 = 35$ $\quad 5 \cdot 7 \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{35}{6}$

$\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$

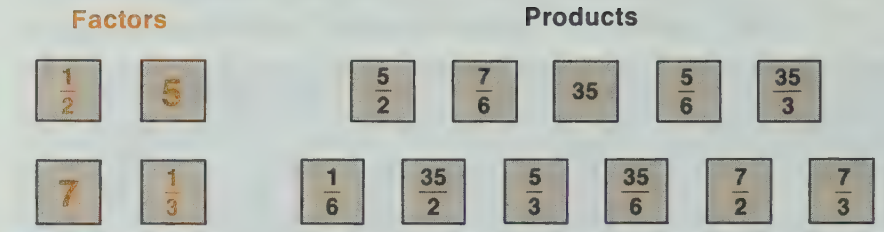
The last case involving all four factors is significant since any student who can compute this product is very near to understanding the multiplication algorithm for fractional numbers.

DISCUSSION

Exercises 1, 2, and 3 refer to the Investigation. Exercise 4 gives a *power skill* method of finding a product by using the basic principles of multiplication for fractional numbers. Exercise 5 illustrates the *speed skill* or shortcut method for multiplying; a flow chart is used to help explain the steps in the algorithm. Once the student understands *how* the basic principles justify the shortcut, the student should be expected to use the algorithm as given in Exercise 5 rather than the power skill method of Exercise 4.

Finding Products

Investigating the Ideas



How many of the products can you find using any of the four factors?

Write an equation for each product that you find.

Discussing the Ideas

1. Which two factors above can you multiply to give the product $\frac{5}{2}$?

2. Which two factors above can you multiply to give the product $\frac{7}{3}$?

3. Which three factors above can you multiply to give the product $\frac{5}{6}$?

4. By using the basic principles, you can find products such as $\frac{5}{2} \cdot \frac{7}{3}$.

   Can you explain what was done in each step?

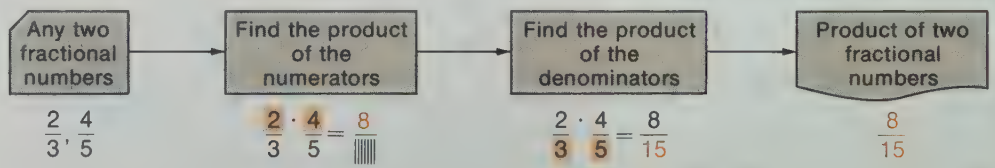
$$\frac{5}{2} \cdot \frac{7}{3} = \left(5 \cdot \frac{1}{2}\right) \cdot \left(7 \cdot \frac{1}{3}\right)$$

$$= \left(5 \cdot 7\right) \cdot \left(\frac{1}{2} \cdot \frac{1}{3}\right)$$

$$= 35 \cdot \frac{1}{6}$$

$$= \frac{35}{6}$$

5. The flow chart below suggests a shortcut for finding products.


Any two fractional numbers
Find the product of the numerators
Find the product of the denominators
Product of two fractional numbers
2/3, 4/5
2/3 · 4/5 = 8/
2/3 · 4/5 = 8/15
8/15

Use the flow chart to find these products.

A $\frac{1}{2} \cdot \frac{3}{5}$ **B** $\frac{4}{5} \cdot \frac{2}{7}$ **C** $\frac{1}{8} \cdot \frac{3}{2}$ **D** $\frac{5}{8} \cdot \frac{3}{4}$ **E** $\frac{7}{8} \cdot \frac{10}{11}$ **F** $\frac{5}{6} \cdot \frac{3}{10}$

Using the Ideas

1. Find the products.

A $\frac{3}{4} \cdot \frac{3}{5}$	**E** $\frac{7}{2} \cdot \frac{5}{3}$	**I** $\frac{53}{17} \cdot \frac{21}{34}$	**M** $\frac{43}{16} \cdot \frac{7}{16}$	**Q** $\frac{17}{10} \cdot \frac{7}{10}$
B $\frac{2}{3} \cdot \frac{4}{5}$	**F** $\frac{7}{4} \cdot \frac{5}{2}$	**J** $\frac{3}{7} \cdot \frac{4}{19}$	**N** $\frac{7}{8} \cdot \frac{5}{6}$	**R** $\frac{23}{100} \cdot \frac{47}{100}$
C $\frac{3}{7} \cdot \frac{5}{6}$	**G** $\frac{5}{6} \cdot \frac{3}{8}$	**K** $\frac{4}{19} \cdot \frac{3}{7}$	**O** $\frac{16}{24} \cdot \frac{27}{36}$	**S** $\frac{5}{21} \cdot \frac{7}{25}$
D $\frac{6}{6} \cdot \frac{4}{3}$	**H** $\frac{3}{11} \cdot \frac{5}{13}$	**L** $\frac{13}{19} \cdot \frac{7}{7}$	**P** $\frac{10}{15} \cdot \frac{50}{100}$	★**T** $\frac{a}{b} \cdot \frac{c}{d}$

2. The examples below show how products are found by first replacing whole numbers and mixed numerals with improper fractions.

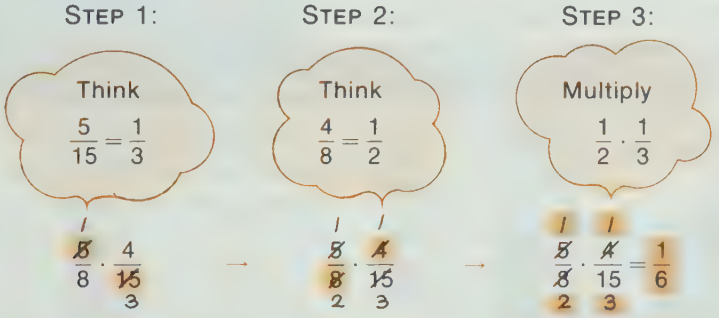
EXAMPLES: **A** $4 \cdot 3\frac{1}{3} = 13\frac{1}{3}$ (↓ $\frac{4}{1} \cdot \frac{10}{3} = \frac{40}{3}$ ↑)

B $3\frac{1}{4} \cdot 2\frac{1}{2} = 8\frac{1}{8}$ (↓ $\frac{13}{4} \cdot \frac{5}{2} = \frac{65}{8}$ ↑)

Find the products as in the examples.

A $1\frac{2}{3} \cdot 3\frac{1}{2}$	**D** $6 \cdot 5\frac{3}{4}$	**G** $12\frac{1}{2} \cdot 2\frac{1}{3}$	**J** $16\frac{2}{3} \cdot 1\frac{4}{5}$
B $1\frac{3}{4} \cdot 1\frac{1}{2}$	**E** $8\frac{1}{5} \cdot 6\frac{2}{3}$	**H** $3\frac{5}{6} \cdot 2\frac{2}{3}$	★**K** $13\frac{3}{5} \cdot 1\frac{3}{17}$
C $2\frac{7}{8} \cdot 1\frac{1}{4}$	**F** $3\frac{1}{3} \cdot 3$	**I** $1\frac{2}{7} \cdot 4$	★**L** $20\frac{2}{5} \cdot 8\frac{1}{3}$

Before trying Exercise 3, study the shortcut demonstrated in the following example.



3. Find the products using the shortcut.

A $\frac{3}{8} \cdot \frac{4}{21}$	**F** $\frac{4}{9} \cdot \frac{15}{10}$	
B $\frac{5}{4} \cdot \frac{2}{15}$	**G** $\frac{14}{63} \cdot \frac{15}{12}$	
C $\frac{8}{21} \cdot \frac{14}{12}$	**H** $\frac{13}{38} \cdot \frac{19}{52}$	
D $\frac{9}{20} \cdot \frac{25}{6}$	**I** $\frac{77}{32} \cdot \frac{12}{35}$	
E $\frac{5}{7} \cdot \frac{21}{10}$	**J** $\frac{50}{42} \cdot \frac{14}{30}$	

Think

Find the smallest number ***y*** (between 100 and 125) so that the sum of all its factors is equal to $y + y + y$. (**Hint:** It must be a number with **many** factors.)

More practice, page S-16, Set 27

UTILIZATION

Exercise 1 offers practice using the standard multiplication algorithm.

You may prefer to discuss the examples given in Exercise 2 before making the assignment. However, most students should be able to find the products after studying the examples on their own.

You may wish to reserve Exercise 3 for only your more able students. The shortcut developed in this problem is one that students have traditionally called "cancelling." (The word "cancellation" has fallen into disrepute due to abuse by students and teachers alike, and we do not recommend its use unless you feel that using it improves communication with your students.) Two important steps involved in the shortcut are: (1) Express all fractions in the multiplication example in lowest terms. (2) Find the GCF of the first numerator and the second denominator and then the second numerator and the first denominator.

EXTENSION

To provide additional appropriate practice, make assignments from Supplementary Exercise Set 27 on page S-16, *Workbook* page 47, *Duplicator Masters* page 30, and *Arithmetic Skill Cards* F-9 and F-11.

Think Solution

The hint suggests that y must be a number with many factors. Consider 117, a number with few factors. The factors are 1, 3, 9, 13, 39, and 117. The sum of these factors can never be three times the number. Therefore, the students might try to think of a number that has factors of 1, 2, 3, 4, 5, 6, and so forth.

They will probably find it necessary to try at least one number other than 120 to establish that they have the smallest number that satisfies the conditions. The number is 120:

$120 + 120 + 120$
$= 1 + 2 + 3 + 4 + 5 + 6 + 8 + 10 + 12 + 15 + 20 + 24 + 30 + 40 + 60 + 120$

Assignments

Minimum 1A–L, 2A–F. Average 1, 2.
Maximum 1–3.

Objective

The student can find the product of two fractional numbers in mixed numeral form by applying the distributive principle.

PREPARATION

Review the method of finding products such as $4\frac{1}{2} \cdot 2\frac{2}{3}$ by expressing the mixed numerals as improper fractions and then multiplying.

It should be noted that in problems involving multiplication with mixed-numeral notation, the mixed numeral can always be expressed as an improper fraction, but sometimes this is undesirable and involves extra labor. Hence, it is useful to learn an alternate method for finding the product of two fractional numbers when mixed-numeral notation is involved.

INVESTIGATION

The Investigation shows two methods of multiplying 4 and $6\frac{1}{3}$. After students have studied the examples, have them try to find the product $8 \cdot 3\frac{2}{5}$ by each method. Some students should be asked to display their work on the chalkboard for others to see and compare their own work. You may want to suggest other examples for the students to try.

DISCUSSION

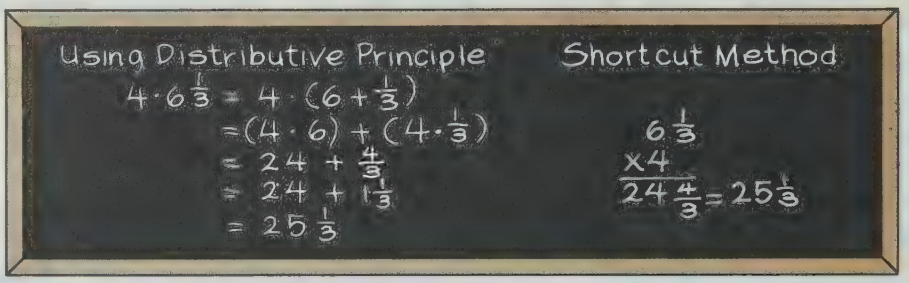
The main difference in the two methods shown in the Investigation is the way in which the student records the partial products. The same steps must be taken in each method, but the first two steps shown under the "Using the Distributive Principle" method are done mentally in the "Shortcut Method."

The most general case of multiplication using mixed numerals is shown in Exercise 2. Step 1 could be broken into the two steps $2 \cdot (4 + \frac{1}{3}) = (2 \cdot 4) + (2 \cdot \frac{1}{3})$.
Step 2 could be broken into the two steps $\frac{1}{2} \cdot (4 + \frac{1}{3}) = (\frac{1}{2} \cdot 4) + (\frac{1}{2} \cdot \frac{1}{3})$.
Then the sum of the four products can be found.

Using the Distributive Principle

Investigating the Ideas

The two examples on the chalkboard show how you can find products using mixed numerals.



Can you find the product $8 \cdot 3\frac{2}{5}$ using each method?

Discussing the Ideas

1. What steps of the distributive principle method are done mentally using the shortcut method?

2. Study the method shown below for finding $2\frac{1}{2} \cdot 4\frac{1}{3}$.

$2\frac{1}{2} \cdot 4\frac{1}{3} = (2 + \frac{1}{2}) \cdot 4\frac{1}{3} = 2 \cdot 4\frac{1}{3} + \frac{1}{2} \cdot 4\frac{1}{3}$

STEP 1:	STEP 2:	STEP 3:
$4\frac{1}{3}$ $\times 2\frac{1}{2}$ $8\frac{2}{3}$	$4\frac{1}{3}$ $\times 2\frac{1}{2}$ $8\frac{2}{3}$ $2\frac{1}{6}$	$4\frac{1}{3}$ $\times 2\frac{1}{2}$ $8\frac{2}{3}$ $2\frac{1}{6}$ $10\frac{5}{6}$
$2 \cdot 4\frac{1}{3} = 8\frac{2}{3}$	$\frac{1}{2} \cdot 4\frac{1}{3} = 2\frac{1}{6}$	$8\frac{2}{3} + 2\frac{1}{6} = 10\frac{5}{6}$

Explain how to find $3\frac{3}{4} \cdot 5\frac{1}{2}$ using this method.

3. Explain how to check your multiplication in Exercise 2 by writing both $3\frac{3}{4}$ and $5\frac{1}{2}$ as improper fractions and then finding the product.

$3\frac{3}{4} \cdot 5\frac{1}{2} = \blacksquare$
$\downarrow \quad \downarrow \quad \uparrow$
$\frac{15}{4} \cdot \frac{11}{2} = \blacksquare$

Using the Ideas

1. Find the products.

A $12\frac{1}{5}$	**B** 8	**C** $6\frac{1}{2}$	**D** $10\frac{1}{6}$	**E** $8\frac{1}{8}$
$\times 4$	$\times \frac{3}{4}$	$\times 4\frac{1}{3}$	$\times 7\frac{1}{2}$	$\times 3\frac{1}{4}$
F $12\frac{1}{5}$	**G** $16\frac{3}{4}$	**H** $93\frac{1}{3}$	**I** $15\frac{1}{3}$	**J** $120\frac{1}{10}$
$\times 7\frac{1}{2}$	$\times 12\frac{1}{2}$	$\times 15\frac{2}{3}$	$\times 12\frac{1}{5}$	$\times 25\frac{1}{3}$

2. Find the products.

A $4 \cdot 6\frac{1}{4}$	**B** $7 \cdot 2\frac{1}{2}$	**C** $3 \cdot 4\frac{1}{6}$	**D** $\frac{1}{2} \cdot 10\frac{1}{4}$	**E** $\frac{1}{8} \cdot 12\frac{1}{2}$
F $\frac{1}{3} \cdot 15\frac{1}{2}$	**G** $\frac{3}{4} \cdot 8\frac{2}{3}$	**H** $10 \cdot 8\frac{3}{10}$	**I** $\frac{2}{3} \cdot 9\frac{3}{5}$	**J** $\frac{3}{5} \cdot 10\frac{5}{6}$

3. There are 12 books on a shelf. Each book is $3\frac{1}{2}$ cm thick. What is the total thickness of the books?

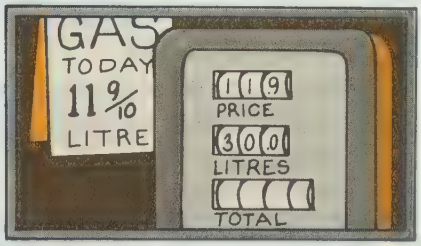
4. Carla earned \$18 a certain week doing parttime work. The next week she earned $1\frac{2}{3}$ times as much. How much did she earn during the second week?

5. Jacqueline saw this road sign. If she walks about $1\frac{3}{5}$ kilometres in half an hour how many hours to Paris?

6. A note pad is $10\frac{3}{5}$ cm wide and $15\frac{1}{2}$ cm long. Find the area of a pad by multiplying the length times the width.

7. Alice had \$24. She spent half of it. in one store and half of her remaining money in another store. How much money did she have left?

8. Raymond lives 3 blocks away from school. Andy lives $4\frac{1}{3}$ times as far from school as Raymond. How far does Andy live from school?



9. Andrea bought 30 litres of gasoline for her car. The price was $11\frac{9}{10}$¢ per litre. How much did she pay for the gasoline?

More practice, page S-16, Set 28

UTILIZATION

If your students are already proficient in computing products using mixed numerals, they may find the word problems in Exercises 3 through 9 more motivating and challenging than Exercises 1 and 2.

EXTENSION

Supplementary Exercise Set 28 on page S-16 and *Workbook* page 48 are designed to help improve computational skills using mixed numerals. *Arithmetic Skill Card* F-10 would also be helpful here.

Enrichment: Challenge the students to design a flow chart which will show the steps in multiplying two mixed numerals such as $3\frac{1}{2} \cdot 5\frac{3}{4}$.

Assignments

Minimum 1A–E, 2A–E, 3. Average 1–5. Maximum 1–9.

Objective

Given a nonzero fractional number, the student can find the reciprocal of the number and state that the product of the number and its reciprocal is one.

PREPARATION

Review the multiplication algorithm for fractional numbers. You might have students work together in small groups to cut up sheets of paper into 15 squares and label them like those shown in the Investigation.

INVESTIGATION

First, have the students study the two examples of multiplication with products of 1. Then have the students use the set of squares they made in the Preparation to locate and record the different pairs of the numbers listed which have a product of 1. Only the number $\frac{3}{2}$ cannot be paired with another factor to give a product of 1.

DISCUSSION

After the students have discovered the pairs of factors in the Investigation which have a product of 1, the definition of reciprocal or multiplicative inverse of a fractional number given in Exercise 1 should be quite meaningful.

Exercise 3 draws together many significant facts about reciprocals. After doing this exercise, the students should realize that the reciprocals of all whole numbers greater than 1 lie between zero and 1. This should further reinforce the idea of density exhibited by the fractional numbers. Since there are infinitely many whole numbers and all their reciprocals lie between zero and 1, there must be infinitely many fractional numbers between zero and 1. Students should also realize that each reciprocal of a number is unique.

If students were using the full set of integers, there would be two answers to Exercise 3E, namely, 1 and $^{-}1$. However, they are working with the positive numbers only, so the answer to problem 3E is the number 1.

For Exercise 4, any reasonable response is acceptable.

Reciprocals

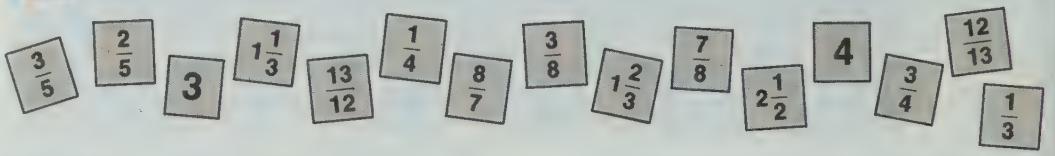
Investigating the Ideas

Some pairs of numbers have a product of 1.

$\frac{3}{2} \cdot \frac{2}{3} = 1$

$8 \cdot \frac{1}{8} = 1$

Using the numbers below, how many pairs can you find that have a product of 1?	Record the pairs that you find.

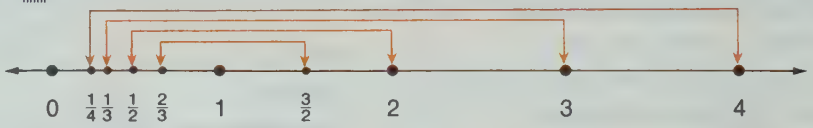


Discussing the Ideas

1. If the product of two fractional numbers is 1, then each number is the **reciprocal** or **multiplicative inverse** of the other.
   - A Since $8 \cdot \frac{1}{8} = 1$, what is the multiplicative inverse of $\frac{1}{8}$?
   - B How can you find the reciprocal of $\frac{3}{2}$?

2. Do you think that zero has a reciprocal? Explain.

3. The number line shows how reciprocals are related. Study the number line and then explain how you can find the number for each ▨.



   - A If a number is between 2 and 3, then its reciprocal will be between ▨ and ▨.
   - B Numbers between $\frac{3}{2}$ and 2 have reciprocals between ▨ and ▨.
   - C Numbers which are very large have reciprocals which are very close to ▨.
   - D Numbers which are very close to 1 have reciprocals which are very close to ▨.
   - E ▨ is the only number which is its own reciprocal.

4. Can you describe an easy method for finding the reciprocal of any fractional number?

Using the Ideas

1. Give the reciprocals of the following numbers.

   A $\frac{5}{8}$ **B** $\frac{8}{5}$ **C** $\frac{1}{9}$ **D** 8 **E** $\frac{4}{11}$ **F** $1\frac{7}{8}$ **G** $8\frac{1}{7}$ **H** $\frac{99}{100}$

2. Solve the equations by finding the reciprocal of the given factor.

   A $n \cdot \frac{3}{8} = 1$ **C** $n \cdot \frac{9}{4} = 1$ **E** $3\frac{1}{2} \cdot n = 1$ **G** $\frac{1}{10} \cdot n = 1$

   B $\frac{5}{6} \cdot n = 1$ **D** $n \cdot \frac{8}{7} = 1$ **F** $7 \cdot n = 1$ **H** $100 \cdot n = 1$

3. Multiply each number by the reciprocal of 2.

   A 8 **B** 14 **C** 7 **D** 1 **E** 15 **F** 20

4. Multiply each number by the reciprocal of 3.

   A 12 **B** 30 **C** 18 **D** 9 **E** 60 **F** 8

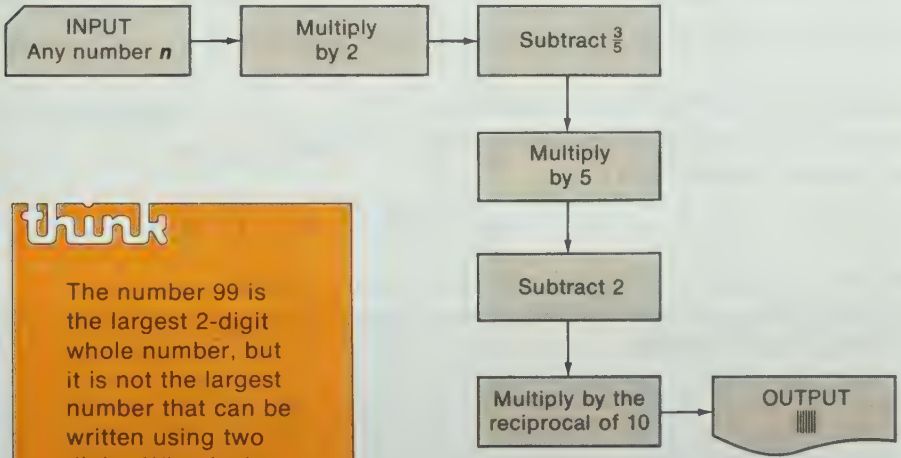
5. Multiply each number by the reciprocal of 5.

   A 10 **B** 25 **C** 100 **D** 40 **E** 15 **F** 7

6. Complete the sentence: Multiplying a whole number by the reciprocal of 2 is the same as dividing the number by ? .

★7. Use each number as an input number for the flow chart to find the output number.

   A 2 **B** 5 **C** $\frac{3}{2}$ **D** $2\frac{1}{4}$ **E** $\frac{5}{8}$ **F** 1



think

The number 99 is the largest 2-digit whole number, but it is not the largest number that can be written using two digits. What is the largest number?

UTILIZATION

Exercises 1 through 6 give practice using reciprocals. In Exercises 3, 4, and 5, students should discover that multiplying by the reciprocal of a whole number is equivalent to dividing by the whole number. Thus,

n times the reciprocal of 2 is the same as $n \cdot \frac{1}{2} = n \div 2$.

The input numbers for the flow chart in Exercise 7 are restricted to numbers greater than $\frac{3}{5}$ in order to avoid any need for work with negative numbers.

EXTENSION

Page 49 of the *Workbook* may be assigned to provide further practice in use of the concepts presented in this lesson.

Think Solution

The largest number that can be formed using 2 digits is 9^9. This power of nine represents 387 420 489.

Assignments

Minimum 1–3. Average 1–6. Maximum 1–7.

Objective

Given a division problem using fractional numbers, the student can solve it by finding a missing factor.

PREPARATION

Review the inverse relation between multiplication and division for whole numbers. Review the fact that the number for n in the multiplication equation $n \cdot 6 = 42$ is the solution to the division equation $42 \div 6 = n$. Use other examples like these with whole numbers. Then remind students that the same relation between multiplication and division also holds for fractional numbers.

INVESTIGATION

The missing factors in the inverse flow charts in the Investigation can be easily found. Thus, in flow chart A, the student can see that since $\frac{2}{5} \cdot \frac{2}{3} = \frac{4}{15}$, $n = \frac{2}{5}$.

At this point, do not attempt to give any rules for division of fractional numbers. Some students may recall the division algorithm and use it here. If they do, have them use the inverse flow charts to check their answers.

DISCUSSION

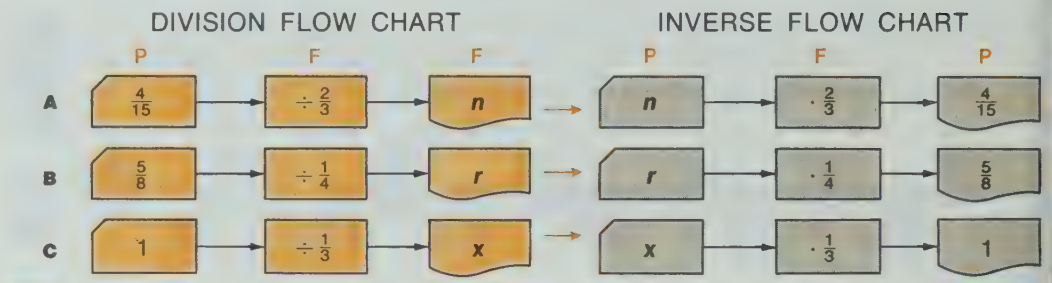
The method illustrated by Mark's work in Exercise 1 is a shortcut that is easily justified by using the inverse relationship between multiplication and division. This method can be used when the numerator and denominator of the dividend are multiples of the numerator and denominator, respectively, of the divisor. All of the exercises in this lesson are of this type. The next lesson will develop the division algorithm which applies in all cases of division of fractional numbers.

Exercises 3, 4, and 5 are all illustrative of the type of problems that the student will be working in the Utilization stage.

Missing Factors and Division

Investigating the Ideas

In order to find the quotients (outputs) for the division flow charts below, you can think about finding the missing factors (inputs) in the inverse flow charts.



How many of the missing numbers in the flow charts can you find?

Discussing the Ideas

1. In division flow chart A, Mark used a shortcut to find the number for ***n***. Can you explain how to use Mark's method for for finding the quotients for flow charts B and C?

Mark

$$\frac{4}{15} \div \frac{2}{3} = \frac{4 \div 2}{15 \div 3} = \frac{2}{5}$$

2. Explain how to find the number for **x** in the flow chart D.

3. Since $\frac{1}{5} \cdot \frac{3}{2} = \frac{3}{10}$, then $\frac{3}{10} \div \frac{3}{2} = \frac{1}{5}$ and $\frac{3}{10} \div \frac{1}{5} = \frac{3}{2}$. What division equations could you write for these multiplication equations?

   A $\frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$ **B** $\frac{2}{7} \cdot \frac{1}{3} = \frac{2}{21}$ **C** $\frac{3}{5} \cdot \frac{2}{3} = \frac{6}{15}$ **D** $\frac{3}{7} \cdot \frac{3}{4} = \frac{9}{28}$

4. Find the missing factor. Then show a division equation so that the quotient will be the missing factor of the multiplication equation.

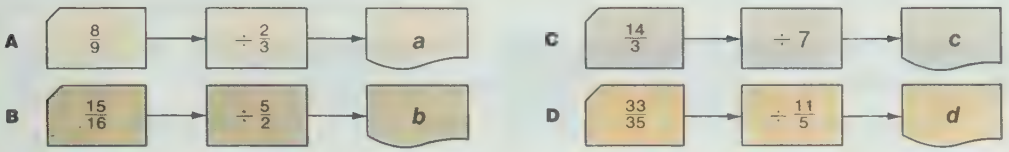
   A $n \cdot \frac{3}{2} = \frac{3}{4}$ **B** $t \cdot \frac{1}{5} = \frac{3}{10}$ **C** $\frac{4}{7} \cdot r = \frac{16}{35}$ **D** $\frac{3}{5} \cdot w = \frac{3}{20}$

5. Explain how to find the quotients by thinking about missing factors.

   A $\frac{10}{21} \div \frac{5}{7}$ **B** $\frac{3}{8} \div \frac{3}{4}$ **C** $\frac{15}{28} \div \frac{3}{4}$ **D** $\frac{35}{8} \div \frac{5}{4}$

Using the Ideas

1. Find the output number for each flow chart.



2. Write division equations for each multiplication equation.

A $\frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4}$ **B** $\frac{5}{6} \cdot \frac{1}{3} = \frac{5}{18}$ **C** $\frac{3}{4} \cdot \frac{5}{2} = \frac{15}{8}$ **D** $\frac{5}{8} \cdot \frac{3}{8} = \frac{15}{64}$

3. Find the missing factor in the first equation. Then find the correct quotient for the screened equation.

A $n \cdot \frac{5}{8} = \frac{15}{32}$ $\frac{15}{32} \div \frac{5}{8} = n$

B $n \cdot \frac{4}{9} = \frac{8}{45}$ $\frac{8}{45} \div \frac{4}{9} = n$

C $\frac{3}{5} \cdot n = \frac{9}{10}$ $\frac{9}{10} \div \frac{3}{5} = n$

D $\frac{1}{2} \cdot n = \frac{3}{8}$ $\frac{3}{8} \div \frac{1}{2} = n$

E $\frac{3}{5} \cdot n = \frac{21}{40}$ $\frac{21}{40} \div \frac{3}{5} = n$

F $n \cdot \frac{3}{2} = \frac{9}{10}$ $\frac{9}{10} \div \frac{3}{2} = n$

G $\frac{3}{10} \cdot n = 1$ $1 \div \frac{3}{10} = n$

H $n \cdot \frac{2}{3} = 1$ $1 \div \frac{2}{3} = n$

4. Find the missing factor. Then write a division equation so that the quotient will be the missing factor of the multiplication equation.

A $\frac{3}{4} \cdot n = \frac{3}{8}$ **B** $n \cdot \frac{5}{9} = \frac{10}{27}$ **C** $\frac{3}{4} \cdot n = \frac{9}{20}$

D $\frac{3}{8} \cdot n = \frac{9}{16}$ **E** $\frac{4}{5} \cdot n = \frac{12}{35}$ **F** $n \cdot \frac{1}{2} = 4$

G $\frac{1}{10} \cdot n = \frac{1}{2}$ **H** $\frac{3}{5} \cdot n = \frac{3}{10}$ **I** $\frac{5}{7} \cdot n = \frac{25}{49}$

J $\frac{4}{7} \cdot n = \frac{9}{7}$ **K** $\frac{5}{12} \cdot n = \frac{10}{24}$ **L** $\frac{3}{2} \cdot n = \frac{3}{2}$

5. Find the quotients.

A $\frac{3}{4} \div \frac{1}{2}$ **B** $\frac{15}{21} \div \frac{3}{7}$ **C** $\frac{1}{10} \div \frac{1}{5}$

D $\frac{7}{10} \div \frac{7}{10}$ **E** $\frac{1}{4} \div \frac{1}{2}$ **F** $\frac{35}{24} \div \frac{5}{4}$

G $\frac{15}{32} \div \frac{3}{8}$ **H** $\frac{3}{10} \div 1$ **I** $\frac{4}{18} \div \frac{2}{3}$

J $\frac{27}{100} \div \frac{9}{10}$ **K** $\frac{18}{25} \div \frac{3}{5}$ **L** $\frac{54}{111} \div \frac{27}{37}$

Think

Each cube represents a room in a hotel constructed according to the pattern shown. How many floors does the hotel have if there are 969 rooms in the hotel?

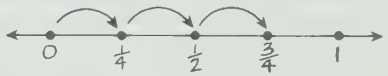
UTILIZATION

Assign the problems according to the ability of your students. If your students can already use the division algorithm for fractional numbers efficiently, you may find it advisable to treat these exercises very lightly and proceed to the next lesson.

EXTENSION

Workbook page 50 and *Duplicator Masters* page 31 offer more practice exercises appropriate for use with this lesson.

Remedial: Use the number line to illustrate division of fractional numbers for students having difficulty understanding the meaning of division. Keep the examples very simple in order not to obscure the ideas. For example, the division problem $\frac{3}{4} \div \frac{1}{4}$ could be interpreted as asking "How many $\frac{1}{4}$'s in $\frac{3}{4}$?" Using the number line (or by counting) the student can see that there are 3.



Hence, $\frac{3}{4} \div \frac{1}{4} = 3$.

Think Solution

The pattern is as follows:

1 floor: $1^2 = 1$ room
2 floors: $1^2 + 3^2 = 10$ rooms
3 floors: $1^2 + 3^2 + 5^2 = 35$ rooms
4 floors: $1^2 + 3^2 + 5^2 + 7^2 = 84$ rooms
5 floors: $1^2 + 3^2 + 5^2 + 7^2 + 9^2 = 165$ rooms
6 floors: $1^2 + 3^2 + 5^2 + 7^2 + 9^2 + 11^2 = 286$ rooms
7 floors: $1^2 + 3^2 + 5^2 + 7^2 + 9^2 + 11^2 + 13^2 = 455$ rooms
8 floors: $1^2 + 3^2 + 5^2 + 7^2 + 9^2 + 11^2 + 13^2 + 15^2 = 680$ rooms
9 floors: $1^2 + 3^2 + 5^2 + 7^2 + 9^2 + 11^2 + 13^2 + 15^2 + 17^2 = 969$ rooms

Answer: 9 floors

Assignments

Minimum 1–3. Average 1, 2, 3, 5.
Maximum 1–5.

A Shortcut for Dividing Fractional Numbers

Objective

Given two fractional numbers, the student can find their quotient by using the division algorithm for fractional numbers.

PREPARATION

You will note that there has been considerable preparation for the development of the division algorithm for fractional numbers. In this section we complete the development of the algorithm. Therefore begin immediately with the Discussion. If your students are already quite familiar with the algorithm, you may want to review it only briefly, but most students will benefit from seeing *why* the algorithm works.

DISCUSSION

You may want to put the division problem $\frac{3}{4} \div \frac{2}{5}$, shown in Exercise 1, on the chalkboard or overhead projector and proceed through the 5 steps. The method shown builds upon the method used in the previous lesson. However, since 3 is not divisible by 2, nor is 4 divisible by 5, we must choose multiples of 3 and 4 which will be divisible by either 2 or 5; hence both the numerator and denominator of $\frac{3}{4}$ are multiplied by $(2 \cdot 5)$ and the remaining steps involve simple division.

Keep in mind that *we do not expect students to use this technique* in division problems. It is used only to show why the division algorithm works. Once the student sees that it is perfectly plausible to multiply the reciprocal of the divisor or, in other words, "to invert and multiply," he should use that rule to work division problems without much thought concerning its development.

In Exercise 2, the student can conclude from the work in Exercise 1 that the division problem $\frac{3}{4} \div \frac{2}{5}$ has the same answer as the multiplication problem $\frac{3}{4} \cdot \frac{5}{2}$. The general case is discussed in Exercise 3, and the algorithm is verbalized in Exercise 4. Use Exercise 5 to ascertain if students are ready to proceed to division problems involving mixed numerals.

Discussing the Ideas

1. Study the steps of the example below. The example may help you understand a shortcut for division problems such as $\frac{3}{4} \div \frac{2}{5}$. Can you explain what is being done in each step of the example?

	Step 1		Step 2		Step 3		Step 4		Step 5
$\frac{3}{4} \div \frac{2}{5}$	$=$	$\frac{3 \cdot (2 \cdot 5)}{4 \cdot (2 \cdot 5)} \div \frac{2}{5}$	$=$	$\frac{(3 \cdot 2 \cdot 5) \div 2}{(4 \cdot 2 \cdot 5) \div 5}$	$=$	$\frac{3 \cdot 5}{4 \cdot 2}$	$=$	$\frac{3}{4} \cdot \frac{5}{2}$	$=$	$\frac{15}{8}$

2. From step 4 in the example above, you can conclude that $\frac{3}{4} \div \frac{2}{5} = \frac{3}{4} \cdot \frac{5}{2}$

A How is the divisor $\frac{2}{5}$ and the second factor $\frac{5}{2}$ related?

B Can you explain how you might write the division problem as a multiplication problem? What is the quotient?

3. The examples above suggest the general statement at the right concerning division of fractional numbers. Illustrate the statement if $\frac{a}{b} = \frac{2}{7}$ and $\frac{c}{d} = \frac{3}{4}$.

> If $\frac{a}{b}$ and $\frac{c}{d}$ are fractional numbers and $\frac{c}{d} \neq 0$, $\quad \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$

4. How would you complete this sentence?
To divide a first number by a second number, multiply the first number by the ? of the second number.

5. In order to divide fractional numbers when they are expressed as mixed numerals, you can rewrite the problem using improper fractions. How would you complete each of these problems?

A $2\frac{1}{2} \div 1\frac{3}{4}$
$\frac{5}{2} \div \frac{7}{4} = \frac{5}{2} \cdot \frac{4}{7} = ?$

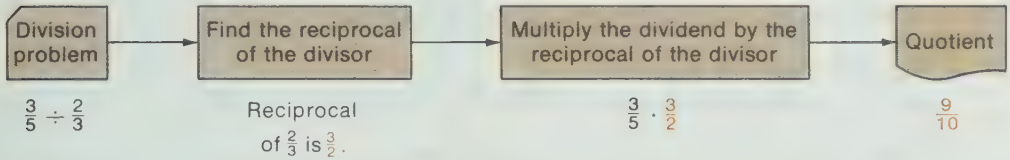
B $3\frac{7}{8} \div 1\frac{2}{3}$
$\frac{31}{8} \div \frac{5}{3} = \frac{31}{8} \cdot \frac{\square}{\square} = ?$

C $4\frac{1}{4} \div 2\frac{1}{3}$
$\frac{\square}{\square} \div \frac{\square}{\square} = \frac{\square}{\square} \cdot \frac{\square}{\square} = ?$

D $5 \div 3\frac{1}{3}$
$\frac{\square}{\square} \div \frac{\square}{\square} = \frac{\square}{\square} \cdot \frac{\square}{\square} = ?$

Using the Ideas

1. Find the quotients. The flow chart for dividing fractional numbers may help you.

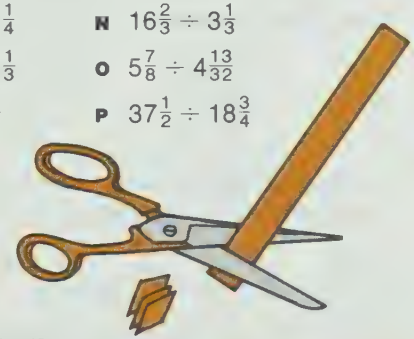


A $\frac{4}{5} \div \frac{2}{3}$	**F** $\frac{9}{10} \div \frac{3}{2}$	**K** $10 \div \frac{3}{8}$	**P** $\frac{3}{10} \div \frac{1}{10}$
B $\frac{1}{6} \div \frac{2}{5}$	**G** $\frac{25}{4} \div \frac{5}{8}$	**L** $8 \div \frac{5}{9}$	**Q** $\frac{4}{9} \div \frac{2}{3}$
C $\frac{3}{7} \div \frac{1}{2}$	**H** $\frac{7}{8} \div \frac{3}{4}$	**M** $5 \div \frac{3}{5}$	**R** $\frac{20}{3} \div 5$
D $\frac{5}{12} \div \frac{5}{3}$	**I** $\frac{3}{4} \div \frac{7}{8}$	**N** $\frac{1}{2} \div \frac{1}{4}$	**S** $\frac{5}{8} \div \frac{5}{4}$
E $8 \div \frac{2}{3}$	**J** $3 \div \frac{1}{4}$	**O** $\frac{1}{2} \div \frac{3}{4}$	**T** $\frac{7}{10} \div \frac{3}{5}$

2. Find the quotients. Rewrite the mixed numerals as improper fractions before dividing.

A $2\frac{1}{2} \div \frac{3}{4}$	**E** $2\frac{3}{4} \div 1\frac{1}{2}$	**I** $\frac{9}{10} \div 1\frac{2}{3}$	**M** $1 \div 2\frac{3}{4}$
B $10\frac{1}{2} \div 6$	**F** $6\frac{2}{3} \div 4$	**J** $15\frac{7}{8} \div 5\frac{1}{4}$	**N** $16\frac{2}{3} \div 3\frac{1}{3}$
C $1\frac{2}{5} \div 1\frac{1}{2}$	**G** $9 \div 3\frac{1}{3}$	**K** $100 \div 3\frac{1}{3}$	**O** $5\frac{7}{8} \div 4\frac{13}{32}$
D $2\frac{2}{3} \div \frac{3}{4}$	**H** $7\frac{1}{2} \div 2$	**L** $10 \div 2\frac{1}{2}$	**P** $37\frac{1}{2} \div 18\frac{3}{4}$

3. If you cut a strip of paper 15 centimetres long into pieces $1\frac{1}{2}$ centimetres long, how many pieces will you have?



4. An airplane travelled 1200 kilometres in $1\frac{2}{3}$ hours. What was the average speed of the airplane?

☆ **5.** Jack wanted to saw a 24-cm strip of plastic into 5 pieces of equal length. If $\frac{3}{10}$ of a cm is lost with each cut, how long will each piece be?

★ **6.** Which number in the set $\left\{1, \frac{4}{9}, \frac{9}{4}, \frac{16}{81}, \frac{81}{16}\right\}$ is the solution for $n \div \frac{4}{9} = \frac{4}{9}$?

Think

Two numbers are reciprocals of each other. One of these numbers is four times as large as the other. Determine both numbers.

More practice, page S-17, Set 29

UTILIZATION

Call students' attention to the flow chart in Exercise 1, which gives a step-by-step presentation of the division algorithm for fractional numbers.

Assign the exercises as needed. Keep in mind that Exercises 1 and 2 contain 36 division problems.

Exercise 6 can easily be solved by thinking of inverse operations. Since $n \div \frac{4}{9} = \frac{4}{9}$, it must be true that $\frac{4}{9} \cdot \frac{4}{9} = n$. Hence $n = \frac{16}{81}$.

EXTENSION

Assign parts of Supplementary Exercises Set 29 on page S-17, *Workbook* page 51, *Duplicator Masters* page 32, and *Arithmetic Skill Card* F-12 for those students who need to improve their skills in dividing with fractional numbers.

Think Solution

Most of your students will approach this problem by trial-and-error methods. There are two conditions that must be satisfied: (1) the product of the numbers must be 1, and (2) one of the numbers must be 4 times as large as the other. In symbols, those conditions are:

$$\frac{1}{n} \cdot n = 1 \qquad \text{and} \qquad \frac{1}{n} = 4n$$

From the second equation,

$$1 = 4n^2$$

$$\text{or} \qquad \frac{1}{4} = n^2$$

$$\text{Thus,} \qquad \frac{1}{2} = n$$

The numbers are $\frac{1}{2}$ and 2.

Assignments

Minimum 1A–G, 2A–D, 3.
Average 1A–M, 2A–H, 3, 4.
Maximum 1K–T, 2I–P, 3–6.

Objective

The student can simplify complex fractions by using the division algorithm for fractional numbers.

PREPARATION

Begin this lesson by writing on the chalkboard problems such as $25 \div 4$, and $32 \div 5$, observing with your students that each quotient can be considered as an improper fraction, or that, by using the procedure below, they can arrive at a mixed-numeral answer.

$$\begin{array}{r} 6 \text{ R1} \\ 4\overline{)25} \end{array} \quad \rightarrow \frac{25}{4} = \frac{24}{4} + \frac{1}{4} = 6 + \frac{1}{4} = 6\frac{1}{4}$$

DISCUSSION

An important aim of this lesson is to develop an understanding of the fact that, if a and b are whole numbers and b is not zero, then a divided by b is equal to the fractional number represented by $\frac{a}{b}$. The symbol written for fractional numbers indicates a division problem for whole numbers. These ideas are developed in Exercise 1.

The remaining Discussion questions deal with the ideas of *complex fractions*. Up to this point in the text, only fractions with whole number numerators and denominators have been considered. Now, fractions with fractional numerators and denominators (called *complex fractions*) will be considered. Thus,

$$\frac{\frac{2}{3}}{\frac{3}{4}}$$

is a fraction and also an indicated division, $\frac{2}{3} \div \frac{3}{4}$.

Exercise 5 illustrates simplification of complex fractions by using division. This method relies on the agreement or definition of complex fractions, that is,

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \div \frac{c}{d}.$$

Exercise 6 suggests an alternate method of simplifying complex fractions. Using this method, one multiplies numerator and denominator by the reciprocal of the denominator in order to get a fraction with a denominator of 1. You may want to reserve this method for your more able students.

Complex Fractions

Discussing the Ideas

1. Give the missing numbers in the sentences below.

   A You can find the quotient for $4 \div 9$ by writing $4 \div 9 = \frac{4}{1} \div \frac{9}{1} = \frac{4}{1} \cdot \frac{1}{9} = \frac{\square}{\square}$.

   B The quotient $2 \div 5$ is $\frac{2}{5}$ because $5 \cdot \frac{\square}{\square} = 2$.

2. Let us agree that a quotient such as $\frac{2}{3} \div \frac{5}{8}$ can be represented by a **complex fraction** $\frac{\frac{2}{3}}{\frac{5}{8}}$.

   A What is the numerator of the complex fraction?

   B What is the denominator of the complex fraction?

3. Give each quotient as a complex fraction.

   A $\frac{1}{2} \div \frac{2}{3}$ **B** $\frac{3}{4} \div \frac{3}{8}$ **C** $\frac{5}{6} \div \frac{1}{10}$ **D** $\frac{5}{4} \div \frac{4}{3}$

4. Write each complex fraction as a quotient of two fractional numbers.

   A $\frac{\frac{2}{5}}{\frac{3}{4}}$ **B** $\frac{\frac{1}{2}}{\frac{5}{6}}$ **C** $\frac{\frac{9}{10}}{\frac{3}{5}}$ **D** $\frac{\frac{17}{32}}{\frac{1}{4}}$

5. You can simplify a complex fraction using division. Complete each equation.

   A $\frac{\frac{26}{5}}{\frac{13}{3}} = \frac{26}{5} \div \frac{13}{3} = \frac{26}{5} \cdot \frac{3}{13} = ?$ **B** $\frac{\frac{3}{5}}{\frac{2}{7}} = \frac{3}{5} \div \frac{2}{7} = \frac{3}{5} \cdot \frac{7}{2} = ?$

   C $\frac{\frac{15}{2}}{\frac{3}{4}} = \frac{15}{2} \div \frac{\square}{\square} = \frac{15}{2} \cdot \frac{\square}{\square} = ?$ **D** $\frac{\frac{5}{8}}{\frac{7}{18}} = \frac{\square}{\square} \div \frac{\square}{\square} = \frac{\square}{\square} \cdot \frac{\square}{\square} = ?$

6. The examples below illustrate another method of simplifying a complex fraction. Explain how this method works and complete the equations.

   A $\frac{\frac{26}{5}}{\frac{13}{3}} = \frac{\frac{26}{5} \cdot \frac{3}{13}}{\frac{13}{3} \cdot \frac{3}{13}} = \frac{\frac{6}{5}}{1} = ?$ **B** $\frac{\frac{3}{5}}{\frac{2}{7}} = \frac{\frac{3}{5} \cdot \frac{7}{2}}{\frac{2}{7} \cdot \frac{7}{2}} = \frac{\frac{21}{10}}{1} = ?$

Using the Ideas

1. Give each quotient as a fraction.

A $4 \div 5$	**D** $11 \div 3$	**G** $17 \div 6$	**J** $\frac{2}{3} \div \frac{1}{5}$
B $7 \div 16$	**E** $19 \div 25$	**H** $42 \div 9$	**K** $\frac{3}{7} \div \frac{7}{3}$
C $8 \div 7$	**F** $23 \div 4$	**I** $13 \div 26$	**L** $\frac{5}{9} \div \frac{10}{19}$

2. Write each quotient as a complex fraction.

A $\frac{5}{6} \div \frac{1}{2}$ **B** $\frac{7}{8} \div \frac{3}{4}$ **C** $2 \div \frac{3}{4}$ **D** $\frac{1}{2} \div 4$ **E** $\frac{17}{3} \div \frac{5}{7}$

3. Write each complex fraction as a quotient of two fractional numbers.

A $\dfrac{\frac{3}{5}}{\frac{1}{2}}$ **B** $\dfrac{\frac{1}{8}}{\frac{5}{2}}$ **C** $\dfrac{\frac{1}{3}}{\frac{3}{7}}$ **D** $\dfrac{\frac{3}{10}}{\frac{5}{4}}$ **E** $\dfrac{\frac{3}{4}}{\frac{3}{2}}$

4. We have agreed that the symbol $\dfrac{\frac{2}{3}}{\frac{5}{7}}$ means $\frac{2}{3} \div \frac{5}{7}$.

Use division to simplify each complex fraction.

A $\dfrac{\frac{1}{2}}{\frac{2}{8}}$	**D** $\dfrac{\frac{7}{6}}{\frac{7}{3}}$	**G** $\dfrac{4\frac{1}{2}}{9}$	**J** $\dfrac{37\frac{1}{2}}{100}$	**M** $\dfrac{2\frac{1}{2}}{10}$	**P** $\dfrac{4\frac{2}{5}}{2\frac{3}{4}}$
B $\dfrac{\frac{5}{9}}{\frac{1}{3}}$	**E** $\dfrac{\frac{3}{4}}{\frac{9}{4}}$	**H** $\dfrac{8\frac{1}{3}}{25}$	**K** $\dfrac{3\frac{1}{3}}{13\frac{1}{3}}$	**N** $\dfrac{7}{4\frac{2}{3}}$	**Q** $\dfrac{2\frac{7}{8}}{3\frac{5}{6}}$
C $\dfrac{\frac{9}{10}}{\frac{9}{100}}$	**F** $\dfrac{\frac{3}{4}}{\frac{3}{2}}$	**I** $\dfrac{\frac{5}{6}}{\frac{5}{18}}$	**L** $\dfrac{1\frac{1}{2}}{1\frac{1}{3}}$	**O** $\dfrac{3\frac{1}{5}}{1\frac{3}{5}}$	**R** $\dfrac{3\frac{1}{3}}{\frac{1}{3}}$

think

Check to see if each equation in Exercises 1 through 4 is correct. Find two more equations of the same kind for each exercise.

1. $1\frac{1}{2} \cdot 3 = 1\frac{1}{2} + 3$
$1\frac{1}{3} \cdot 4 = 1\frac{1}{3} + 4$
$1\frac{1}{4} \cdot 5 = 1\frac{1}{4} + 5$

2. $1 \cdot \frac{1}{2} = 1 - \frac{1}{2}$
$2 \cdot \frac{2}{3} = 2 - \frac{2}{3}$
$3 \cdot \frac{3}{4} = 3 - \frac{3}{4}$

3. $1\frac{1}{3} \div \frac{2}{3} = 1\frac{1}{3} + \frac{2}{3}$
$2\frac{1}{4} \div \frac{3}{4} = 2\frac{1}{4} + \frac{3}{4}$
$3\frac{1}{5} \div \frac{4}{5} = 3\frac{1}{5} + \frac{4}{5}$

4. $4\frac{1}{2} \div 3 = 4\frac{1}{2} - 3$
$5\frac{1}{3} \div 4 = 5\frac{1}{3} - 4$
$6\frac{1}{4} \div 5 = 6\frac{1}{4} - 5$

More practice, page S-17, Set 30

UTILIZATION

The exercises provide practice in finding fractional-number quotients for pairs of whole numbers and in simplification of complex fractions by means of division. Exercise 2 involves only writing the indicated quotient as a complex fraction. Exercise 3 is the reverse of Exercise 2; the student writes the complex fraction as the quotient of two fractional numbers. If more practice is desired, you may instruct students to find all the quotients as fractional numbers of the form $\frac{a}{b}$, where a and b are whole numbers.

EXTENSION

Practical applications involving complex fractions are somewhat limited, because in such applications decimals rather than fractions are ordinarily used. However, work with complex fractions involves development of some useful pre-algebra skills, so additional practice exercises are provided to help develop these skills. (Supplementary Exercise Set 30 and *Workbook* page 52).

Think Solution

By observing the number patterns involved, the students should be able to write further examples of each case.

1. $1\frac{1}{5} \cdot 6 = 1\frac{1}{5} + 6$
   $1\frac{1}{6} \cdot 7 = 1\frac{1}{6} + 7$
2. $4 \cdot \frac{4}{5} = 4 - \frac{4}{5}$
   $5 \cdot \frac{5}{6} = 5 - \frac{5}{6}$
3. $4\frac{1}{6} \div \frac{5}{6} = 4\frac{1}{6} + \frac{5}{6}$
   $5\frac{1}{7} \div \frac{6}{7} = 5\frac{1}{7} + \frac{6}{7}$
4. $7\frac{1}{5} \div 6 = 7\frac{1}{5} - 6$
   $8\frac{1}{6} \div 7 = 8\frac{1}{6} - 7$

Each pattern can be explained quite easily by means of algebra, but this would be beyond the ability of most students at this level.

Assignments

Minimum 1A–F, 3, 4A–F. Average 1–4. Maximum 1–4.

Objective

The student will be able to solve problems involving function rules with fractional numbers and the operations of multiplication and division.

PREPARATION

Review the use of the function machine by giving simple function rules and whole number inputs and outputs. Another useful warm up would be to play the guess-the-function-rule game. One student selects a simple function rule. A second student gives an input number and the first student gives the output number by applying his rule. After a few input and output numbers have been given, the second student must guess the function rule.

UTILIZATION

Have the students read the introductory material. You may find it necessary to review the meaning of the functional notation $f(n) = n \cdot \frac{2}{3}$. The symbol $f(n)$ simply refers to the output number for the function when the input number is n. The symbol $n \cdot \frac{2}{3}$ is the rule for computing the output number. Thus,

$$f(5) = 5 \cdot \tfrac{2}{3} = \tfrac{10}{3} = 3\tfrac{1}{3}.$$

There are no new mathematical ideas developed in the exercises. The functions provide a motivating way of strengthening multiplication and division skills using fractional numbers.

EXTENSION

Additional exercises designed to strengthen the skills developed in this lesson are provided on page 53 of the *Workbook*.

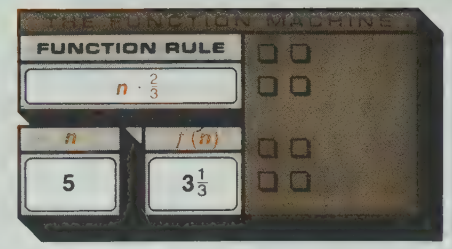
Assignments

Minimum 1A–F, 2, 3, 4. Average 1–5.
Maximum 1–7.

Fractional Number Functions

The function machine shows the function rule $f(n) = n \cdot \frac{2}{3}$. When the input number is 5, the output number is $5 \cdot \frac{2}{3}$ or $3\frac{1}{3}$.

We write, $f(5) = 3\frac{1}{3}$ and we say "f of 5 equals $3\frac{1}{3}$."



1. Use the function rule $f(n) = n \cdot \frac{2}{3}$ to complete each exercise.

A $f(2) =$ ▒	D $f(0) =$ ▒	G $f(3\frac{1}{2}) =$ ▒	J $f($▒$) = 2\frac{1}{5}$
B $f(3) =$ ▒	E $f(\frac{3}{2}) =$ ▒	H $f($▒$) = 1$	K $f(\frac{2}{3}) =$ ▒
C $f(1) =$ ▒	F $f(1\frac{1}{5}) =$ ▒	I $f($▒$) = \frac{3}{2}$	L $f($▒$) = 100$

In Exercises 2 through 7, complete the function tables.

2. Function Rule $n \cdot \frac{1}{4}$

	n	$f(n)$
A	4	▒
B	$\frac{2}{5}$	▒
C	1	▒
D	$\frac{5}{4}$	▒
E	$1\frac{2}{3}$	▒

3. Function Rule $\frac{2}{5} \cdot n$

	n	$f(n)$
A	$\frac{1}{5}$	▒
B	6	▒
C	▒	$3\frac{1}{2}$
D	▒	1
E	$2\frac{3}{5}$	▒

4. Function Rule $n \cdot 1\frac{2}{3}$

	n	$f(n)$
A	0	▒
B	$\frac{2}{7}$	▒
C	▒	$2\frac{2}{5}$
D	▒	2
E	$3\frac{3}{4}$	▒

5. Function Rule $\left(\frac{3}{2} \cdot n\right) + \frac{1}{2}$

	n	$f(n)$
A	4	▒
B	$\frac{7}{5}$	▒
C	▒	$\frac{1}{2}$
D	▒	$\frac{3}{2}$
E	$4\frac{1}{7}$	▒

★6. Function Rule

	n	$f(n)$
A $n \cdot$ ▒	$\frac{1}{3}$	$\frac{1}{6}$
	$\frac{2}{5}$	$\frac{1}{5}$
	4	2
	$\frac{8}{3}$	$\frac{4}{3}$
B	$2\frac{3}{5}$	▒

★7. Function Rule $n \div \frac{1}{2}$

	n	$f(n)$
	$\frac{2}{3}$	$\frac{4}{3}$
A	$\frac{3}{5}$	▒
B	3	▒
C	$3\frac{2}{7}$	▒
D	$4\frac{1}{5}$	▒

Problem Solving-Division

Write a division equation for each exercise and then solve the equation. For some exercises, it may be helpful to write a multiplication equation first. Study the example before proceeding to the exercises.

EXAMPLE: $\frac{2}{5}$ of Janet's age is 6. What is Janet's age?

SOLUTION: Let x represent Janet's age.

Then $\frac{2}{5} \cdot x = 6$.

$x = 6 \div \frac{2}{5} = 6 \cdot \frac{5}{2} = 15$

Janet is 15 years old.

1. $\frac{3}{4}$ of Jim's age is 12. How old is Jim?

2. $\frac{2}{3}$ of a number is $\frac{16}{3}$. What is the number?

3. $\frac{2}{3}$ of a number is $\frac{3}{4}$. What is the number?

4. Harry's height is $\frac{4}{5}$ of Paul's height. Harry is 155 cm tall. How tall is Paul?

5. A jet flying 903 km/h is flying at about $\frac{6}{7}$ the speed of sound at sea level. What is the speed of sound at sea level?

6. Joe walked $12\frac{7}{10}$ km in $2\frac{1}{2}$ hours. What was his average walking speed in km/h?

7. A car travelled 140 km in $1\frac{3}{4}$ hours. What was the average speed of the car?

8. John can walk about $5\frac{1}{2}$ km/h. About how long will it take him to walk 22 km?

9. A centimetre is only $\frac{2}{5}$ as long as segment AB. How long is the segment?

10. The product of two numbers is $\frac{8}{3}$. If one of the numbers is $\frac{1}{3}$, what is the other number?

11. The quotient of two numbers is $\frac{8}{3}$. The larger number is $\frac{1}{3}$. Find the other number.

12. The product of two numbers is $\frac{2}{3}$ One of the numbers is $\frac{5}{4}$. Find the other number.

13. If the quotient of two numbers is $\frac{2}{3}$ and the smaller number is $\frac{5}{4}$, what is the other number?

Objective

Given word problems with fractional numbers, the student can write and solve an equation for the problems.

UTILIZATION

Discuss the example at the top of the page with the class. Students should be encouraged to use a style of solution somewhat like the example, although you should feel free to alter the format to your own choice. In solving an equation such as the one in the example, we prefer that the student think of inverse operations rather than the more formal algebraic technique of multiplying each side of the equation by the reciprocal of the coefficient of the variable. However, you should again tailor the approach to your own preferences and the ability of your students.

Exercises 11, 12, and 13 are most suitable for your more able students.

Solution, Exercise 11

Let x = the smaller number.

Equation:

$\frac{1}{3} \div x = \frac{8}{3} \rightarrow \frac{8}{3} \cdot x = \frac{1}{3} \rightarrow x = \frac{1}{3} \div \frac{8}{3} = \frac{1}{8}$

(Observe that the larger number must be divided by the smaller number in order for the quotient to be greater than 1.)

EXTENSION

Page 33 of the *Duplicator Masters* provides additional problems appropriate for use in conjunction with this lesson.

Assignments

Minimum 1–4.

Average Even-numbered Exercises 2–10.

Maximum Odd-numbered Exercises 1–13.

Objective

The student can solve word problems involving fractional numbers.

UTILIZATION

These thematic exercises can be assigned in conjunction with the exercises of the previous page if you so desire.

EXTENSION

Enrichment: This lesson could be utilized with a social studies unit involving a study of the Northwest Territories. Many other geographical and mathematical facts about the Northwest Territories could be researched by students. Some practice in constructing maps to scale could be involved in such a unit.

Assignments

Minimum 1–4. Average 1–6. Maximum 1–7.

THE NORTHWEST TERRITORIES

Area	3 380 620 km²
Population	34 805
Capital (named Jan. 1967)	Yellowknife
Largest Lake	Great Bear Lake (31 790 km²)
Longest River	Mackenzie River (4240 km)

The Northwest Territories extend east from the Yukon to Hudson Bay including the islands of the Bay and north from the 60th parallel of latitude excluding that portion of the mainland of Quebec. Originally including much of the prairie provinces and the northern sections of Ontario and Quebec, the Northwest Territories' present boundaries were established in 1912.

1. The area of the Northwest Territories is about $\frac{1}{3}$ the area of all Canada. Use this fact to estimate the area of Canada. Round your answer to the nearest 100 000 km².

2. The population of the Northwest Territories was about $\frac{1}{4}$ of the population of the Prince Edward Island in 1971. About how many people lived in Prince Edward Island in 1971 to the nearest 1 000?

3. The Great Bear Lake is 12 660 km² larger than Lake Ontario. How large is Lake Ontario?

4. The area of the Northwest Territories is about $6\frac{3}{10}$ times the area of the Yukon. Estimate the area of the Yukon, rounding to the nearest 1 000 km².

5. The 1971 population of the Northwest Territories was $\frac{15}{8}$ as large as the 1946 population. About how many people lived in the Northwest Territories in 1946?

6. The area of the Northwest Territories is 29 820 km² more than twice the area of Quebec. What is the area of Quebec?

7. The longest river in the Northwest Territories is approximately two thirds the length of the Mississippi River. About how long is the Mississippi?

THE PROVINCES

Newfoundland	
Land Area	370 582 km²
Population "73"	537 000

Prince Edward Island	
Land Area	5 658 km²
Population "73"	114 000

Nova Scotia	
Land Area	52 854 km²
Population "73"	802 000

New Brunswick	
Land Area	72 111 km²
Population "73"	648 000

Quebec	
Land Area	1 357 150 km²
Population "73"	6 068 000

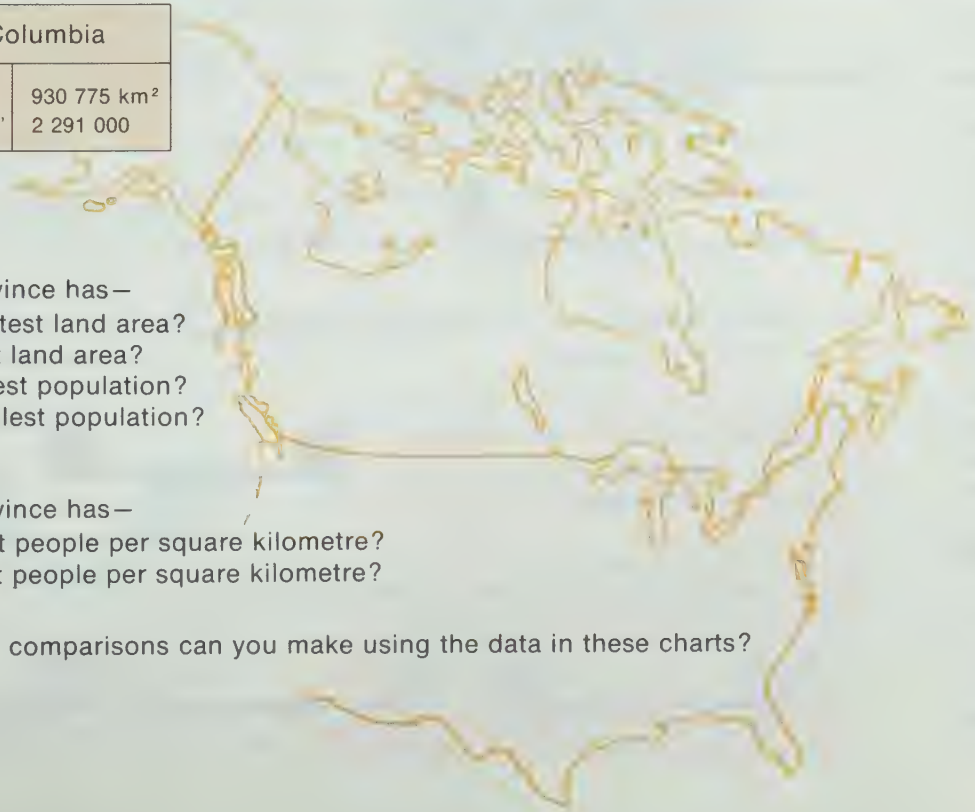
Ontario	
Land Area	891 430 km²
Population "73"	7 893 000

Manitoba	
Land Area	548 640 km²
Population "73"	993 000

Saskatchewan	
Land Area	570 420 km²
Population "73"	910 000

Alberta	
Land Area	644 560 km²
Population "73"	1 671 000

British Columbia	
Land Area	930 775 km²
Population "73"	2 291 000



1. Which province has—
   - A the greatest land area?
   - B the least land area?
   - C the largest population?
   - D the smallest population?

2. Which province has—
   - A the most people per square kilometre?
   - B the least people per square kilometre?

3. What other comparisons can you make using the data in these charts?

Objective

The student can use his knowledge of fractions to simplify comparative statistics.

UTILIZATION

These exercises should be assigned along with the exercises on the previous page.

Assignments

Minimum 1–2. Maximum 1–3.

Objective

The student can solve simple balance problems.

PREPARATION

Materials: A mathematical balance, or home-made equipment to demonstrate balance problems.

If a mathematical balance is available, you can use it to good advantage in this lesson. The usual balance has pegs at unit intervals from the fulcrum, from which unit weights can be hung. Thus 2 masses hung 6 units from the fulcrum will balance with 3 masses hung 4 units on the other side of the fulcrum because $2 \cdot 6 = 3 \cdot 4$. Each of these products is called the *torque* about the fulcrum.

INVESTIGATION

It would be helpful to use the mathematical balance to demonstrate the kind of balance problems given in the Investigation.

Have the students study example A before trying to find the numbers for x in diagrams B and C. Do not expect every student to complete the Investigation. However, after allowing sufficient time, have one or more students explain their method of solving the problems for diagrams B and C.

DISCUSSION

The important idea of this lesson is illustrated by the following diagrams, where m_1 and m_2 are masses and d_1 and d_2 are the distance of the masses from the fulcrum.

If the system is to balance, that is, to be in *equilibrium*, it must be true that

$$m_1 d_1 = m_2 d_2.$$

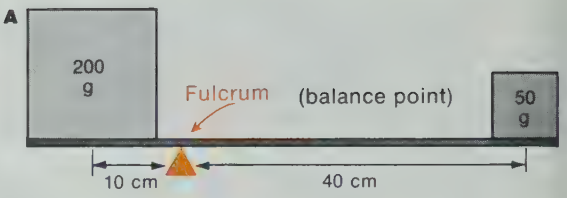
In the solution of the equation for Exercise 1, the remaining steps are

$$12 \cdot x = 420$$
$$x = 420 \div 12$$
$$x = 35 \text{ cm}.$$

Balance Problems

Investigating the Ideas

In diagram A, the 200-gram mass balances with the 50-gram mass because the product of one mass and its distance from the fulcrum is equal to the product of the other mass and its distance from the fulcrum.



mass · distance = mass · distance

$200 \cdot 10 = 50 \cdot 40$

Can you find the missing numbers for diagrams B and C if the masses balance?

Discussing the Ideas

1. In order for the masses of diagram B to balance,

   the equation $12 \cdot x = 28 \cdot 15$ must hold.

   How can you find the number for **x** in the equation?

2. **A** What equation can you write for diagram C?
   B What is the mass **x**?

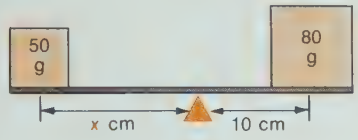
3. Which object in the diagram at the right is heavier? How can you tell?

4. In which of these diagrams will the masses balance?

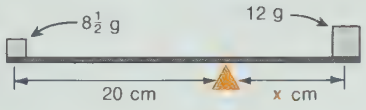
Using the Ideas

1. Write an equation for the diagram if the masses balance. Then solve the equation for the distance **x**.

A

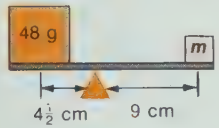


B

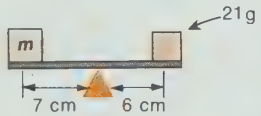


2. The masses shown in each diagram balance. Write an equation for each diagram and then solve it to find the mass ***m***.

A

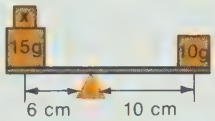


B

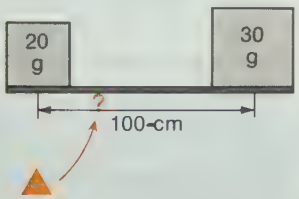


3. Tom weighs 56 kilograms and sits 160 centimetres from the balance point (fulcrum) of a seesaw. Lora weighs 40 kilograms and is able to balance with Tom. How far is Lora from the balance point?

★ **4.** What is the mass **x** if the 10-gram mass balances the others?



★ **5.** Where must the fulcrum be placed so that the masses will balance?



think

Fill in the empty squares so that they will be **magic squares**.

A

$\frac{1}{2}$		$\frac{1}{6}$
$\frac{1}{12}$	$\frac{5}{12}$	
$\frac{2}{3}$		$\frac{1}{3}$

B

$6\frac{1}{4}$		$11\frac{1}{4}$
	$7\frac{1}{2}$	
		$8\frac{3}{4}$

UTILIZATION

Exercises 1, 2, and 3 employ the basic ideas discussed on the previous page. Exercises 4 and 5 are more difficult and should be assigned only to capable students.

Solution, Exercise 4

Let x = unknown mass in grams.

$$6 \cdot (15 + x) = 10 \cdot 10$$
$$90 + 6 \cdot x = 100$$
$$6 \cdot x = 10$$
$$x = \tfrac{10}{6} = 1\tfrac{2}{3}$$

Solution, Exercise 5

Let x = distance from fulcrum (cm) for the 20 g mass.

$100 - x$ = distance from the fulcrum for the 30 g mass.

$$20 \cdot x = 30 \cdot (100 - x)$$
$$20 \cdot x = 3000 - 30 \cdot x$$
$$30 \cdot x + 20 \cdot x = 3000$$
$$(30 + 20) \cdot x = 3000$$
$$50 \cdot x = 3000$$
$$x = 60 \text{ cm (distance for 20 grams mass)}$$
$$100 - x = 40 \text{ cm (distance for 30 grams mass)}$$

EXTENSION

This would be an opportune time to encourage some of your students to undertake Research Project B on page B-63.

Think Solution

In both diagrams, the necessary sum can be found by adding the numbers in the given diagonals. In part B, where there are two addends missing in some rows and columns, the student should first find the missing addends for the rows and columns in which only one addend is missing and then find the others.

This Think problem offers an interesting means of reviewing addition and subtraction of fractional numbers, and all students should be encouraged to try it.

Assignments

Minimum 1–3. Average 1–3. Maximum 1–5.

Objective

The student can demonstrate the ability to work with the concepts presented in this module.

UTILIZATION

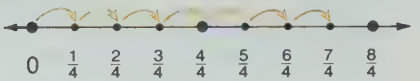
Observe that Exercises 2 through 9 have multiple parts. It is not necessary that all parts of these exercises be assigned, unless extra practice is needed by some students. If a student can correctly solve one or two parts, this should demonstrate his understanding of that particular kind of problem.

EXTENSION

For additional review exercises, refer to *Workbook* page 54.

REVIEWING THE IDEAS

1. Give the multiplication fact suggested by the number line.



2. Solve the equations.

A $\frac{1}{4} \cdot \frac{1}{3} = n$ **C** $5 \cdot \frac{1}{6} = n$ **E**

B $\frac{1}{5} \cdot \frac{1}{2} = n$ **D** $\frac{1}{7} \cdot 3 = n$

3. Find the part of each number.

A $\frac{1}{4}$ of 16 **D** $\frac{1}{3}$ of 27

B $\frac{1}{8}$ of 24 **E** $\frac{1}{9}$ of 63

C $\frac{1}{6}$ of 54 **F** $\frac{1}{12}$ of 84

4. Solve the equations.

A $\frac{1}{2} \cdot \frac{3}{8} = n \cdot \frac{1}{2}$

B $\frac{5}{6} \cdot y = \frac{5}{6}$

C $\left(\frac{7}{8} \cdot x\right) \cdot \frac{3}{4} = \frac{7}{8} \cdot \left(\frac{5}{3} \cdot \frac{3}{4}\right)$

D $\frac{3}{4} \cdot w = 1$

E $\frac{1}{4} \cdot m = \frac{1}{20}$

F $12 \cdot \frac{1}{2} \cdot \frac{1}{5} = 12 \cdot t$

5. Find the products.

A $\frac{2}{3} \cdot \frac{5}{8}$ **D** $\frac{7}{8} \cdot \frac{4}{7}$

B $\frac{1}{4} \cdot \frac{3}{7}$ **E** $\frac{5}{9} \cdot \frac{3}{10}$

C $\frac{3}{10} \cdot \frac{1}{3}$ **F** $\frac{5}{6} \cdot \frac{12}{25}$

6. Find the products.

A $1\frac{1}{2} \cdot 6$ **D** $\frac{13}{5} \cdot 1\frac{2}{3}$

B $8 \cdot 5\frac{1}{4}$ **E** $2\frac{1}{3} \cdot 3\frac{1}{2}$

C $3 \cdot 2\frac{2}{3}$ **F** $2\frac{2}{5} \cdot 4\frac{1}{6}$

7. Give the reciprocal of each number.

A $\frac{11}{13}$ **C** 5 **E** $3\frac{1}{2}$

B $\frac{17}{3}$ **D** $\frac{1}{6}$ **F** $4\frac{3}{7}$

8. Find the quotients.

A $\frac{3}{5} \div \frac{3}{8}$ **D** $2\frac{3}{4} \div \frac{4}{5}$

B $\frac{5}{7} \div \frac{15}{14}$ **E** $5\frac{3}{8} \div 4$

C $\frac{27}{17} \div \frac{12}{7}$ **F** $4\frac{2}{3} \div 1\frac{3}{4}$

9. Simplify each complex fraction.

A $\frac{\frac{3}{4}}{\frac{3}{8}}$ **B** $\frac{\frac{10}{11}}{\frac{5}{22}}$ **C** $\frac{1\frac{1}{4}}{3\frac{1}{2}}$

10. A special kind of colored glass used in making stained glass windows costs 10¢ per gram or fraction of a gram. What does a piece weighing $2\frac{3}{4}$ grams cost?

11. A dealer paid $2400 for a car. He added on $\frac{1}{5}$ of this cost in order to determine his selling price. What was the selling price of the car?

12. A car was driven 90 km/h for 1 hour 45 minutes. How far did it go?

13. A man spent $\frac{3}{5}$ of his money for a watch. If the watch cost $60, how much money did he have originally?

14. What is the distance **x** if the masses balance.

TEST YOURSELF

Solve the equations.

A $\frac{1}{2} \cdot \frac{1}{6} = n$ B $5 \cdot \frac{1}{4} = n$

How much is

A $\frac{1}{3}$ of 63? B $\frac{1}{6}$ of 102?

Give the number for n.

A $\frac{3}{4} \cdot \frac{6}{7} = n \cdot \frac{3}{4}$

B $\left(\frac{2}{3} \cdot \frac{5}{8}\right) \cdot \frac{7}{4} = \frac{2}{3} \cdot \left(n \cdot \frac{7}{4}\right)$

Find the products.

A $\frac{5}{6} \cdot \frac{3}{8}$ B $\frac{2}{3} \cdot \frac{4}{5}$

Find the products.

A $1\frac{1}{2} \cdot 2\frac{2}{3}$ B $12\frac{2}{3} \times 3\frac{1}{2}$

6. What is the reciprocal of $\frac{9}{10}$?

7. What is the reciprocal of $5\frac{3}{4}$?

8. Find the quotients.

A $\frac{1}{2} \div \frac{3}{4}$ C $2\frac{1}{4} \div 1\frac{7}{8}$

B $\frac{2}{5} \div \frac{4}{15}$ D $1 \div 3\frac{2}{5}$

9. Simplify the complex fractions.

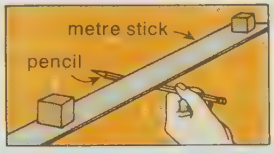
A $\frac{\frac{2}{3}}{\frac{5}{4}}$ B $\frac{\frac{3}{4}}{1\frac{1}{3}}$

10. A car averaged 88 kilometres per hour for $2\frac{3}{4}$ hours. How far did it travel in that time?

RESEARCH PROJECTS

Choose a stock from the listings in the financial section of a daily newspaper. Keep a record of the closing price of the stock each day for a period of two weeks. Then make a line-segment graph to show the "ups and downs" of your stock. Pretend that you bought 1000 shares of the stock at the beginning of the two-week period and sold them at the end of the period. Find out how much money you would have gained or lost on the stock.

B Make a working model for a few equilibrium problems similar to those on pages B–60 and B–61 of this module. You might use a metre stick to hold objects to be balanced on a fulcrum. A pencil might be used for a fulcrum.



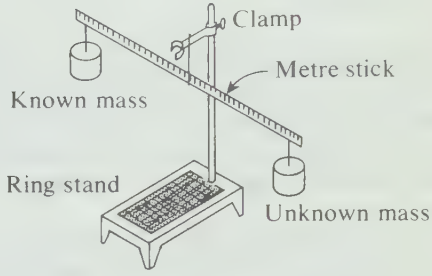
Compare known masses with unknown masses using the model. Do not expect your results to be very accurate using this simple equipment. Your science teacher may have more sophisticated equipment for you to use.

TEST YOURSELF

The ten self-evaluation problems should enable students to determine their level of proficiency with the ideas of this module and prepare them for an achievement test covering the same material of this module.

RESEARCH PROJECTS

A more accurate model for Research Project B can be made by obtaining a ringstand and clamp and suspending a metre stick from the clamp by means of a string as shown in the diagram below.



Hang the known mass from one side of the metrestick with string at some fixed distance from the fulcrum. Slide the unknown mass along the metre stick until the system is in equilibrium. Reading the distance from the fulcrum for each mass will provide the information needed for writing and solving an equation to find the unknown mass.

MATHEMATICAL RECREATION

Encourage as many students as possible to make congruent copies of the Reuleaux triangles. If feasible, attempt to have some wooden models made.

Among the interesting properties of the Reuleaux triangle is that all of the squares circumscribing a given Reuleaux triangle will be congruent to each other.

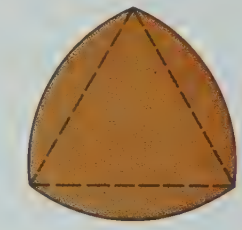
This means that the Reuleaux triangle can turn freely inside the square without having any room to spare. This can be shown by cutting a square hole from cardboard with each edge the length and the breadth of the Reuleaux triangle.

For additional discussion of Reuleaux triangles and other curves of constant width, refer to *The Enjoyment of Mathematics* by H. Rademacher and O. Toeplitz; Princeton, New Jersey: Princeton University Press, 1957, pp. 163–177.

MATHEMATICAL RECREATIONS

When a flat strip is placed on two circular rollers with the same diameter and pushed, the strip will move along the rollers and remain parallel to the surface on which the rollers move.

It is surprising that there are other rollers which are not circular which have the same property. These noncircular rollers are sometimes called *curves of constant width*. One of the simplest of these curves is called a **Reuleaux Triangle**. The Reuleaux Triangle is constructed from an equilateral triangle. The diagram at the right suggests how to make an Reuleaux Triangle. The center of each arc is the vertex of the triangle opposite that arc.



If you wish to make the curve less pointed and easier to roll, you may extend the sides of the triangle as shown at the right. The small arc at each vertex has its center at the nearer vertex of the triangle.

Make two Reuleaux Triangles that are the same size from cardboard. See if your curves are good noncircular rollers. You can make more permanent models by cutting them out of wood.

TEST YOURSELF Answers

1. A $\frac{1}{12}$; B $\frac{5}{4}$ or $1\frac{1}{4}$ **2.** A 21; B 17 **3.** A $\frac{6}{7}$; B $\frac{5}{8}$
4. A $\frac{5}{16}$; B $\frac{8}{15}$ **5.** A 4; B $44\frac{1}{3}$ **6.** $\frac{10}{9}$ or $1\frac{1}{9}$ **7.** $\frac{4}{23}$
8. A $\frac{2}{3}$; B $\frac{3}{2}$ or $1\frac{1}{2}$; C $\frac{6}{5}$ or $1\frac{1}{5}$; D $\frac{5}{17}$ **9.** A $\frac{8}{15}$; B $\frac{9}{16}$ **10.** 242 km

UNIT B: THE ARITHMETIC OF FRACTIONAL NUMBERS

Module 4: Decimals

General Objectives

To show the relation between decimal notation and fractional notation.
To extend the ideas of place value to decimal notation.
To strengthen the student's ability to add, subtract, multiply, and divide using decimals.
To provide experience in estimating and rounding decimals.
To enable the student to write decimals for fractions.
To extend work with scientific notation.
To provide additional work with metric units using decimals.
To provide a variety of word problems using decimal notation.

	Performance Objectives	Pupil Text	Reteach-Reinforce	Related Activities
58	**RED** The student can give the place value of each digit in a decimal.	B-66, 67	**WB** 55	**SP** h-1 **SWM 1** 270, 271
59	**RED** The student can express fractions (whose denominators are multiples of 2, 5, or 10) in decimal notation or write a fraction for a given decimal.	B-66, 67	**DM** 34	**SP** h-1 **SWM 1** 269, 271
60	**RED** Given two decimals, the student can use place value to decide which one represents the larger number.	B-68, 69	**WB** 56	
61	**RED** Given an addition problem using decimals, the student can find the sum.	B-70, 71	**WB** 57 **DM** 35	**ASC** D-1 **PD** h-1 **SP** h-2 **DS** g-2
62	**RED** Given a subtraction problem using decimals, the student can find the difference.	B-70, 71	**WB** 57 **DM** 35	**ASC** D-2 **PD** h-1 **SP** h-2 **DS** g-2
63	**RED** Given a metric measure, the student can express the measure in larger or smaller metric units.	B-72, 73	**WB** 58 **DM** 36	**PD** b-1
64	**RED** Given a multiplication problem using decimals, the student can find the product.	B-74, 75 B-76, 77	**WB** 59, 60 **DM** 37	**ASC** D-3 **PD** h-1 **SP** h-3, 4 **DS** g-2, 3
65	**RED** The student can use rounding to estimate answers to problems involving decimals.	B-78, 79 B-80, 81	**WB** 61 **DM** 39	
66	**RED** Given a division problem using decimals, the student can find the quotient.	B-82, 83	**WB** 62 **DM** 40	**ASC** D-4, 5, 6 **SWM 1** 300 **SP** h-3, 4 **PD** h-1 **DS** g-3
67	**YELLOW** Given a fraction, the student can find a terminating or a repeating decimal for it.	B-86, 87 B-88, 89	**WB** 63 **DM** 42, 43	**ASC** D-7, 8 **DS** g-2
68	**YELLOW** Given word problems involving decimals, the student can solve them.	B-75, 80, 81 84, 85, 93	**DM** 41	
69	***GREEN*** *The student can write a mixed decimal numeral for a given fraction or a fraction for a mixed decimal numeral.*	B-90, 91		**ASC** D-10
70	***GREEN*** *Given a decimal greater than 1, the student can express the number in scientific notation, and vice versa.*	B-92	**DM** 44	**ASC** AP-3, 4 **SWM 3** 196, 197 **SP** a-3
	Reviewing the Ideas	B-94	**WB** 64	

MATHEMATICS

In any discussion of decimals there are two concepts to be considered:

(1) The notation (the *decimal symbols* that we write).

(2) The set of *numbers* represented by the decimal symbols.

When we use *decimal notation,* we are merely choosing an efficient way of writing symbols for *fractional numbers.* Decimal notation can be thought of as a way of simplifying the use of fractions. Fractions whose denominators are any whole number except zero may be written in fractional notation, whereas only fractions whose denominators are 1, 10, 100, 1000, . . . may be written in decimal notation. Thus, $\frac{1}{10} = 0.1$, $\frac{1}{100} = 0.01$, $\frac{1}{1000} = 0.001$, and so on.

The following equation illustrates a specific example of the general definition of decimal notation:

$$2.618 = 2 + \tfrac{6}{10} + \tfrac{1}{100} + \tfrac{8}{1000}$$

In order to read 2.618 as *"two and six hundred eighteen thousandths,"* we must show that $2 + \frac{6}{10} + \frac{1}{100} + \frac{8}{1000} = 2\frac{618}{1000}$, which we can easily do by means of addition of fractional numbers.

Once we understand decimal notation, we can derive rules for computing with these symbols *because we understand the fractional number system.* To develop skill with decimal notation it is necessary to have

(1) an understanding of computation with fractional numbers, using fractional notation, and

(2) an understanding of the relationship between decimal notation and fractional notation.

Another important concept that is introduced in this module is that every fractional number has a repeating decimal representation. Furthermore, every repeating decimal can be represented by a fractional number. This idea assumes added significance in *School Mathematics 2* when the irrational numbers are introduced as numbers which have *nonrepeating decimal representations.*

TEACHING THE MODULE

Materials

Graph paper, posterboard, metre sticks, centimetre rulers, metric units of mass, standard litre measure, scientific balance scale.

Vocabulary

decimal
decimal point
equivalent decimals
estimation
gram
litre
metre
metric system
mixed decimal numerals
power
repeating decimal
scientific notation
terminating decimal

In teaching this module two important objectives should be kept in mind:

(1) Students should clearly understand decimal notation and the relation between fractions and decimals.

(2) Students should be able to compute efficiently with decimals.

Students who have good computational skills with whole numbers ordinarily will do well in computing with decimals. Students who have good skills with decimals should be encouraged to try some of the special Research Projects or Extension activities suggested in the module. On the other hand, there is adequate review and practice material for those students who need improvement in their computational skill.

Lesson Schedule

Some parts of this module will be review, while other parts will contain almost completely new material. Therefore, adjust your time schedule to the background of your students. If your class is strong in the computational aspects of the work, spend more time on such topics as repeating decimals, the metric system, and scientific notation. About 16 days should be sufficient for adequate coverage of this module. However, if you have a particularly slow class that has had little prior success in working with decimals, more than 16 days may be needed.

Evaluation

Although most students have had previous experience with decimals, it is not uncommon for some students to lack understanding of decimal notation or to be weak in computational skills with decimals. By keeping close contact with the students on a daily basis, you should be able to determine whether they are mastering the various computational algorithms for decimals and diagnose their difficulties.

Conferences with students who do not do well on the self-evaluation test may help you determine which objectives of the chapter were not reached by the student and what remedial steps may need to be taken. If you wish to include an achievement test in your evaluation, such tests are a part of the testing package that is available from the publisher, or you might create a test, using the self-test as a model.

RESOURCES FOR ACTIVE LEARNING

Activities

Chip Trading Activities, Set V, 19–22, Scott Resources

Developmental Math Cards, L13, Addison-Wesley

Fractional Numbers, Geocard 27, Creative Publications

Let's Play Games in Mathematics, Vol. 7, "Concentration . . ," pp. 26–27; "Decimal In Between," p. 79, National Textbook Co.

Nuffield Project: *Computation and Structure 5* (decimal numbers), pp. 1–6, 45–57, Wiley

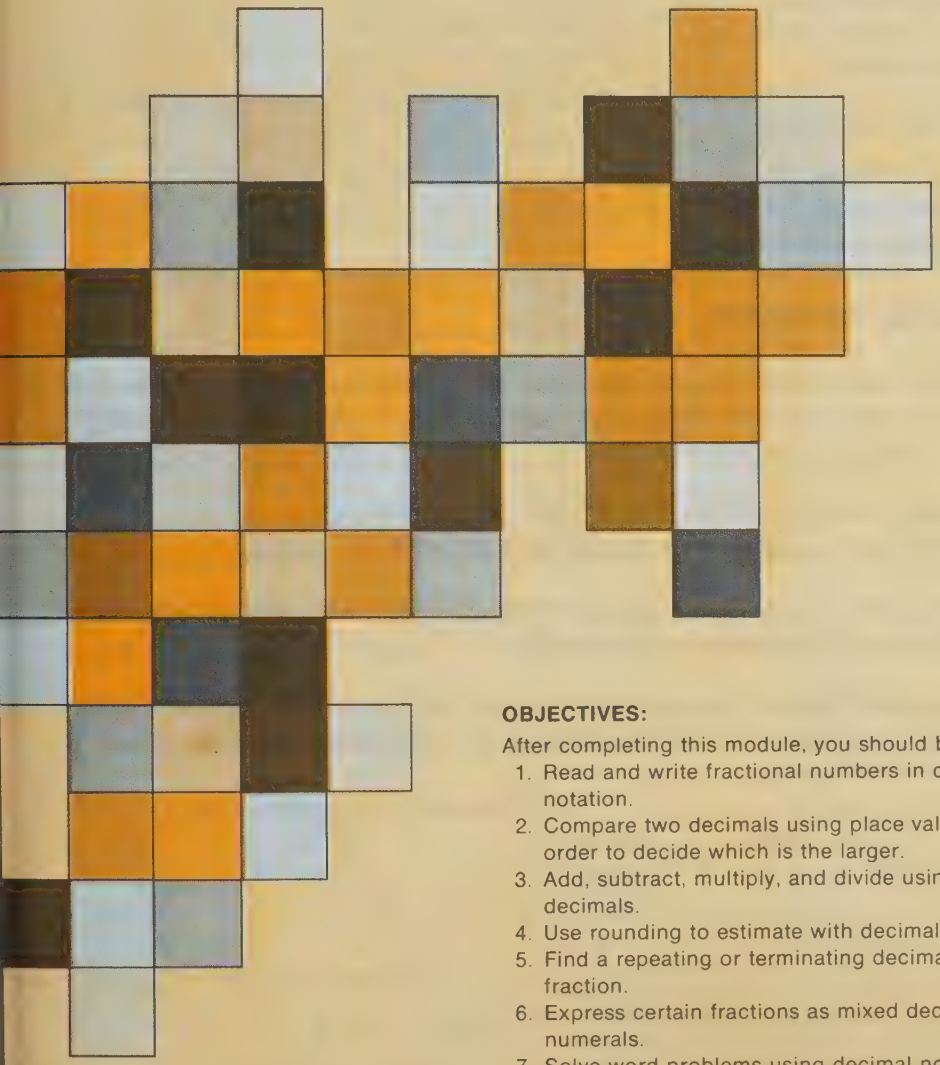
Manipulative Devices

Base Ten Blocks (Addison-Wesley)
Chip Trading materials (Scott Resources)
Geoboards (Addison-Wesley)
Metric aids (Addison-Wesley; Balla; Creative Publications; Dick Blick; Educational Teaching Aids)

Games and Puzzles

Decimal Fraction Dominoes (Mind/Matter)
In-order Game Kit (Midwest Publications)

MODULE 4: *Decimals*



OBJECTIVES:

After completing this module, you should be able to:

1. Read and write fractional numbers in decimal notation.
2. Compare two decimals using place value in order to decide which is the larger.
3. Add, subtract, multiply, and divide using decimals.
4. Use rounding to estimate with decimals.
5. Find a repeating or terminating decimal for a fraction.
6. Express certain fractions as mixed decimal numerals.
7. Solve word problems using decimal notation.

Objectives

Given a decimal, the student can give (a) the name of each place of the decimal and (b) a fraction for the decimal.

Given a fraction with a denominator which is a power of ten, the student can write a decimal for the fraction.

PREPARATION

Materials: 1-centimetre graph paper (*Duplicator Masters*, page 83).

In order to make the most effective use of the Investigation, it would be best to begin work on it immediately. Since completion of the Investigation involves only an understanding of fractions, any prior discussion of decimals might detract from the Investigation.

INVESTIGATION

In order to shade $\frac{1}{10}$, $\frac{1}{100}$, and $\frac{1}{1000}$ of the rectangular region, the students must determine how many small squares are in the region. As soon as they discover that there are 1000 small squares, they can readily complete the Investigation.

DISCUSSION

It is very important for students to realize that most decimals represent fractional numbers and that the only difference between the kind of decimals used in this module and fractions is the notation used. When this is fully understood, the students will find it much easier to understand and work with operations involving decimal notation.

In discussing Exercise 2, refer back to the Investigation so that the student can get a concrete interpretation of 0.1, 0.01, and 0.001 of the rectangular region.

If you wish, you can mention other place names for decimals (such as hundredths, thousandths, and millionths) when discussing Exercise 4. Also, refer again to the Investigation and point out that $\frac{1}{10} + \frac{1}{100} + \frac{1}{1000}$ of the region has been shaded, and this sum expressed as a decimal is 0.111.

What are Decimals?

Investigating the Ideas

Use a sheet of graph paper to draw a rectangular region that is 40 squares long and 25 squares wide

Can you shade or color $\frac{1}{10}$, $\frac{1}{100}$, and $\frac{1}{1000}$ of the rectangular region?

Discussing the Ideas

1. A How many small squares are in the entire rectangular region above?
   B How many squares must you shade or color in order to show $\frac{1}{10}$ of the entire rectangular region?

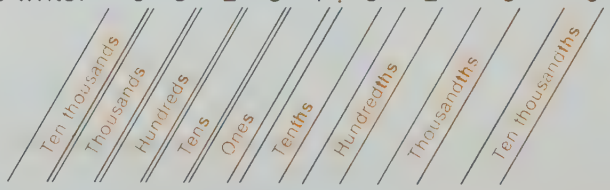
2. Instead of writing the fraction $\frac{1}{10}$, you can write the **decimal** 0.1. For $\frac{1}{100}$, you can write 0.01. What decimal could you write for $\frac{1}{1000}$?

3. A How many of the small squares represent $\frac{237}{1000}$ of the entire region?
   B What decimal can you write for $\frac{237}{1000}$?

4. The diagram below shows how we extend our place-value system to give meaning to decimals. The dot between the ones' and the tenths' place is a **decimal point**.

   For the sum $6\ 3\ 2\ 8\ 4 + \frac{8}{10} + \frac{2}{100} + \frac{3}{1000} + \frac{5}{10\,000}$

   we write: 6 3 2 8 4 . 8 2 3 5



   Give the missing fractions.

   A The 8 in the tenths' place means ? .
   B The 5 in the ten thousandths' place means ? .
   C The 2 in the hundredths' place means ? .

Using the Ideas

1. Copy the sentence. Give the missing word and number.

EXAMPLE: 43.38. The 8 in the hundredths' place represents $8 \cdot \frac{1}{100}$

A 476.835: The 4 in the ? place represents ▒.

B 476.835: The 3 in the ? place represents ▒.

C 9476.835: The 9 in the ? place represents ▒.

D 9476.835: The 5 in the ? place represents ▒.

E 0.4568: The 8 in the ? place represents ▒.

F 5.87654: The 4 in the ? place represents ▒.

G 0.123457 The 7 in the ? place represents ▒.

2. Write a decimal for each fraction.

A $\frac{9}{10}$ B $\frac{17}{100}$ C $\frac{247}{1000}$ D $\frac{3}{100}$ E $\frac{18}{1000}$ F $\frac{1637}{10\,000}$

3. Write each number as shown in the example.

$3.759 = \frac{7}{10} + \frac{5}{100} + \frac{9}{1000} = \frac{700}{1000} + \frac{50}{1000} + \frac{9}{1000} = 3\frac{759}{1000}$

We read 3.759 as "three **and** seven hundred fifty-nine thousandths" or simply "three **point** seven five nine."

A 7.52 B 0.364 C 9.452 D 0.2763 E 27.584 F 8.073

4. Write a decimal for each number represented.

A $8 + \frac{4}{10}$ D $8 + \frac{472}{1000}$ G $614 + \frac{3}{100}$ J $0 + \frac{37\,019}{100000}$

B $21 + \frac{23}{100}$ E $15 + \frac{407}{1000}$ H $50\frac{47}{1000}$ K $4 + \frac{612}{1000000}$

C $0 + \frac{69}{100}$ F $29 + \frac{2152}{10000}$ I $50\frac{471}{10000}$ L $25\frac{5607}{1000000}$

5. Write a mixed numeral for each number.

A 6.4	D 1.5	G 919.7154	J 286.11111
B 17.31	E 2.003	H 40.0352	K 27.5680
C 5.03	F 81.426	I 2.1746	L 3.000123

★ **6.** Give the missing exponent for each ▒.

A $0.1 = \frac{1}{10} = \frac{1}{10^{\blacksquare}}$ B $0.01 = \frac{1}{100} = \frac{1}{10^{\blacksquare}}$ C $0.001 = \frac{1}{1000} = \frac{1}{10^{\blacksquare}}$

★ **7.** Write each number in expanded notation as in the example.

$634.265 = (6 \cdot 10^2) + (3 \cdot 10^1) + (4 \cdot 10^0) + \left(2 \cdot \frac{1}{10^1}\right) + \left(6 \cdot \frac{1}{10^2}\right) + \left(5 \cdot \frac{1}{10^3}\right)$

A 35.42	C 562.34	E 0.0307	G 8.374	I 860.079
B 1.254	D 503.208	F 27.64	H 973.56	J 0.506

UTILIZATION

Exercises 1 and 3 might be completed orally as part of a general class discussion. In addition the numbers in these exercises could be used to review how decimals are read.

Answers, Exercise 7

A. $(3 \cdot 10^1) + (5 \cdot 10^0) + \left(4 \cdot \frac{1}{10^1}\right) + \left(2 \cdot \frac{1}{10^2}\right)$

B. $(1 \cdot 10^0) + \left(2 \cdot \frac{1}{10^1}\right) + \left(5 \cdot \frac{1}{10^2}\right) + \left(4 \cdot \frac{1}{10^3}\right)$

C. $(5 \cdot 10^2) + (6 \cdot 10^1) + (2 \cdot 10^0) + \left(3 \cdot \frac{1}{10^1}\right) + \left(4 \cdot \frac{1}{10^2}\right)$

D. $(5 \cdot 10^2) + (0 \cdot 10^1) + (3 \cdot 10^0) + \left(2 \cdot \frac{1}{10^1}\right) + \left(0 \cdot \frac{1}{10^2}\right) + \left(8 \cdot \frac{1}{10^3}\right)$

E. $\left(0 \cdot \frac{1}{10^1}\right) + \left(3 \cdot \frac{1}{10^2}\right) + \left(0 \cdot \frac{1}{10^3}\right) + \left(7 \cdot \frac{1}{10^4}\right)$

F. $(2 \cdot 10^1) + (7 \cdot 10^0) + \left(6 \cdot \frac{1}{10^1}\right) + \left(4 \cdot \frac{1}{10^2}\right)$

G. $(8 \cdot 10^0) + \left(3 \cdot \frac{1}{10^1}\right) + \left(7 \cdot \frac{1}{10^2}\right) + \left(4 \cdot \frac{1}{10^3}\right)$

H. $(9 \cdot 10^2) + (7 \cdot 10^1) + (3 \cdot 10^0) + \left(5 \cdot \frac{1}{10^1}\right) + \left(6 \cdot \frac{1}{10^2}\right)$

I. $(8 \cdot 10^2) + (6 \cdot 10^1) + (0 \cdot 10^0) + \left(0 \cdot \frac{1}{10^1}\right) + \left(7 \cdot \frac{1}{10^2}\right) + \left(9 \cdot \frac{1}{10^3}\right)$

J. $\left(5 \cdot \frac{1}{10^1}\right) + \left(0 \cdot \frac{1}{10^2}\right) + \left(6 \cdot \frac{1}{10^3}\right)$

EXTENSION

See *Workbook* page 55 and *Duplicator Masters* page 34 for further practice exercises.

Remedial: If students have difficulty with naming and writing decimals, present the spelled-out number names for decimals and have the students write the decimals, and conversely, write some decimals and have students read the decimals aloud.

Assignments

Minimum 1, oral; 2; 4. Average 1, oral; 2–5. Maximum 1, oral; 2–7.

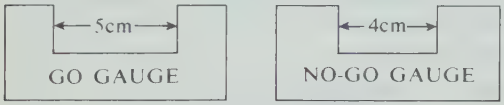
Objective

Given two numbers in decimal notation, the student can identify the larger number and express the comparison by a statement of inequality.

PREPARATION

Materials: Posterboard.

The Investigation will be enhanced if you prepare a cardboard model to illustrate the idea of the Go Gauge and No-Go Gauge, as illustrated below.



Now cut several strips of cardboard – some between 4 and 5 cm in width, some less than 4 cm wide, and some more than 5 cm wide. Then demonstrate that the pieces having the "correct" width are those that will fit in the 5 cm slot but not in the 4 cm slot, that is, those that are between 4 and 5 cm wide. The pieces that are more than 5 cm wide will not fit either gauge, while those which are less than 4 cm wide will fit in both gauges.

INVESTIGATION

After the demonstration suggested in the Preparation section, have the students list the pieces which are the correct size. Note that this involves either place value consideration or thinking about fractions for the decimals. If some students have difficulty, you might suggest that they are to find those pieces that are "between 0.290 and 0.312 units in width."

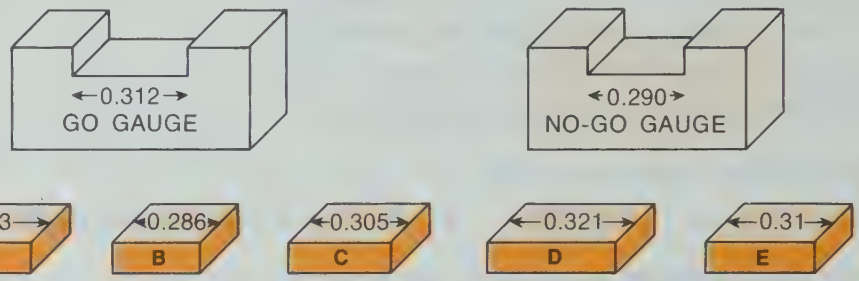
DISCUSSION

In the Discussion, bring out the fact that comparing two decimals is much like comparing whole numbers: that is, given two decimals such as 3.176 and 3.174 we compare the whole number parts, then tenths, then hundredths, then thousandths. Since 3.176 has a greater number of thousandths, we write $3.176 > 3.174$. If necessary, review the meaning of the symbols $>$ and $<$.

Using Decimals to Compare Numbers

Investigating the Ideas

The two gauges below are to be used to see if the pieces A, B, C, D, and E have the correct width. Each piece must be small enough to slide through the GO GAUGE yet too large to fit into the slot of the NO-GO GAUGE.



Can you find which pieces are the correct size?

Discussing the Ideas

1. Which pieces did you think were too large? Why?

2. Which pieces did you think were too small? Why?

3. A Since $\frac{286}{1000} < \frac{290}{1000}$, how does 0.286 compare with 0.290?
   B How does 0.286 compare with 0.312?
   C Does piece B fit into the slots of both gauges?

4. A Since $0.3 = \frac{3}{10} = \frac{30}{100} = \frac{300}{1000} = 0.300$, will piece A fit into the GO GAUGE?
   B Will it fit into the NO-GO GAUGE?

5. Which symbol ($<$, $=$, or $>$) goes in each ▨?
   A $0.305 \square 0.312$ C $0.31 \square 0.310$ E $0.31 \square 0.305$
   B $0.305 \square 0.290$ D $0.3 \square 0.31$ F $0.290 \square 0.29$

6. If the pieces A, B, C, D, and E were arranged from the smallest to the largest in width, what would the order be?

Using the Ideas

1. Which pieces below will fit into the slot of this gauge?

2. Give the correct symbol (<, =, or >) for each ◍.

A 1.372 ◍ 1.732
B 1.372 ◍ 1.273
C 1.372 ◍ 1.237
D 1.372 ◍ 1.723
E 1.372 ◍ 1.327
F 1.732 ◍ 1.723

3. Give the correct symbol (<, =, or >) for each ◍.

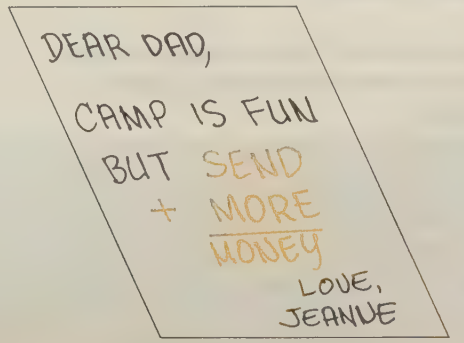
$\frac{89}{100} < \frac{98}{100}$ $\quad \frac{67}{100} = \frac{670}{1000}$ $\quad \frac{200}{1000} \; \frac{199}{1000}$

A 0.89 ◍ 0.98
B 0.67 ◍ 0.670
C 0.2 ◍ 0.199
D 1.010 ◍ 0.999
E 0.54 ◍ 0.45
F 8.93 ◍ 8.930
G 5.001 ◍ 4.999
H 0.0988 ◍ 0.1
I 0.9 + 0.1 ◍ 1
J 0.009 + 0.1 ◍ 0.010
K 0.999 + 0.001 ◍ 1
L 0.09999 ◍ 0.1
M 5.654 ◍ 5.0654
N 98.999 ◍ 98.99
O 0.1 + 0.9 ◍ 0.9 + 1
P 0.99 + 0.99 ◍ 0.99 + 0.9
Q 0.056 + 1 ◍ 0.06 + 1
R 0.7362 ◍ 0.73620

think

Jeanne sent her father this message from summer camp.

Can you help Jeanne's father figure out this addition puzzle: SEND + MORE = MONEY? Each letter stands for one of the digits 0, 1, 2, . . . , 9, and no two letters may represent the same digit.



UTILIZATION

The exercises give practice in comparing numbers in decimal notation and writing inequalities for pairs of numbers. *Workbook* page 56 offers additional practice exercises.

Think Solution

(1) Clearly, $M = 1$. Therefore, S must be either 8 or 9 in order to give us a sum requiring that a digit be carried. If S is 8 or 9, then $O = 0$ (again, because we know $M = 1$). If $S = 8$, we would have

$$\begin{array}{r} 8\ E\ N\ D \\ +\ 1\ 0\ R\ E \\ \hline 1\ 0\ N\ E\ Y \end{array}$$

and $E + 0$ would have to equal 10 and N would equal 0, which is not possible. Hence, $S = 9$.

(2) $E + O = N$; so 1 must have been carried to the hundreds' place, which tells us that N is 1 more than E: $E + 1 = N$.

(3) If no tens were carried, $N + R = E$, or, if one ten was carried, $1 + N + R = E$. Let $N + R = E$, which is the same as $(E + 1) + R = E$ (Step 2). Thus R would equal 9. But $S = 9$, so $1 + N + R$ must equal E.

(4)
$$\begin{array}{r} 9\ E\ N\ D \\ +\ 1\ 0\ R\ E \\ \hline 1\ 0\ N\ E\ Y \end{array}$$

If $1 + N + R = E$, then $1 + (E + 1) + R = E$, because $E + 1 = N$. $R + 2$ is zero (plus 1 to carry to the hundreds); thus $R = 8$.

(5) Now trial and error must be used, but some numbers (0, 1, 8, 9) have already been assigned. Since $E + 1 = N$, E cannot be 7. If $E = 6$ and $N = 7$, then $D + 6 = Y + 10$; thus D would have to be 4 or 5, and Y either zero or 1. Since zero and 1 are already assigned, these values for E and N are incorrect. Consider $E = 5$ and $N = 6$. Then $D + 5 = Y + 10$, and the choices for D are 7, 4, or 3. If D equals 4 or 3, the values for Y are again already assigned. Thus $D = 7$ is the only choice. Substituting: $7 + 5 = 2 + 10$, which is a solution. Thus $E = 5$, $N = 6$, $D = 7$, and $Y = 2$.

Assignments

Minimum 1, oral; 2. Average 1, oral; 2; 3. Maximum 1–3.

Objective

The student can find sums or differences of fractional numbers using decimal notation.

PREPARATION

On the chalkboard write several addition and subtraction problems using fractions with denominators of 10, 100, and 1000. Have students find the sums or differences and then give decimals for each fraction in the equations. For example:

A. $\frac{3}{10} + \frac{4}{10} = n$
$0.3 + 0.4 = n$

B. $\frac{9}{10} - \frac{3}{10} = n$
$0.9 - 0.3 = n$

C. $\frac{41}{100} - \frac{26}{100} = n$
$0.41 - 0.26 = n$

D. $\frac{425}{1000} + \frac{101}{1000} = n$
$0.425 + 0.101 = n$

INVESTIGATION

The Investigation will require only a brief amount of time, but it will serve to illustrate both correct and incorrect examples of addition using decimals.

DISCUSSION

In discussing the Investigation and Exercises 1 and 2, point out that alignment of the decimal point in addition or subtraction problems enables us to add tenths to tenths, hundredths to hundredths, and so on. Annexing zeros to the decimals as shown in Exercise 2 is equivalent to writing all decimals as fractions with a common denominator of 1000.

The answers for Exercise 4 are given below.

A. $0.4 = \frac{4}{10} = \frac{400}{1000} = 0.400$

B. Using fractions, we can see that regrouping is necessary because we cannot subtract $\frac{937}{1000}$ from $\frac{400}{1000}$.

$$\begin{array}{r} 57\frac{400}{1000} = \\ -\ 8\frac{937}{1000} = \end{array} \quad \begin{array}{r} 56\frac{1400}{1000} \\ -\ 8\frac{937}{1000} \end{array}$$

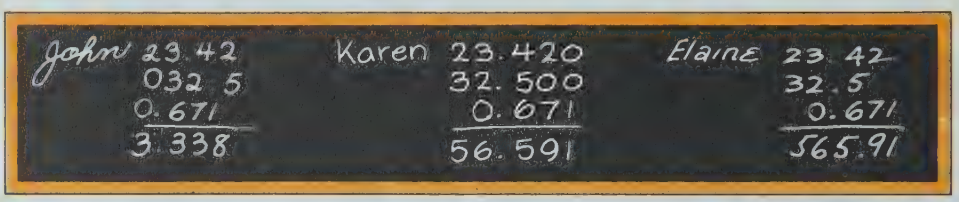
C. After regrouping, the subtraction can be completed as if working with whole numbers and aligning the decimal point in the difference with those in the problem.

D. The difference is added to the number subtracted to get the third number.

Adding and Subtracting with Decimals

Investigating the Ideas

Three students tried to find the sum $23.42 + 32.5 + 0.671$.



Can you find which student found the correct sum?

Discussing the Ideas

1. What is wrong with the way John arranged his addition problem?

2. Karen used the fact that 23.42 and 23.420 are **equivalent decimals** because $23.42 = 23\frac{42}{100} = 23\frac{420}{1000} = 23.420$.

   A. Explain why 32.5 and 32.500 are equivalent decimals?

   B. How did this help Karen to arrange her work for the problem?

   C. What is the sum of all the thousandths in Karen's problem?

$$\begin{array}{r} {\scriptstyle 1} \\ 23.420 \\ 32.500 \\ +\ 0.671 \\ \hline 56.591 \end{array}$$

$$\frac{420}{1000} + \frac{500}{1000} + \frac{671}{1000} = \frac{}{1000}$$

3. What did Elaine do that caused her to get an incorrect answer to the problem?

4. The figure shows how a Ken arranged his work for the subtraction problem $57.4 - 8.937$.

   A. Using fractions, explain why 0.4 can be written as 0.400.

   B. Can you explain the regrouping in the example?

   C. Explain how to complete the subtraction problem.

   D. How can you check your answer to the subtraction problem?

$$1\tfrac{400}{1000} = \tfrac{1400}{1000}$$

$$\begin{array}{r} {\scriptstyle 6} \\ 57.400 \\ -\ 8.937 \\ \hline \end{array}$$

Using the Ideas

1. Write equivalent decimals in **hundredths** as needed to add or subtract.

A $2.3 + 4 + 5.36$
B $14.21 + 0.6 + 7.9$
C $0.54 + 4.3 + 0.8$
D $5 + 2.12 + 0.4$
E $28.4 - 9.28$
F $1.6 - 0.42$
G $0.9 - 0.76$
H $237.43 - 0.967$

EXAMPLE: $1.4 + 2 + 3.14$

SOLUTION:
$$\begin{array}{r} 1.40 \\ 2.00 \\ +\,3.14 \\ \hline 6.54 \end{array}$$

2. Find the sums without changing to hundredths.

A $8 + 2.3 + 1.54$
B $3.5 + 0.62 + 11$
C $4.6 + 0.25 + 0.8$
D $8.7 + 14 + 1.62$
E $17.04 + 1.3 + 25$
F $9 + 0.63 + 0.2$

EXAMPLE: $7 + 1.4 + 0.62$

SOLUTION:
$$\begin{array}{r} 7 \\ 1.4 \\ +\,0.62 \end{array}$$

3. Write equivalent decimals in **thousandths** as needed for finding each sum or difference.

A $1.412 + 86 + 2.3$
B $76.2 + 0.34 + 0.079$
C $2.3 + 17 + 4.265$
D $13.8 + 7.243 + 11.63$
E $49.5 - 4.7$
F $58.267 - 0.19$
G $91 - 0.143$
H $3.06 - 0.999$

EXAMPLE: $5.21 - 4.218$

SOLUTION:
$$\begin{array}{r} 5.210 \\ -\,4.218 \end{array}$$

4. Find the sums and differences.

A $0.48 + 31$
B $3.2 + 4.7$
C $0.5 + 3 + 0.02$
D $2.54 + 37.3$
E $13.354 + 423.4568$
F $1.05 + 0.75 + 21.5$
G $6.563 + 0.278 + 37.937$
H $4.763 - 0.321$
I $5.346 + 35.27 + 161.2 + 4.387$
J $146.2 - 21.462$
K $1.002 + 0.0102$
L $10.2 + 0.34 + 4$
M $90 + 9 + 0.9 + .09$
N $(3.1416 + 1.4142) - 2.8$
O $(0.306 - 0.21) + 0.067$
P $0.01 - 0.001$
Q $100 - 0.01$
R $10 - 0.00011$

think

Find the two products in part **A** and the two products in part **B**.

Can you find some other pairs of multiplication problems like these?

A
$$\begin{array}{rr} 32 & 23 \\ \times 69 & \times 96 \end{array}$$

B
$$\begin{array}{rr} 96 & 69 \\ \times 46 & \times 64 \end{array}$$

More practice, page S-18, Set 31

UTILIZATION

Exercises 1, 2, and 3 give specific examples showing how to write the decimals. Exercise 4 allows students to make their own choice in writing the numerals to find the sums.

Temper your assignment according to the needs of your students. It would be best to allow slower students to concentrate on fewer exercises and strive for greater accuracy.

EXTENSION

For students who would benefit from additional practice, suggest assignments from Supplementary Exercise Set 31 on page S-18, *Workbook* page 57, *Duplicator Masters* page 35, and *Arithmetic Skill Cards* D-1 and D-2.

Enrichment: Provide some reconstruction problems involving addition and subtraction of decimals, such as the ones given below.

1. $\begin{array}{r} 2\,\square\,.\,6\,\square \quad (5, 6) \\ +\,3\,5\,.\,\square\,7 \quad (6) \\ \hline 6\,1\,.\,3\,3 \end{array}$

2. $\begin{array}{r} 0\,.\,\square\,2 \quad (8) \\ -\,0\,.\,3\,3\,\square \quad (4) \\ \hline 0\,.\,4\,\square\,6 \quad (8) \end{array}$

Think Solution

A 2-digit by 2-digit multiplication problem will have the same product if the digits in each factor are reversed, provided the product of the units' digits is equal to the product of the tens' digits.

Let $(10a + b)$ be one 2-digit number and $(10c + d)$ be a second 2-digit number.
Then $(10a + b)(10c + d) = 100ac + 10bc + 10ad + bd$.

If the tens' and the units' digits are reversed in both factors, we have
$(10b + a)(10d + c) = 100bd + 10bc + 10ad + ac$.

Clearly the two products are equal if $bd = ac$, that is, if the products of the tens' digits equals the product of the units' digits.

Here are other examples which have this property: $(14 \cdot 82)$, $(46 \cdot 32)$, $(93 \cdot 13)$, and $(63 \cdot 48)$.

Assignments

Minimum 1A–D, 2A–C, 3A–D. Average 1–3. Maximum 1–4.

Objective

The student can solve metric system problems by using decimals.

PREPARATION

Materials: Metric stick; standard units of mass (grams) and scientific scale to measure in grams; litre measure. (page 87 of the *Duplicator Masters* provides a pattern for a 1 metre tape measure.)

This lesson will be greatly enhanced if the measuring devices listed can be available to the student. The science department of your school may have equipment you can borrow for demonstration purposes.

INVESTIGATION

As students carry out the Investigation, have them record both their estimate and the actual measurement in decimal notation. Some students may have had very little experience working with metric units of mass and capacity and hence may have difficulty in making meaningful estimates. However, the purpose of the Investigation is to have students work strictly in terms of the metric system rather than attempting to convert from English measures to metric. Thus a student might estimate his height to be 1.5 metres but find that he is 1 metre 68 cm or 1.68 m tall.

For the activity pertaining to capacity, you may prefer to have students use rice, sand, or small beans rather than water.

DISCUSSION

Knowing the meaning of the metric prefixes will greatly enhance understanding of the relationship between the various units of the metric system. Such an understanding of the various names of the subunits and multiples of the units should be a continuing objective.

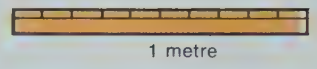
In discussing Exercise 2, a student response for part 2A might be, "Since *deci* means *tenths*, 0.3 metres is 3 decimetres." For 2B, students might explain: "Since *centi* means *hundredths*, 25 centimetres is 0.25 metres." And so on.

The Metric System

Investigating the Ideas

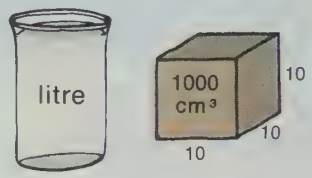
Three of the basic units of the **metric system** are the metre, the kilogram, and the litre.

Unit of length: **Metre**(m)
A metre is 100 centimetres (cm).



Unit of mass: **Kilogram**(kg)
A mass of 1 kilogram is 1000 grams (g).

Unit of capacity: **Litre**(ℓ)
A litre container holds 1000 cubic centimetres (cm^3).



Can you use the information in the chart above to make the estimates below?	Find a way to check your estimates.

1. Estimate your height in metres.
2. Estimate the mass of your mathematics book in kilograms.
3. Estimate the capacity of a soft drink bottle in litres.

Discussing the Ideas

1. Other units in the metric system are related to the basic three. These prefixes and the use of decimals can help you understand their relationship.

milli- means *one thousandth*	**deca**- means *ten*
centi- means *one hundredth*	**hecto**- means *one hundred*
deci- means *one tenth*	**kilo**- means *one thousand*

EXAMPLES: A **centi**metre is 0.01 metre; a **kilo**gram is 1000 grams.

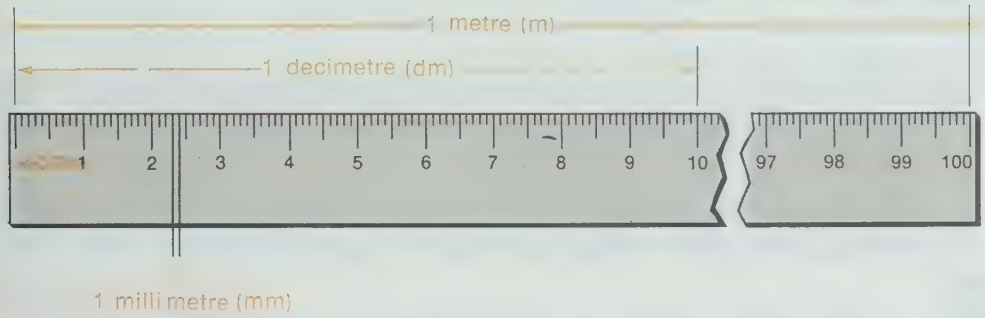
A	Which means 10 metres?	decimetre or decametre
B	Which means 0.01 gram?	hectogram or centigram
C	Which means 0.001 metre?	kilometre or millimetre
D	Which means 0.1 litre?	decilitre or decalitre

2. Explain how to determine the missing numbers.

A	0.3 metre = ▒▒ decimetres	D	12 centigrams = ▒▒ milligrams
B	25 centimetres = ▒▒ metre	E	2000 litres = ▒▒ kilolitres
C	0.341 litre = ▒▒ millilitres	F	2.3 centimetres = ▒▒ millimetres

Using the Ideas

1. Study this diagram. Then give the missing numbers in each part below using decimal notation.



A 1 m = ▒ dm	F 1 mm = ▒ m	K 8 dm = ▒ m
B 1 m = ▒ cm	G 1 cm = ▒ mm	L 25 cm = ▒ m
C 1 m = ▒ mm	H 1 dm = ▒ cm	M 59 cm = ▒ m
D 1 dm = ▒ m	I 1 mm = ▒ cm	N 4 mm = ▒ m
E 1 cm = ▒ m	J 1 cm = ▒ dm	O 29 mm = ▒ cm

2. Give the missing numbers.

A 1 kilogram = ▒ g
B 1 g = ▒ kg
C 1 hectolitre = ▒ ℓ
D 1 decilitre = ▒ ℓ
E 1 ℓ = ▒ ml
F 1 ℓ = ▒ hectolitre (hl)

3. The body of an average man contains 5.7 litres of blood. The body of an average woman contains about 3.3 litres. How much greater is the blood capacity of a man?

4. Barbara weighed 36.29 kilograms. Caroline weighed 45.8 kilograms. How much more did Caroline weigh?

5. William is 1.72 metres tall. Henry is 1.59 metres tall.
   A How much taller is William?
   B How much taller in centimetres is William?

6. The brain of a man weighs about 1.1 kilograms. The brain of a woman weighs about 0.05 kilograms less. What is the weight of a woman's brain?

UTILIZATION

Most students can supply the missing numbers in Exercise 1 by referring to the diagram. However, metre sticks should be available for students to use if necessary. In both Exercises 1 and 2 students should be encouraged to refer to the metric prefixes given in the Discussion section.

Exercises 3 through 6 involve only simple addition and subtraction with metric units.

EXTENSION

Page 58 of the *Workbook* and page 36 of the *Duplicator Masters* provide exercises appropriate for further practice.

Enrichment: Have your students construct a metric tape measure on strips of adding machine tape, or see page 87 of the *Duplicator Masters*. (Vinyl cloth would make more durable tapes, if it is available.) Then have students use their tapes to measure and record various body measurements. Included in these measurements might be height, waist, chest, hips, neck, biceps, wrist, foot, and head measurements.

This would also be an appropriate time to encourage interested students to initiate work on Research Project C, page B-95.

Assignments

Minimum 1–3. Average 1–6. Maximum 1–6.

Objective

Given two numbers in decimal notation, the student can find their product.

PREPARATION

Review writing decimals in fractional notation and the reverse. Be sure to use only fractions whose denominators are powers of ten.

INVESTIGATION

The Investigation is relatively short but quite significant. Many students will recall the method of multiplying with decimals, but some may have forgotten or lack understanding of the rule for placing the decimal point in products. For such students, the Investigation will provide a basis for developing an understanding of the rule.

DISCUSSION

The Discussion Exercises are designed so that your students can discover the basic rules for placement of the decimal point in multiplication problems. For those students who are already familiar with the rules, the discovery sequence provides an opportunity to gain additional insight into the meaning of the rules.

Exercise 3 calls for a statement of the rule for placing a decimal point in the product. You should accept any statement which indicates an understanding of the rule. It is more important that students be able to place the decimal point correctly than that they be able to verbalize a rule. The general rule is, the number of decimal places in the product is the sum of the number of places in the factors. However, students who say, "A one-place decimal times a 2-place decimal gives a 3-place decimal" or "Tenths times hundredths gives thousandths" are indicating a good working understanding of the rule.

Multiplying with Decimals

Investigating the Ideas

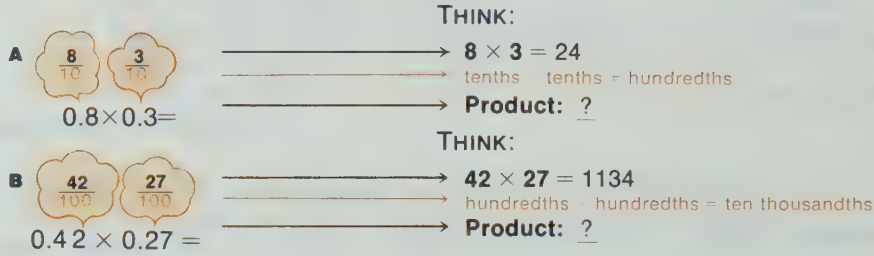
Decimals are symbols for fractions whose denominators are powers of ten. If you can multiply using this type of fraction, then you should be able to understand how to multiply using decimals.

A	B	C
$\frac{1}{10} \times \frac{1}{10} = a$	$\frac{1}{10} \times \frac{1}{100} = a$	$\frac{1}{10} \times \frac{1}{1000} = a$
$0.1 \times 0.1 = b$	$0.1 \times 0.01 = b$	$0.1 \times 0.001 = b$
tenths × tenths = ?	tenths × hundredths = ?	tenths × thousandths = ?

Can you give a fraction for ***a***, a decimal for ***b***, and a word for the blank in each example above?

Discussing the Ideas

1. Study the examples. Then give the product as a decimal.



2. Multiplying numbers represented by decimals is much like multiplying whole numbers. Study example A. Then tell where the decimal point should be placed in example B.

A First find 76 × 3, then think:

7.6 tenths (one-place decimal)
times
× 0.3 tenths (one-place decimal)
equals
228 hundredths (two-place decimal)

Place decimal point here

B First find 942 × 31, then think:

94.2 tenths (one-place decimal)
times
× 0.31 hundredths (two-place decimal)
equals
942
2826
29202 thousandths (three-place decima

3. Can you state a rule for placing the decimal point in a product of two decimals?

Using the Ideas

1. Find the products.

A 0.4×0.3	C 0.8×0.6	E 0.9×0.08	G 0.7×0.300	I 0.07×0.0004
B 0.5×0.05	D 0.5×0.07	F 0.08×0.06	H 0.9×0.006	J 0.003×0.005

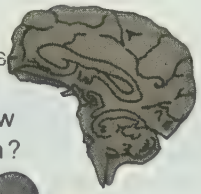
2. Find the products.

A 56.3×2	B 43.5×0.2	C 3.6×4.5	D 4.5×9.8	E 0.15×66
F 0.68×1.8	G 5.9×0.77	H 0.41×0.26	I 0.507×4.9	J 0.379×47.6
K 9.763×0.07	L 8.462×0.76	M 0.346×0.571	N 0.408×0.930	O 4.67×0.394

SHORT STORIES – The Human Body

1. The amount of water in a person's body is 0.58 times body weight. If body weight is 69.2 kilograms, how many kilograms of water?

2. A man's brain is 0.02 of his body weight. If his body weight is 64 kilograms, how much does his brain weigh?



3. Man's muscles are 0.4 of his body weight. If a man weighs 61.3 kilograms, how much do his muscles weigh?

4. We breathe about 12 kilolitres of air per day. The oxygen into the blood is 0.05 of the total air. How many litres of oxygen into the blood each day?

5. Human bones: 0.18 of body weight. Body weight: 67.9 kilograms. What is the weight of the bones?

6. The human heart pumps 45 litres of blood per minute during strenuous exercise. During relaxation, it pumps 0.125 as many litres. How many litres per minute does the heart pump during relaxation?

7. Man takes 18 breaths per minute. When resting, he takes in 0.75 litres of air in each breath. When doing light work, 1.62 litres. When doing heavy work, 2.14 litres. How many litres of air used if a man rested 15 minutes, worked lightly for 15 minutes, and then worked hard for 15 minutes?

More practice, page S-18, Set 32

UTILIZATION

Exercises 1 and 2 provide straightforward drill in finding products. The short stories based on the human body give additional practice in multiplication of decimals and in using metric units of mass and capacity.

EXTENSION

To provide additional practice, make assignments from Supplementary Exercise Set 32 on page S-18, *Workbook* page 59, *Duplicator Masters* page 37, and *Arithmetic Skill Card* D-3.

Remedial: Prepare problems such as the ones below and have students multiply mentally and give the product as a decimal.

A. 0.9×0.8 *(0.72)*
B. 0.5×0.8 *(0.40)*
C. 0.6×0.3 *(0.18)*
D. 0.02×0.06 *(0.0012)*
E. 0.08×0.7 *(0.056)*
F. 0.004×0.4 *(0.0016)*
G. 0.9×0.1 *(0.09)*
H. 0.5×0.5 *(0.25)*
I. 0.9×0.9 *(0.81)*
J. 0.09×0.06 *(0.0054)*
K. 0.006×0.7 *(0.0042)*
L. 0.4×0.0008 *(0.00032)*

Enrichment: Tell students that for each problem, they should find one number that can serve as both quotient and divisor. (Give them this *hint:* Estimate the answer, then use multiplication to check your guess.)

A. 2.25 *(1.5)*

B. 5904.49 *(24.3)*

C. 0.0625 *(0.25)*

D. 17.0569 *(4.13)*

Assignments

Minimum 1, 2A–E, Short Stories 1–4.
Average 1, 2A–J, Short Stories, 1–6.
Maximum 1, 2, Short Stories 1–7.

Objective

Given one number in decimal notation and a second number which is a power of ten, the student can find their product.

PREPARATION

Materials: Centimetre ruler scaled in millimetres.

Review multiplication of decimals and the placement of the decimal point.

INVESTIGATION

Students should be able to answer the first two questions because they deal with multiplication of a whole number by a power of ten. The segment ten times as long as $\overline{AB}$ is 120 cm long, or 1.2 metres long. The segment 100 times as long as $\overline{AB}$ is 1200 cm long, or 12 metres long.

Do not insist on great accuracy in students' drawings that are 0.1 and 0.01 as long as $\overline{AB}$. The important thing for the students to do is to find the products $0.1 \times 12 = 1.2$ and $0.01 \times 12 = 0.12$ and be able to interpret the numbers as lengths of segments.

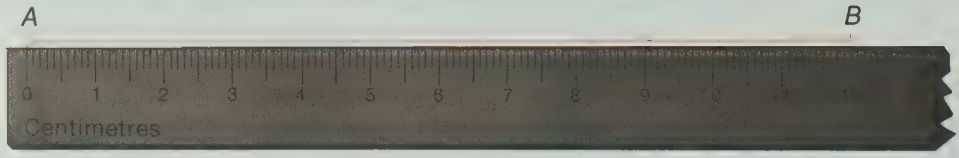
Since 0.1 of 12 cm is 1.2 cm or 12 mm, this segment can be easily drawn. But 0.01 of 12 cm is 0.12 cm or 1.2 mm, which is somewhat more difficult to represent. The students should draw a segment just slightly more than 1mm to represent this length.

DISCUSSION

In order to see the pattern when multiplying by powers of ten in Exercise 3, place a chart similar to the one shown in the text on the chalkboard or overhead projector and have students complete each row of the table. If a number is multiplied by 10, 100, or 1000, the decimal point is shifted 1, 2, or 3 places, respecitively, to the *right*. When multiplied by 0.1, 0.01, or 0.001, the decimal point is shifted 1, 2, or 3 places, respectively, to the *left*. Do not expect a statement as formal as this for Exercise 4; accept any statement which simply indicates good understanding. Keep in mind that ability to use and understand a rule is more valuable than the ability to verbalize the rule.

Multiplying with Powers of Ten

Investigating the Ideas



1. How long is a segment that is ten times as long as $\overline{AB}$?
2. How long is a segment that is one hundred times as long as $\overline{AB}$?

Can you use a centimetre ruler to draw segments that are 0.1 and 0.01 as long as $\overline{AB}$?

Discussing the Ideas

1. **A** How many centimetres long is the segment that is 0.1 as long as $\overline{AB}$?

B What is 0.1×12?

2. **A** How many centimetres long is the segment that is 0.01 as long as $\overline{AB}$?

B What is 0.01×12?

3. It is important for you to be able to multiply a decimal by 0.1, 0.01, 0.001, and so forth, as well as by 10, 100, and 1000. Study part **A** in the table below. Try to find the products. You may discover a shortcut for finding them.

	× 1000	× 100	× 10	Number	× 0.1	× 0.01	× 0.001
A				3.71			
B				42.193			
C				0.04			
D				264.38			
E				6394.7125			

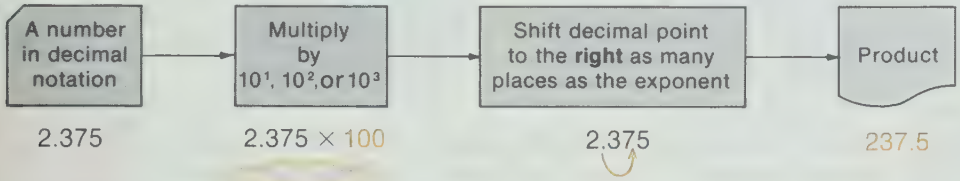
4. Can you describe a shortcut for multiplying a decimal by each of the factors 1000, 100, 10, 0.1, 0.01, and 0.001?

Using the Ideas

1. Find the products.

A 10×8.5	D 100×5.03	G 1000×5.468	J 10×9.358
B 10×0.64	E 100×2.7	H 1000×3.14	K 100×93.58
C 10×9.037	F 100×29.1	I 1000×0.82	L 1000×9.358

2. Find the products. The flow chart may help you use a shortcut.

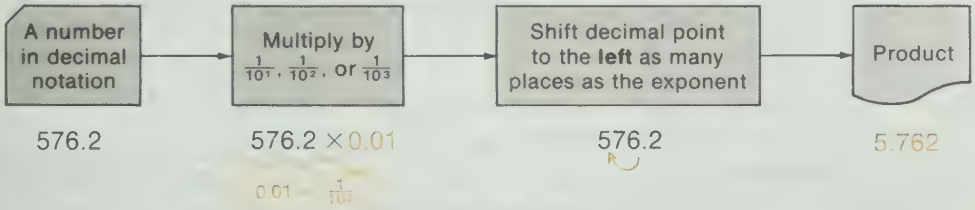


A $10^1 \times 45.62$	D 10×0.937	G $10^3 \times 0.376$	J $10^2 \times 5.63$
B 0.496×10^2	E 8.462×10^2	H 1000×0.84	K 8.891×10
C 8.437×10^3	F $10^2 \times 0.762$	I 53.762×10^3	L $10^3 \times 0.0057$

3. Find the products.

A 0.1×8.3	D 0.01×327	G 0.001×964.3	J 0.1×1.05
B 0.1×25.17	E 0.01×42.6	H 0.001×87.56	K 0.01×3.9
C 0.1×432.08	F 0.01×5.79	I 0.001×6684.2	L 0.001×8.2

4. Find the products. The flow chart may help you use a shortcut.



A 156.2×1	C 8427.4×0.01	E $8.3 \times \frac{1}{10^1}$	G $846.4 \times \frac{1}{10^2}$
B 7.26×0.01	D 839.7×0.001	F $67.7 \times \frac{1}{10^2}$	H 7.4×0.001

think

What is the greatest amount of money you can have (using pennies, nickels, dimes, quarters, and half dollars) and still not be able to give someone change for a dollar?

More practice, page S-19, Set 33

UTILIZATION

The exercises for this lesson are numerous, but they are fairly short and provide students with practice in the traditional "move-the-decimal-point" rule for multiplying by 10, 100, and 1000, and by 0.1, 0.01, and 0.001.

Students should be encouraged to do the calculations mentally. The flow charts in Exercises 2 and 4 will aid in improving the students' skills with factors which are powers of ten.

EXTENSION

Assign Supplementary Exercise Set 33 and, perhaps, *Duplicator Masters* page 38 for those students who would benefit from additional practice.

Enrichment: Present this partially completed 4 by 4 magic square.

	0.2	0.3	1.3
0.5		1.0	
	0.7		
0.4	1.4		0.1

Have the students complete the square. (The magic sum is 3.4.) Then have the students multiply each number in the completed magic square by some power of ten: 0.1, 0.01, 0.001, 10, 100, or 1000, and ask: "Is the resulting square still a magic square?" (Yes) Finally, have students find the new magic sum.

Think Solution

You could have 1 half dollar, 1 quarter, 4 dimes, and 4 pennies. Another possibility is 1 quarter, 9 dimes, and 4 pennies. The amount of money would be $1.19 in both cases.

Assignments

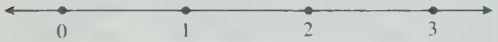
Minimum 1A–F, 2A–F, 3A–F.
Average 1–3. Maximum 1–4.

Objective

Given two decimal factors, the student can estimate the product by rounding one or more of the factors.

PREPARATION

Display a number line scaled in units on the chalkboard or overhead projector.



Then present the following decimals: 0.87, 1.24, 2.718, 1.96, 2.066, and 3.19.

Ask some students to show the approximate location of points for these numbers on the line. Then pose questions such as: "Is the point for 0.87 nearer to 0 or to 1?" "Is the point for 2.718 nearer to 2 or to 3?" and so on. Such examples will help to review and develop understanding of rounding using decimals.

INVESTIGATION

In introducing the Investigation suggest that the students attempt to associate each problem with the estimated product without using pencil or paper. Students should definitely be discouraged from attempting to multiply the factors in their given form, because the purpose of the Investigation is to have students use rounded factors in estimating the products.

DISCUSSION

In Exercise 1, students might reason as follows:

A. $7 \times 12 = 84$ C. $8 \times 7 = 56$
B. $33 \times 3 = 99$ D. $630 \times 0.1 = 63$

In reviewing rounding of decimals, point out that a number which is halfway between 2 numbers is rounded "up." Thus, if 3.65 is to be rounded to the nearest tenth, it is rounded to 3.7 even though it is halfway between 3.6 and 3.7.

Investigating the Ideas

You can often use estimation to find approximations for products involving decimals.

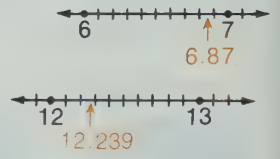
MULTIPLICATION PROBLEMS	ESTIMATED PRODUCT
A 6.87×12.239	56
B 33.75×2.93	63
C 8.367×6.72	84
D 629.6×0.09	99

Can you find which of the estimated products is nearest to the exact product for each of the multiplication problems?

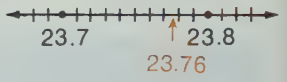
Discussing the Ideas

1. Explain how you decided which estimated product was nearest to the answer for each multiplication problem.

2. Rounding decimals is much like rounding whole numbers.
   - A To **round** 6.87 **to the nearest whole number**, you can think: "Is 6.87 nearer to 6 or to 7?" Which do you think it is?
   - B Which whole number is nearest 12.239, 12 or 13?
   - C Which two whole numbers can be used to estimate 6.87×12.239?



3. Often you must be able to round a decimal to the nearest tenth or to the nearest hundredth.
   - A Is 23.76 nearer to 23.8 or 23.7?
   - B What is 23.76 rounded to the nearest tenth?



4. Estimate the product shown by rounding each number to the nearest tenth.

$$\begin{array}{r} 3.196 \\ \times\ 0.296 \\ \hline \end{array}$$

5. Estimate the product by rounding the first factor to the nearest hundredth and the second factor to the nearest whole number.

$$0.07926 \times 3.129$$
$$\downarrow \qquad \downarrow$$
$$0.08 \times 3$$

Using the Ideas

1. Round each number to the nearest whole number. **When a number is halfway between two numbers, round it to the larger number.**

A 63.42 **B** 341.849 **C** 9.50 **D** 5.4999 **E** 60.099

2. Round each factor to the nearest whole number. Then use the rounded factors to estimate the product.

A 5.9×4.67
B 8.3×7.1
C 5.697×9.83
D 7×30.9573
E 7.85×29.87
F 9.39×29.32
G 59.42×98.943
H 0.94×202.78

EXAMPLE: 4.73×8.49
Rounded factors → 5×8
Estimated product → 40

3. Round the first factor to the nearest tenth. Round the second factor to the nearest whole number. Use the rounded factors to estimate the product.

A 0.297×8.2
B 0.189×7.34
C 0.4297×9.771
D 0.783×29.8
E 0.5013×5.23
F 1.199×2.909

EXAMPLE: 0.348×9.24
Rounded factors → 0.3×9
Estimated product → 2.7

4. Round the first factor to the nearest hundredth. Round the second factor to the nearest whole number. Then estimate the product.

A 0.034×7.2
B 0.06419×7.9
C 0.3023×6.33
D 0.07895×3.29
E 0.02507×9.3
F 0.075×6.5

EXAMPLE: 0.0576×7.21
Rounded factors → 0.06×7
Estimated product → 0.42

5. Estimate the products. Round the factors so that you can estimate the products quickly.

A 10.001×54.39
B 10.1×99.98
C 10.113×153.71
D 0.512×10.0037
E 100.11×0.5032
F 102×0.0305
G 45.21×98.532
H 5.324×1004.376
I 45.113×20.974
J 3.5432×9998.879
K 0.101×59.91
L 0.201×59.91
M 0.021×48.04
N 0.021×486.31
O 0.009×462

UTILIZATION

Exercises 2, 3, and 4 give specific examples for rounding factors to find estimates of problems. In Exercise 5 no instructions for rounding are given; students should simply round each factor in any way that will enable them easily to give a good estimate of the product. Thus, estimates will vary somewhat. For example, in 5H if the factors are rounded to 5 and 1000, the estimate will be 5000. However, some students may round only the second factor to 1000, think 5.324×1000, and give an estimate of 5324.

EXTENSION

Workbook page 60 and *Duplicator Masters* page 39 offer additional practice exercises.

Enrichment: This would be an opportune time to suggest that capable students explore Research Project B on page B-95, dealing with the use of slide rules for multiplying and dividing decimals.

Assignments

Minimum 1, 2A–D, 3, 4.
Average 1, 2, 3, 4, 5A–H.
Maximum 1–5.

Objective

Given word problems, the student can solve them by adding and multiplying using decimals.

PREPARATION

Materials: Centimetre rulers, metre sticks, or metric tape measures.

The materials will be needed for Exercises 4 through 7.

UTILIZATION

This lesson provides experiences in measurement, in multiplying, and in adding and rounding with decimals.

If you prefer active student involvement in this lesson, use the first three exercises to help familiarize the student with the use of the table in estimating heights. Then use Exercises 4 through 7 as laboratory experiments. You should expect considerable variation in heights computed from the table and actual heights. Encourage students to work in pairs or in small groups as they do the measurement problems and to record all results in a table.

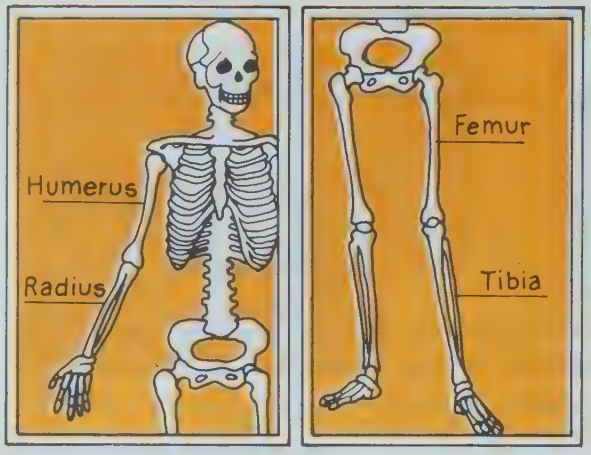
A variety of formulas may be created by the students for Exercise 7. For example, a student whose foot is 25 cm long and whose height is 162 cm might develop a formula such as the following:

Height = (6 × length of foot) + 12 cm.

Assignments

Minimum 1, 4. Average 1, 2, 4, 5.
Maximum 1–7.

ESTIMATING BODY HEIGHTS



Height (centimetres)	
Male	
(2.89 × length of humerus)	+ 70.64
(3.27 × length of radius)	+ 85.93
(1.88 × length of femur)	+ 81.31
(2.38 × length of tibia)	+ 78.66
Female	
(2.75 × length of humerus)	+ 71.48
(3.34 × length of radius)	+ 81.22
(1.95 × length of femur)	+ 72.85
(2.35 × length of tibia)	+ 74.78

Anthropologists, using a single bone from a human skeleton, can estimate quite accurately the height of a man or woman who lived many centuries ago. For example, if an anthropologist finds a [illegible]-cm radius for a caveman, he can estimate the height of the caveman as follows: (3.27 × 27.9) + 85.93 = 177.163 Rounding the answer, the man was about 177 cm or 1.77 meters tall.

1. A 45-cm femur was found. About how tall was the man?

2. The length of a humerus bone from a woman who lived 20 000 years ago is 32.3 cm. Estimate the woman's height to the nearest cm.

3. The skeleton of a Neanderthal man who lived about 50 000 years ago contained a tibia 38 cm long. Estimate the man's height.

4. Measure, as accurately as you can, the length of your radius and humerus and then use the table to find your height. How does your result compare with your actual height?

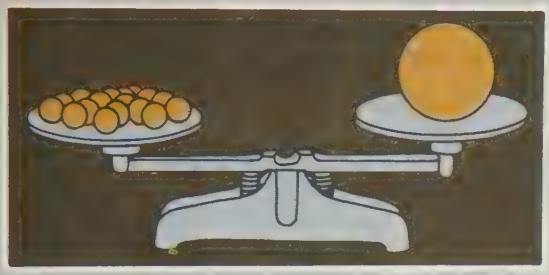
5. Measure your femur and tibia as accurately as you can. Then use the table to determine your height.

6. One student's height in centimetres was about 20 times the length of his middle finger measured to the nearest tenth of a centimetre. How much does your height differ from this comparison?

☆ 7. Measure the length of your foot to the nearest tenth of a centimetre. Can you derive a formula that will give you an estimate of your height by using the length of your foot?

Planet Masses

The table compares the mass of the planets with the mass of Earth. When we say that the **mass** of Neptune is 17.20 compared with the mass of Earth, think of a giant balance in which 17.20 "earths" were needed to balance one "Neptune."

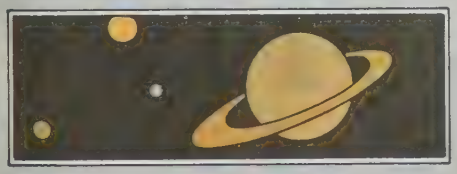


Mass of each planet compared with Earth	
Earth	1.000
Mercury	0.0543
Mars	0.1069
Venus	0.8136
Pluto	0.18
Uranus	14.54
Neptune	17.20
Saturn	95.3
Jupiter	318.35

The mass of Earth has been estimated to be about 6×10^{24} kilograms or 6 septillion kilograms.

1. A Round the number in the table for Neptune to the nearest whole number.
   B Use the rounded number to estimate the mass of Neptune.

2. Round the appropriate number in the table to the nearest whole number to estimate the mass of Saturn.



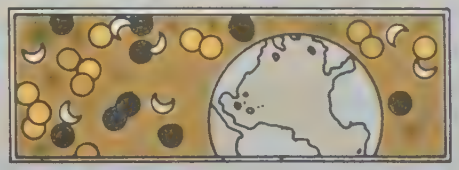
3. Round the appropriate number in the table to the nearest tenth to estimate the mass of Venus.

4. Round the appropriate number in the table to the nearest hundredth to estimate the mass of Mars.

5. Round the appropriate number in the table to the nearest hundredth to estimate the mass of Mercury.

6. About how many planets the size of Uranus would it take to "balance" with Saturn?

7. About how many planets the size of Mercury would it take to "balance" with Earth?



☆ 8. The mass of the moon is only 0.012 as that of Earth. About how many moons would be needed to "balance" the mass of Earth?

Objective

The student can use estimation and multiplication to solve word problems about masses of the planets.

UTILIZATION

In Exercises 1 through 6 the student must round the factors in the table and then estimate the mass of the various planets.

In Exercise 1, knowing the mass of Earth to be 6×10^{24} kilograms, the estimate would be

$$17 \times (6 \times 10^{24}) = (17 \times 6) \times 10^{24}$$
$$= 102 \times 10^{24},$$
or 1.02^{26} kilograms.

In Exercise 3, the estimate could be made as follows:

$$0.8 \times (6 \times 10^{24}) = 4.8 \times 10^{24}$$

To solve Exercises 6, 7, and 8 some students may think it necessary to divide using decimals. However, these problems may be solved by thinking about a missing factor in multiplication. Since 14.54 Earths "balance" with Uranus while it would take 95.3 Earths to "balance" with Saturn, we can solve the problem by finding approximately how many times 14.54 equals 95.3. Rounding each number to the nearest whole number, we see that the estimate should be a number between 6 and 7. Trying 6.5 as an estimate, we can find that $6.5 \times 14.54 \approx 95$; hence 6.5 is an excellent estimate. Similar methods can be used to make estimates for exercise 7 and 8.

Assignments

Minimum 1, 2, 3. Average 1–5.
Maximum 1–8.

B-81

Objective

Given two fractional numbers in decimal notation, the student can find their quotient.

PREPARATION

Plan a brief review of estimation of products using decimals such as those in Exercises 2, 3, and 4 on page B-79 of this module.

INVESTIGATION

The Investigation should give you some insight into how well your students understand the inverse relationship between multiplication and division. Students may estimate quotients for each part as follows.

A. Since $5 \times 2 < 12.85$ and $5 \times 3 > 12.85$, the quotient must be 2.57.

B. Since 0.9 is almost 1, the quotient should be about the same as the dividend; hence 5.24 must be the quotient.

C. Round the divisor to 3. The quotient cannot be 271 because $271 \times 3 > 92.14$. The quotient cannot be 2.71 because 3×2.71 is only about 9. Hence 27.1 is the quotient and $3.4 \times 27.1 = 92.14$.

DISCUSSION

For many of your students, this will be review of material that they already understand. Therefore, tailor the discussion to the needs of your students.

In discussing Exercise 2, bring out the fact that if the divisor is a whole number, the decimal point can be placed directly above the decimal point in the dividend.

Exercises 3 and 4 are designed to explain the rationale underlying the shortcut for placing the decimal in the quotient. Multiplying both divisor and dividend by the same power of ten does not change the quotient. Thus, we can always rewrite a given division problem so that we have an equivalent problem but with a whole number divisor.

Dividing with Decimals

Investigating the Ideas

Dividing numbers named by decimals is much like dividing whole numbers. Often you can use estimation and multiplication to help you place the decimal point correctly in the quotient.

Can you place the decimal point correctly in the three quotients below and prove that you are correct?

A $5\overline{)12.85}$ → 257

B $0.9\overline{)4.716}$ → 524

C $3.4\overline{)92.14}$ → 271

Discussing the Ideas

1. What can you do to check to see if you placed the decimal point correctly in the quotients above?

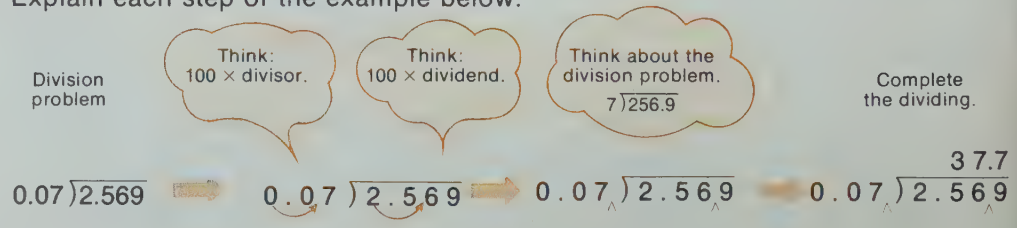
2. **A** Where does the decimal point go in this quotient? $8\overline{)58.48}$ → 731

   B If the divisor is a whole number, can you give a rule for placing the decimal point in the quotient?

3. Every division problem involving decimals can be replaced by a division problem that has a whole number divisor and yet has the same quotient as the original problem.
   Explain how the diagram shows that the quotient for $0.9\overline{)0.3924}$ is the same as the quotient for $9\overline{)3.924}$.

$$0.9\overline{)0.3924} \rightarrow \boxed{\frac{0.3924}{0.9} = \frac{0.3924 \times 10}{0.9 \times 10} = \frac{3.924}{9}} \rightarrow 9\overline{)3.924}$$

4. Explain each step of the example below.



Using the Ideas

1. Find the quotients.

A $2\overline{)67.6}$	C $4\overline{)15.16}$	E $63\overline{)0.504}$	G $58\overline{)21.46}$
B $7\overline{)23.59}$	D $9\overline{)58.5}$	F $29\overline{)17.4}$	H $84\overline{)21.00}$

2. Study the example to see how zeros must be annexed in some problems. Then find each quotient.

EXAMPLE: $0.09\overline{)47.7}$

$$\begin{array}{r} 530. \\ 09\overline{)47.70} \\ 45 \\ \hline 27 \\ 27 \\ \hline 0 \\ 0 \\ \hline 0 \end{array}$$

A $0.4\overline{)29.6}$	I $0.008\overline{)3.568}$	Q $0.23\overline{)0.851}$
B $5.8\overline{)0.406}$	J $3.45\overline{)24.15}$	R $5.6\overline{)3416}$
C $0.5\overline{)0.675}$	K $0.029\overline{)0.493}$	S $0.34\overline{)2.856}$
D $0.7\overline{)305.9}$	L $61.8\overline{)5.562}$	T $0.029\overline{)28.71}$
E $0.2\overline{)538}$	M $0.06\overline{)3.708}$	U $54.2\overline{)27.642}$
F $0.6\overline{)0.5616}$	N $0.08\overline{)756.8}$	V $0.41\overline{)1.599}$
G $0.09\overline{)47.7}$	O $0.05\overline{)0.1385}$	W $8.7\overline{)5.568}$
H $0.25\overline{)0.625}$	P $0.07\overline{)15.218}$	X $0.643\overline{)263.63}$

3. Study the example. Then find each quotient, rounded to the nearest hundredth. Annex zeros, as in the example, when you need them.

EXAMPLE: $2.3\overline{)1.669}$

We say: "the quotient, rounded to the nearest hundredth is .73."

$$\begin{array}{r} 0.725 \\ 2.3\overline{)1.6690} \\ 161 \\ \hline 59 \\ 46 \\ \hline 130 \\ 115 \\ \hline 15 \end{array}$$

A $8\overline{)3.0}$	G $0.58\overline{)0.4683}$	
B $7\overline{)32.54}$	H $0.004\overline{)20.593}$	
C $0.9\overline{)42.345}$	I $9\overline{)4}$	
D $0.05\overline{)2.4730}$	J $0.41\overline{)0.3457}$	
E $6.1\overline{)31.76}$	K $2.9\overline{)1.672}$	
F $0.39\overline{)2654}$	L $52\overline{)0.3854}$	

think

Draw an array of nine dots like the one at the right. Place your pencil at one dot, and without lifting your pencil from the paper and without retracing, draw four line segments so that each dot lies on one of the segments.

More practice, page S-19, Set 34

UTILIZATION

Students should study the examples in Exercises 2 and 3 before trying the exercises. Some parts of Exercise 2 will involve annexation of zeros in the dividend in order to find the quotients. You may use the fraction form shown in Discussion Exercise 3 to help explain why we annex zeros.

EXTENSION

Additional practice exercises may be assigned from Supplementary Exercise Set 34 on page S-19, *Workbook* page 61, and *Duplicator Masters* page 40. *Arithmetic Skill Cards* D-4, D-5, and D-6 also offer appropriate practice.

Remedial: If some students have difficulty understanding division with decimals, present fraction notation in simple examples such as these:

$0.2\overline{)8} \rightarrow \frac{8}{0.2} = \frac{8 \times 10}{0.2 \times 10} = \frac{80}{2} = 40$

$0.03\overline{)0.6} \rightarrow \frac{0.6}{0.03} = \frac{0.6 \times 100}{0.03 \times 100} = \frac{60}{3} = 20$

Enrichment: Ask students to solve this decimal division problem, in which each letter stands for a digit.

$$\begin{array}{r} a.b \\ a.b\overline{)a.cd} \\ a\ b \\ \hline bd \\ bd \\ \hline \end{array}$$

Solution: $a = 1$, $b = 3$, $c = 6$, $d = 9$

Think Solution

The Think problem can be solved by trial-and-error methods. Students will discover after a few trials that some of the segments must extend beyond the array of dots.

Assignments

Minimum 1A–D, 2A–H, 3A–F.
Average 1, 2A–P, 3A–F.
Maximum 1, 2, 3G–L.

Objective

The student can read and solve word problems involving decimals and metric units.

UTILIZATION

In keeping with the philosophy of the text, the importance of problem-solving skills is again emphasized.

The Short Stories provide students with some interesting material on space and space travel. Metric units are used throughout the problem set.

EXTENSION

Enrichment: Have students find out some facts about any recent space explorations and then create their own problems based on them. They can then give the problems to other students to solve, but they must be sure that they themselves can solve each of the problems they create.

Assignments

Minimum 1–5.
Average Odd-numbered problems.
Maximum 1–13.

SPACE TRAVEL

1. Satellite orbiting Earth:
28 080 kilometres in an hour.
Averaged how many kilometres per second?

2. Rocket to the moon: About 104 kilometres in 9 seconds.
How many kilometres per second (to the nearest tenth)?

3. Diameter of Mars: About 0.53 times the diameter of Earth.
Diameter of Earth: 12 754 kilometres.
What is the diameter of Mars (to the nearest hundred kilometres)?

4. Rocket to leave Earth:
11.2 kilometres per second.
Rocket to leave Mars: 0.6 kilometres per second less than half for Earth.
How fast does the rocket travel to leave Mars?

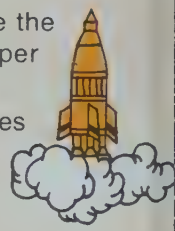
5. Objects weigh about 6 times as much on Earth as on the moon.
Boy on Earth: 42.5 kilograms.
Dog on Earth: 1.9 kilograms.
What is the total "moon weight" of the boy and dog?

6. Weight of an object on the moon is 0.16 times Earth weight.
Moon weight of astronaut:
11.6 kilograms.
What is his Earth weight?

7. Rock sample on the moon:
1.2 kilograms.
What is the Earth weight?
(See Exercise 6.)

8. A "year" on Mars: 1.9 Earth years.
How many Mars years is 18.05 Earth years?

9. Speed needed to leave the moon: 2.4 kilometres per second.



A How many kilometres per hour?

B Speed needed to leave Saturn is 14.7 times as fast as for the moon.
How fast in kilometres per hour?

10. A "day" on the moon:
About 27.3 Earth days.
A day on Saturn is 10.5 hours.
How many times greater is the moon day (to the nearest hundredth)?

11. Time needed to orbit Earth:
About 95.6 minutes.
How many complete orbits in 8 hours?

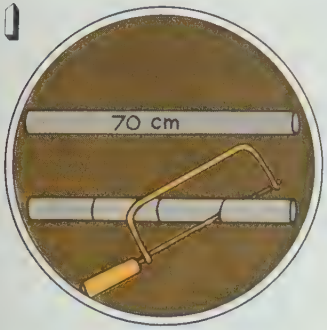
12. Weight of an object on Mars is 0.39 times its weight on Earth.
Astronaut's weight on Mars:
30.42 kilograms.
What is his Earth weight?

13. Steak: 1.12 kilograms.
Potatoes: 4.5 kilograms.
Flour: 2.25 kilograms.
Grapes: 1 kilogram.
Find the total "Mars weight" of the groceries.
(See Exercise 12.)

Wordless Problems

Each picture suggests a problem. Study the picture carefully.
Then write and solve your own problem for the picture.

1



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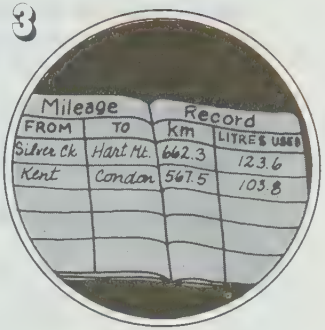

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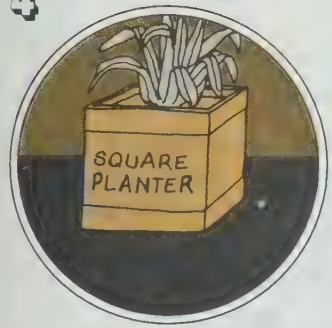

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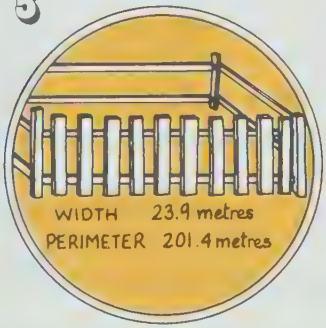

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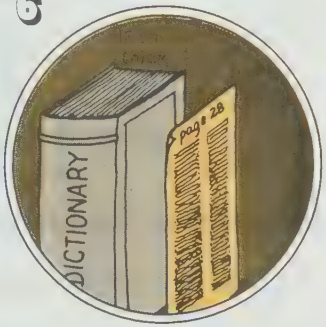

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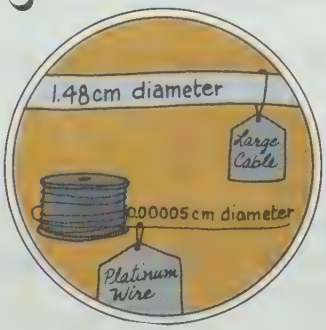


9

Objective

Given a picture or diagram, the student can write a word problem for it and solve the problem.

UTILIZATION

The Wordless Problems provide the students with an opportunity to construct their own problems from given situations.

Sample Problems and Solutions

1. A pipe 70 cm long is cut into 4 pieces of the same length. How long is each piece? (17.5 cm)
2. If a car is driven from the city to the lake, a distance of 260 km, in 3 hours, what is the average speed? (86.7 km/h, to nearest tenth.)
3. What was the total distance of two trips, one of 662.3 km and the other of 567.5 km? (1229.8 km)
4. What is the length of one side of a square whose perimeter is 209.6 cm? (52.4 cm)
5. A field with perimeter 201.4 metres has a width of 23.9 metres. What is its approximate length? (76.8 metres)
6. A page from a dictionary is about 0.008 cm thick. The dictionary is about 12 cm thick. About how many pages does the dictionary have? (About 1500)
7. After sailing 448 km, the Flynn family found that they had traveled about 0.7 of their trip. How far was their trip? (640 km)
8. One wire has a diameter of 1.48 cm and a second fine wire has a diameter of 0.00005 cm. How many times greater is the diameter of the first wire than that of the second? (29 600 times)
9. It is 152 km to Toledo and the speed limit is 95 km/h. How long will it take to drive to Toledo at the maximum speed? (1.6 h)

EXTENSION

Page 41 of the *Duplicator Masters* provides problems that would be appropriate for additional practice.

Assignments

Minimum 1, 2, 4, 5. Average 1–6.
Maximum 1–9.

Objective

Given a fraction, the student can write an equivalent decimal for the fraction.

PREPARATION

Review the method of finding fractions for decimals and finding decimals for single fractions such as $\frac{3}{4}$, $\frac{2}{5}$, and $\frac{9}{10}$. If you desire, you can move immediately into the Investigation and discuss the two methods shown for finding a decimal for $\frac{3}{8}$.

INVESTIGATION

After the students have studied the two methods of finding a decimal for $\frac{3}{8}$, give them sufficient time to try to find decimals for the three fractions in the question box. The first two fractions can be expressed as decimals by using the multiplication method:

$$\frac{7}{20} = \frac{7 \cdot 5}{20 \cdot 5} = \frac{35}{100} = 0.35;$$

$$\frac{11}{25} = \frac{11 \cdot 4}{25 \cdot 4} = \frac{44}{100} = 0.44$$

For $\frac{13}{16}$, most students will probably use the division method, although

$$\frac{13}{16} = \frac{13 \cdot 625}{16 \cdot 625} = \frac{8125}{10\,000} = 0.8125.$$

The difficulty with the multiplicative method for $\frac{13}{16}$ is that it is difficult for most students to discover the proper number, 625, by which to multiply both numerator and denominator.

The method used by most of your students will probably be the division method, in which the numerator is divided by the denominator. This is an acceptable method, but you should help students understand why it works.

DISCUSSION

Work through the Discussion Exercises as a class activity. As you do so, be sure to bring out points such as the following: 0.33 is not exactly $\frac{1}{3}$, but it is the decimal representation of $\frac{1}{3}$ to the nearest hundredth. Demonstrate this fact by dividing 1 by 3.

Writing Decimals for Fractions

Investigating the Ideas

When you are given a decimal, it is usually easy to write a fraction that names the same number. But, when you are given a fraction whose denominator is not a power of ten, the problem is more difficult.

Study the two methods below.

Decimal		Fraction
0.17	⇒	$\frac{17}{100}$
?	⇐	$\frac{3}{8}$

MULTIPLICATION METHOD

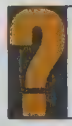
$$\frac{3}{8} = \frac{3 \cdot 125}{8 \cdot 125} = \frac{375}{1000}$$

Therefore, $\frac{3}{8} = \frac{375}{1000} = 0.375$

DIVISION METHOD

$$\frac{3}{8} = 3 \div 8 = 8\overline{)3.000}$$

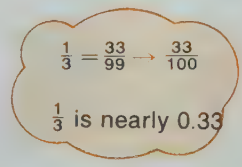
$$\begin{array}{r} 0.375 \\ 8\overline{)3.000} \\ \underline{24} \\ 60 \\ \underline{56} \\ 40 \\ \underline{40} \\ 0 \end{array}$$



Can you use one of these methods to find decimals which name the same number as $\frac{7}{20}$, $\frac{11}{25}$, and $\frac{13}{16}$?

Discussing the Ideas

1. Explain how to use each of the methods to find a decimal for $\frac{4}{5}$.
2. Explain how to use each of the methods to find a decimal for $\frac{7}{8}$.
3. Which method do you think is easier?
4. Explain how to use each method to find the decimal for $\frac{3}{25}$.
5. This picture shows how Mike thought in order to find an approximate decimal for $\frac{1}{3}$. Do you think Mike has found a decimal that represents the same number as the fraction $\frac{1}{3}$? Explain your answer.



6. Can you use either method shown above to find a decimal that represents the same number as the fraction $\frac{1}{3}$? Explain your answer.

Using the Ideas

1. Give the number for n. Then give the correct decimal.

A $\frac{23}{50} = \frac{n}{100} = ?$ **D** $\frac{23}{200} = \frac{n}{1000} = ?$ **G** $\frac{3}{4} = \frac{n}{100} = ?$

B $\frac{4}{5} = \frac{n}{10} = ?$ **E** $\frac{9}{500} = \frac{n}{1000} = ?$ **H** $\frac{17}{125} = \frac{n}{1000} = ?$

C $\frac{7}{25} = \frac{n}{100} = ?$ **F** $\frac{64}{250} = \frac{n}{1000} = ?$ **I** $\frac{23}{80} = \frac{n}{10\,000} = ?$

2. Find the quotients. In each problem, continue dividing until the remainder is zero. Then give the correct decimal for the fraction.

A $8\overline{)5} \rightarrow \frac{5}{8} = ?$ **C** $2\overline{)27} \rightarrow \frac{27}{2} = ?$ **E** $40\overline{)9} \rightarrow \frac{9}{40} = ?$

B $25\overline{)6} \rightarrow \frac{6}{25} = ?$ **D** $16\overline{)1} \rightarrow \frac{1}{16} = ?$ **F** $80\overline{)7} \rightarrow \frac{7}{80} = ?$

3. Write a decimal for each number. Use any method you choose.

A $\frac{3}{5}$	**E** $\frac{7}{20}$	**I** $\frac{29}{40}$	**M** $\frac{7}{200}$	**Q** $\frac{372}{500}$
B $\frac{1}{4}$	**F** $\frac{19}{25}$	**J** $\frac{43}{80}$	**N** $\frac{9}{400}$	**R** $\frac{98}{200}$
C $\frac{7}{8}$	**G** $\frac{27}{50}$	**K** $\frac{99}{150}$	**O** $\frac{569}{500}$	**S** $\frac{120}{150}$
D $\frac{11}{16}$	**H** $\frac{3}{2}$	**L** $\frac{147}{160}$	**P** $\frac{4}{25}$	★ **T** $\frac{964}{1250}$

★ **4.** Study the examples. Then write a decimal that represents a number equal to or approximately equal to the number given. The symbol $\approx$ means "is approximately."

EXAMPLES: **A** $\frac{37}{99} \approx \frac{37}{100} = 0.37$

B $\frac{56}{300} = \frac{56 \div 3}{300 \div 3} \approx \frac{19}{100} = 0.19$

C $\frac{43}{65} = \frac{4300}{6500} = \frac{4300 \div 65}{6500 \div 65} \approx \frac{66}{100} = 0.66$

A $\frac{17}{50}$	**F** $\frac{73}{999}$	**K** $\frac{17}{90}$	**P** $\frac{4672}{6000}$	**U** $\frac{79}{101}$
B $\frac{7}{99}$	**G** $\frac{63}{501}$	**L** $\frac{412}{2000}$	**Q** $\frac{72}{13}$	**V** $\frac{38}{499}$
C $\frac{12}{200}$	**H** $\frac{41}{300}$	**M** $\frac{614}{1001}$	**R** $\frac{1}{13}$	**W** $\frac{840}{5000}$
D $\frac{23}{600}$	**I** $\frac{41}{700}$	**N** $\frac{46}{250}$	**S** $\frac{27}{75}$	**X** $\frac{63}{99}$
E $\frac{7}{49}$	**J** $\frac{11}{30}$	**O** $\frac{217}{3000}$	**T** $\frac{14}{37}$	**Y** $\frac{19}{33}$

★ **5.** In each exercise, solve the equation for n. Then find a fraction for x.

A $7 \times 0.42857 = n$, $0.42857 \approx x$ **B** $19 \times 0.8421 = n$, $0.8421 \approx x$ **C** $9 \times 0.22222 = n$, $0.22222 \approx x$

More practice, page S-20, Set 35

UTILIZATION

All of the fractions in Exercises 1, 2, and 3 have terminating decimal representations; that is, if the division method is used to find a decimal for the fraction, a remainder of zero will eventually occur. A fraction can be represented by a terminating decimal only when there is a fraction having a denominator of 10, 100, 1000, and so forth, in the set of equivalent fractions for the given fraction. It is easy to determine whether or not such a fraction exists within a set of equivalent fractions by examining the denominator of the lowest-terms fraction. This denominator must contain only factors of 2, 5, or both in its complete factorization. This is clear when you consider that the numbers 10, 100, 1000, 10 000 and so forth contain only twos and fives as factors. Therefore, if the denominator has a factor of 3, for example, the set of equivalent fractions will not contain one having a denominator of 100.

Point out the introduction of the symbol for "approximately" ($\approx$) in Exercise 4. This symbol will be used frequently in the future, so the student should become familiar with it.

EXTENSION

Additional practice exercises are provided in Supplementary Exercise Set 35 on page S-20, on *Workbook* page 62, on *Duplicator Masters* page 42, and on *Arithmetic Skill Cards* D-7, D-8, and D-9.

Enrichment: Give additional practice in estimating decimals for fractions like those in Exercise 4:

A. $\frac{11}{23}$ B. $\frac{27}{101}$ C. $\frac{3}{51}$ D. $\frac{49}{99}$ E. $\frac{4}{15}$
F. $\frac{1}{11}$ G. $\frac{11}{32}$ H. $\frac{56}{101}$ I. $\frac{4}{200}$ J. $\frac{33}{330}$

Assignments

Minimum 1A–F, 2A–D, 3A–H.
Average 1, 2, 3A–P.
Maximum 1–5, alternate parts.

Objective

Given a fraction, the student can represent it by a repeating decimal.

PREPARATION

Since this lesson is closely related to the previous lesson, little preparation is necessary. If you wish, however, review the division method for finding a decimal for a fraction.

INVESTIGATION

Allow time for the students to study the two examples in the Investigation and to find the decimals for the fractions.

You may expect that some students will not easily see the repeating pattern for $\frac{1}{7}$ because its decimal has a *period* of 6; that is, the decimal goes six places before repeating itself. The remainder for each step is a different number and less than 7, hence the division process leads to a previous remainder after 6 steps.

DISCUSSION

The important mathematical concept associated with this section is that each fractional number can be represented by a repeating decimal and, conversely, each repeating decimal represents some fractional number. The converse statement is not discussed in this book.

It is convenient to have your students understand the two different types of notation used for repeating decimals. Three dots may be used to show that the indicated pattern continues to repeat, or a bar over the digits may show the repeating period. Note also that when a decimal terminates, it can be thought of as a decimal which repeats zeros. Thus,

$\frac{1}{4} = 0.25 = 0.250 = 0.25000 \ldots = 0.250$

Repeating Decimals

Investigating the Ideas

The decimal for $\frac{3}{4}$ **terminates.**

$\frac{3}{4} = 0.75$

$$\begin{array}{r} 0.75 \\ 4\overline{)3.00} \\ \underline{28} \\ 20 \\ \underline{20} \\ 0 \end{array}$$

The decimal for $\frac{6}{11}$ **repeats** endlessly!

$$\begin{array}{r} 0.5454\ldots \\ 11\overline{)6.0000\ldots} \\ \underline{55} \\ 50 \\ \underline{44} \\ 60 \\ \underline{55} \\ 50 \\ \underline{44} \\ 6 \\ \vdots \end{array}$$

Can you use division to show which of the fractions have **terminating** or **repeating** decimals?

A. $\frac{1}{4}$ B. $\frac{5}{6}$ C. $\frac{7}{8}$ D. $\frac{4}{7}$

Discussing the Ideas

1. If the decimal for a fraction terminates, what must be the last remainder in the division process?

2. If a fraction has a repeating decimal representation, what can you say about the remainders in the division process after the repeating part begins?

3. **A** Study the example for finding a decimal for $\frac{1}{3}$. Will the remainder ever be zero?

Dividing tenths	Dividing hundredths	Dividing thousandths
$0.3, \text{R } 1$	$0.33, \text{R } 1$	$0.333, \text{R } 1$
$3\overline{)1.0}$	$3\overline{)1.00}$	$3\overline{)1.000}$ . . .

B What kind of decimal names the fraction for $\frac{1}{3}$?

4. The three dots in 0.33. . . indicate that the threes repeat indefinitely. Another way of representing a repeating decimal is to write a bar over the group of digits that repeats.

$\frac{1}{3} = 0.33\ldots = 0.\overline{3}$ $\quad \frac{5}{11} = 0.454545\ldots = 0.\overline{45}$ $\quad \frac{1}{27} = 0.037037\ldots = 0.\overline{037}$

Can you use both the dot and the bar notations to express the repeating decimals that you found in the Investigation?

Using the Ideas

1. Use division to find the repeating decimal that represents each fractional number.

A $\frac{2}{3}$	**C** $\frac{7}{11}$	**E** $\frac{5}{27}$	**G** $\frac{4}{11}$	**I** $\frac{7}{15}$	**K** $\frac{1}{7}$	**M** $\frac{5}{12}$
B $\frac{1}{6}$	**D** $\frac{1}{9}$	**F** $\frac{5}{6}$	**H** $\frac{2}{37}$	**J** $\frac{5}{13}$	**L** $\frac{5}{9}$	☆ **N** $\frac{5}{17}$

2. Round each repeating decimal that you found in Exercise 1 to the nearest hundredth.

3. Use bar notation to represent each repeating decimal.

A 0.232323 . . .	**D** 0.20672067 . . .	**G** 0.010101 . . .
B 0.4122222 . . .	**E** 63.7777 . . .	**H** 0.434343 . . .
C 0.617617 . . .	**F** 5.61222222 . . .	**I** 0.70517051 . . .

4. Use the "three dot" notation to indicate these repeating decimals.

A $0.\overline{7}$	**C** $7.4\overline{8}$	**E** $0.8\overline{3}$	**G** $0.\overline{7892}$	**I** $0.\overline{46}$
B $0.\overline{21}$	**D** $0.2\overline{13}$	**F** $56.24\overline{9}$	**H** $3.\overline{0}$	**J** $8.6\overline{3}$

5. Find a repeating decimal for each fraction.

A $\frac{1}{9}$	**B** $\frac{1}{99}$	**C** $\frac{1}{999}$	**D** $\frac{1}{9999}$	**E** $\frac{1}{99\,999}$

6. Use the results of Exercise 5 to give a repeating decimal for each fraction. Try to do this without dividing.

A $\frac{7}{9}$	**B** $\frac{8}{99}$	**C** $\frac{5}{999}$	**D** $\frac{2}{9999}$	**E** $\frac{1000}{99\,999}$

7. Write a fraction that represents the same number as $0.\overline{347}$. Check by dividing.

8. Find the repeating decimal that represents $\frac{1}{81}$. What interesting fact do you notice about this decimal?

think

In the magic square to the right, the sum of each row, column, diagonal, and the four corners is the same.

1. To complete the magic square, use each of the numbers 1 through 16 just once.

2. What is the sum?

1			15
	13		6
7		16	9
14	11		

UTILIZATION

Permit the students to use either the bar notation or the dot notation in expressing their answers to Exercise 1. Note the relation between exercises 5 and 6. Students should observe that $\frac{1}{9} = 0.111\ldots$, then $\frac{7}{9} = 7 \cdot \frac{1}{9}$, so $\frac{7}{9} = 0.777\ldots$. Similarly, since $\frac{1}{99} = 0.010101\ldots$, $\frac{8}{99} = 8 \cdot \frac{1}{99} = 0.080808\ldots$.

From these examples students can generalize so that in Exercise 7, they will see that $0.\overline{347}$ must have a fractional representation of $\frac{347}{999}$. This can be verified by carrying out the division indicated by the fractions.

EXTENSION

To provide more practice, choose assignments from *Workbook* page 63, *Duplicator Masters* page 43, and *Arithmetic Skill Card* D-8.

Enrichment: Ask students to find some sums using repeating decimals and then use fractions to check the problems. Sample problems:

A. $0.11111\ldots = \frac{1}{9}$
$+0.11111\ldots = +\frac{1}{9}$

B. $0.44444\ldots = \frac{4}{9}$
$+0.22222\ldots = +\frac{2}{9}$

C. $0.3333\ldots = \frac{1}{3}$
$+0.0202\ldots = +\frac{2}{99}$

D. $0.777777\ldots = \frac{7}{9}$
$+0.002002\ldots = \frac{2}{999}$

Think Solution

The problem will be solved by trial-and-error methods. A good place to begin is the square in the lower right hand column. The only numbers possible for this square are 2, 3, 4, 5, 8, 10, and 12. Only the number 4 in this square is possible. Hence the magic sum is found to be 34 and the remaining squares can be filled in quickly.

Assignments

Minimum 1A–H, 2A–H, 3A–E, 4A–D.
Average 1–6. Maximum 1–8.

Objective

Given an appropriate fraction, the student can write an equivalent mixed decimal numeral and conversely.

PREPARATION

This lesson is closely related to the previous lesson. You might again review the division process of finding a decimal for a fraction before beginning the Investigation.

INVESTIGATION

An effective way to introduce the Investigation is to go through the step-by-step development for each of the three mixed numerals in the Investigation. In the first step, note that the fraction is expressed in lowest terms $\frac{5}{12} = 0.4\frac{2}{12} = 0.4\frac{1}{6}$. Similarly, lowest-terms fractions were used in expressing the other two mixed numerals. After carrying out the division to thousandths, ask the students what would be the mixed numeral in ten thousandths for $\frac{5}{12}$. From their work with repeating decimals, they should see that it would be $0.4166\frac{2}{3}$.

DISCUSSION

The Discussion Exercises bring out the converse of the problem given in the Investigation. Thus, given a mixed decimal numeral, students are to find the common fraction for the decimal. As an example other than the one given in Exercise 3, you might use one of the mixed numerals in the Investigation:

$$0.4\frac{1}{6} = \frac{4\frac{1}{6}}{10} = \frac{25}{6} \div 10 = \frac{25}{6} \cdot \frac{1}{10} = \frac{25}{60} = \frac{5}{12}$$

Work through the example in Exercise 3, and then have the students try to find the fraction for $.12\frac{1}{2}$ in this manner:

$$0.12\frac{1}{2} = \frac{12\frac{1}{2}}{100} = \frac{25}{2} \div 100 = \frac{25}{2} \cdot \frac{1}{100} = \frac{25}{200} = \frac{1}{8}$$

The number line interpretation of mixed decimal numerals in Exercise 4 is designed to give students a less abstract notion of such numerals.

Mixed Decimal Numerals

Investigating the Ideas

Often a combination of a decimal and a fraction called a mixed decimal numeral is written for a fraction. The examples in the table show how you can find several mixed decimal numerals for the fraction $\frac{5}{12}$.

$12\overline{)5.0}$ → 0.4, 48, 2 $\frac{5}{12} = 0.4\frac{1}{6}$	$12\overline{)5.00}$ → 0.41, 48, 20, 12, 8 $\frac{5}{12} = 0.41\frac{2}{3}$	$12\overline{)5.000}$ → 0.416, 48, 20, 12, 80, 72, 8 $\frac{5}{12} = 0.416\frac{2}{3}$
4 and $\frac{1}{6}$ **tenths**	41 and $\frac{2}{3}$ **hundredths**	416 and $\frac{2}{3}$ **thousandths**

Can you give mixed decimal numerals for $\frac{5}{7}$ in **tenths**, **hundredths**, and **thousandths**?

Discussing the Ideas

1. What mixed decimal numeral in tenths does the division problem suggest?

$$13\overline{)8.0} \rightarrow 0.6,\ 7.8,\ 2 \qquad \frac{8}{13} = 0.$$

2. How could you find a mixed decimal numeral in hundredths for $\frac{8}{13}$?

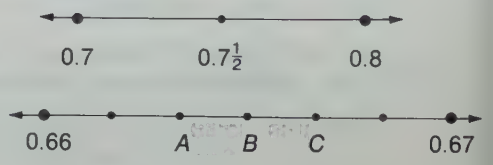
3. The example shows how you can find a fraction for a mixed decimal numeral.

   Explain how you could find a fraction for $0.12\frac{1}{2}$.

EXAMPLE:

$$0.3\frac{1}{4} = \frac{3\frac{1}{4}}{10} = \frac{13}{4} \div 10$$
$$= \frac{13}{4} \cdot \frac{1}{10}$$
$$= \frac{13}{40}$$

4. You can think of $0.7\frac{1}{2}$ as a name for the point on the number line that is halfway between the points for 0.7 and 0.8. Can you give mixed decimal numerals for points *A*, *B*, and *C*?



Using the Ideas

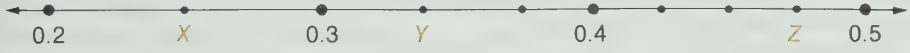
1. Write a mixed decimal numeral for each number name.
   - **A** 7 and $\frac{2}{3}$ tenths
   - **B** 16 and $\frac{1}{4}$ hundredths
   - **C** 37 and $\frac{1}{2}$ thousandths
   - **D** 66 and $\frac{2}{3}$ hundredths
   - **E** 785 and $\frac{1}{5}$ thousandths
   - **F** 83 and $\frac{5}{6}$ hundredths

2. Copy and complete the table.

	FRACTION	MIXED DECIMAL NUMERALS tenths	hundredths	thousandths
	$\frac{1}{3}$	$0.3\frac{1}{3}$	$0.33\frac{1}{3}$	$0.333\frac{1}{3}$
A	$\frac{5}{6}$			
B	$\frac{3}{16}$			
C	$\frac{4}{9}$			
D	$\frac{13}{6}$			
E	$\frac{3}{7}$			
F	$\frac{5}{18}$			

3. Write a fraction for each mixed decimal numeral.
   A $0.1\frac{1}{4}$ **B** $0.06\frac{1}{2}$ **C** $0.83\frac{1}{3}$ **D** $0.66\frac{2}{3}$ **E** $0.031\frac{3}{4}$ **F** $0.37\frac{1}{2}$

4. Give a mixed decimal numeral for each lettered point on the number line.

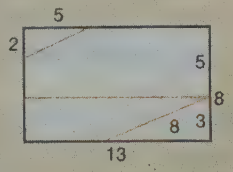


5. Write a fraction for each mixed decimal numeral of Exercise 4.

think

Cut out an 8 by 13-cm rectangle and cut it into pieces as shown in the figure. Reassemble the pieces to form what seems to be a 5 by 21 cm rectangle.

1. What is the area of the 8 by 13-cm rectangle?
2. What is the area of the 5 by 21-cm rectangle?
3. Can you explain where the extra square centimetre is?



More practice, page S-21, Set 36

UTILIZATION

Assign the exercises according to the ability of your students. Those students needing additional practice should be directed to Supplementary Exercise Set 36 on page S-21.

EXTENSION

Pose this question for the students: "If the fractional part of a mixed decimal numeral is $\frac{1}{2}$, how can you write the numeral as a terminating decimal?" Give an example such as $0.7\frac{1}{2}$. Students can solve this problem as follows:

$$0.7\frac{1}{2} = \frac{7}{10} + \frac{\frac{1}{2}}{10} = \frac{7}{10} + \frac{1}{20} = \frac{7}{100} + \frac{5}{100} = \frac{75}{100} = 0.75$$

Alternately,

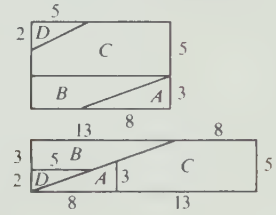
$$0.7\frac{1}{2} = \frac{7\frac{1}{2}}{10} = \frac{15}{2} \cdot \frac{1}{10} = \frac{15}{20} = \frac{3}{4} = 0.75$$

Thus, $\frac{1}{2}$ in a mixed decimal numeral can always be replaced by the digit 5 in the next decimal place. Similarly, $0.6\frac{1}{4} = 0.625$, $0.08\frac{1}{4} = 0.0825$, etc.

Arithmetic Skill Card D-10 would be appropriate for use in reinforcing these ideas.

Think Solution

The pieces do not quite fit together to make a rectangle that is 5×21. However, they come so close to forming a 5×21 rectangle that it is somewhat astonishing that the area of the two rectangles differs by 1 square unit. It can be proved mathematically that the pieces of an 8 by 13 rectangle cannot be reassembled to form a 5 by 21 rectangle, but it is sufficient if students see that if they cut very carefully, the pieces will not fit exactly to form a rectangle. If the pieces are labeled as below, it might be easier to explain the reassembling.



Assignments

Minimum 1–3. Average 1–4. Maximum 1–5.

Objective

Given a decimal for a number that is greater than 1, the student can represent the number in scientific notation.

PREPARATION

Scientific notation was introduced in Unit A on page A-10, in the work with whole numbers. You could introduce this lesson by reviewing the use of scientific notation with simple multiples of powers of ten such as 600, 8000, 9 000 000, and so on.

UTILIZATION

Discuss the material in the table with the students. Point out that just as 800 can be written in scientific notation as 8×10^2 or 8.00×10^2, so also can we write 864 in scientific notation by placing the decimal point in the numeral 864 so that it represents a number between 1 and 10 and then multiplying it by the appropriate power of 10. Discuss the other examples in the table, and then assign the exercises.

When the students have finished, have the answers checked and have troublesome problems placed on the chalkboard for discussion.

EXTENSION

Additional practice in scientific notation is provided by Supplementary Exercise Set 37 on page S-21, *Duplicator Masters* page 44, and *Arithmetic Skill Cards* AP-3 and AP-4.

Assignments

Minimum 1A–E, 2A–D, 4A–D.
Average 1A–G, 2, 3A–E, 4.
Maximum 1–4.

scientific notation—DECIMALS

A number other than a power of ten is represented using **scientific notation** when it is written as the product of a **number between 1 and 10 and a power of ten**. In the table, each number in Column A is represented in Column B using scientific notation.

A Number		B Number between 1 and 10	×	Power of ten
864	=	8.64	×	10^2
3785.3	=	3.7853	×	10^3
57	=	5.7	×	10^1
89600	=	8.96	×	10^4

For a power of ten such as 1 000 000, we might write 1×10^6

1. Find the number for ***n*** in each row of the table. →

Number		Number between 1 and 10	×	Power of ten
A 37	=	***n***	×	10^1
B 467	=	4.67	×	***n***
C 6843	=	***n***	×	10^3
D ***n***	=	4.756	×	10^3
E 4800	=	***n***	×	10^3
F 96 000	=	9.6	×	***n***
G ***n***	=	5.83726	×	10^2
H 9463.21	=	***n***	×	10^3
I 580 000	=	5.8	×	***n***
J ***n***	=	3.7	×	10^6

2. Represent the numbers in scientific notation.
   - A 700
   - B 900
   - C 8000
   - D 70 000
   - E 600 000
   - F 100 000

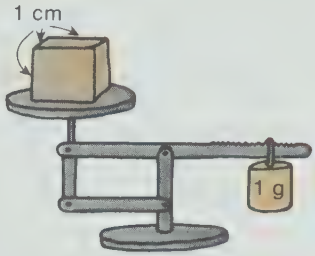
3. Represent the numbers in scientific notation. Each number is a period of time given in seconds.
   - A Hour: 3600
   - B Day: 86400
   - C Month: 2 550 000
   - D Solar year: 31 556 900
   - E Average life of man: 2 000 000 000
   - F Revolution of Pluto about the sun: 7 820 000 000
   - G Half-life of radium 226: 52 400 000 000
   - H One rotation of the Milky Way: 6 000 000 000 000 000

4. Write each numeral in the usual way.
   - A 4.32×10^3
   - B 5.4×10^4
   - C 6.16×10^2
   - D 7.0×10^5
   - E 8.2×10^{10}
   - F 1.53×10^8
   - G 5.9×10^7
   - H 1.02×10^0

More practice, page S-21, Set 37

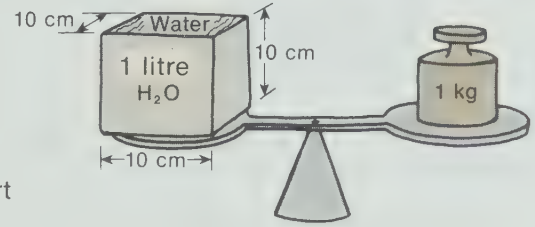
Comparing Units within the Metric System

In the metric system there is an important relationship between some of the units for length (metre), mass (gram), and volume (litre). The problems below deal with this relationship.



1. The mass of a cube of water that is 1 centimetre on each edge (1 cubic centimetre) is 1 gram. You probably drink about 1800 cubic centimetres of water each day. How many grams of water is this?

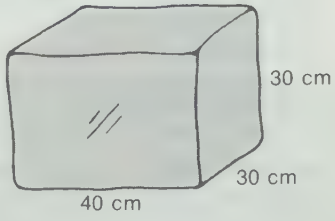
2. Use the diagram at the right to answer the questions.
   - **A** How many grams is in a litre of water (H_2O)?
   - **B** How many cubic centimetres are in a litre?
   - **C** One gram of water is what part of a litre of water?



3. A cubic centimetre is 1 millilitre (ml); that is 0.001 litre. What is the mass in grams of 25 millilitres of water?

4. The mass of a given volume of sea water is about 1.03 times that of an equal volume of fresh water. What is the mass in grams of one litre of sea water?

5. The mass of a given volume of gasoline is only 0.68 times that of an equal volume of water.
   - **A** How many grams is one litre of gasoline?
   - **B** How many litres of gasoline would be needed to make one kilogram?



6. Ice has a mass that is about 0.92 times that of an equal volume of water. What is the mass of the block of ice shown?

7. Gold is about 19.3 times as massive as water. How many millilitres (to the nearest tenth) of gold would have the same mass as one litre of water?

Objective

The student can solve problems involving comparisons of metric units.

UTILIZATION

Metric units of length, mass, and volume have been used frequently in other lessons of this module and in earlier modules. This lesson shows the close relationship between these three units.

A gram is the mass of 1 cubic centimetre of water at standard pressure and temperature. Since a litre contains 1000 cubic centimetres, one gram of water is 0.001 of a litre. This fundamental relationship between the units makes conversion from one unit to another quite simple. For example, given 250 grams of water, its mass is 250 grams and its volume is 250 cubic centimetres, or 250 millilitres or 0.25 litres.

If possible, have standard metric units available for students to see and use in connection with this lesson. Students may be interested in facts such as that a Canadian nickel weighs about 5 grams and a half dollar weighs about 12.5 grams.

Medical dosages are often given in terms of cubic centimetres, which is symbolized by cm^3.

Assignments

Minimum 1–3. Average 1–6. Maximum 1–7.

Objective

The student can demonstrate the ability to work with the concepts presented in this module.

UTILIZATION

Many of the review exercises can be effectively used as discussion exercises. Other problems, which are computational in nature, could be assigned to selected students and solutions could be placed on the chalkboard and discussed.

Following the completion of the review exercises, the self-evaluation test could be taken by the student as preparation for a module achievement test.

More module review exercises are provided on page 64 of the *Workbook*.

REVIEWING THE IDEAS

1. In the decimal **0.347516**, which digit tells how many

- **A** hundredths
- **B** tenths
- **C** millionths
- **D** ten thousandths
- **E** thousandths
- **F** hundred thousandths

2. Write a decimal for $5 + \frac{376}{1000}$.

3. Write a mixed numeral for 72.0936.

4. Write a decimal for $7 \cdot 100 + 5 \cdot 10 + 3 \cdot 1 + 2 \cdot \frac{1}{10} + 8 \cdot \frac{1}{100}$

5. Write 26.3745 in expanded notation (exponent form).

6. Give the correct symbol ($<$, $=$, or $>$) for each ▒.

- **A** 0.9 ▒ 0.900
- **B** 3.09 ▒ 3.90
- **C** 6.2370 ▒ 6.0237
- **D** 7.099 ▒ 7.10

7. Find the sums.

- **A** $2.6 + 7.1 + 7.39$
- **B** $3.7 + 59.642 + 463.24 + 0.803$

8. Find the differences.

- **A** $97.5 - 36.9$
- **B** $23.7 - 9.341$

9. Find the products.

- **A** 0.7×4.9
- **B** 0.03×0.0057
- **C** 42.8×0.006
- **D**
$$\begin{array}{r} 3.59 \\ \times\ 4.3 \\ \hline \end{array}$$

10. Find the products.

- **A** 10×5.736
- **B** 100×0.856
- **C** $10^3 \times 5.428$
- **D** 0.1×87.54
- **E** $\frac{1}{10^2} \times 956$
- **F** 0.001×87.6

11. Round **57.5649** to the nearest

- **A** whole number
- **B** tenth
- **C** hundredth
- **D** thousandth

12. **A** Estimate the product 7.96×49.843.

B Find the product in part **A**. How close was your estimate?

13. Find the quotients.

A $8\overline{)2.776}$ **B** $0.7\overline{)0.526}$ **C** $0.04\overline{)1.9}$

14. Use the information in the chart to answer the following questions.

Day	Kilometres travelled	Litres used	Cents per litre
Sat.	477.6	53.1	11.9
Sun.	256.4	30.5	11.9

- **A** Find the total number of kilometres travelled.
- **B** Find the total number of litres used.
- **C** What was the total cost of the gasoline for the weekend?
- **D** How many kilometres per litre (to the nearest tenth) did the car average on Saturday?

15. Write a decimal for each number.

A $\frac{4}{5}$ **B** $\frac{17}{20}$ **C** $\frac{9}{32}$

16. **A** Represent $\frac{9}{11}$ as a repeating decima

B Round the decimal in part **A** to the nearest hundredth.

17. Give a mixed decimal numeral in hundredths for $\frac{4}{7}$.

18. The distance from the Earth to the sun is 149 000 000 kilometres. Expres this distance in scientific notation.

19. Give the correct number for each ▒.

- **A** 1 litre = ▒ millilitres
- **B** 1 millilitre of water weighs ▒ gram
- **C** 10 millimetres = ▒ centimetre
- **D** 1 litre of water weighs ▒ kilogram

TEST YOURSELF

1. **A** Write a fraction for 0.217.
   B Write a decimal for $5\frac{37}{100}$.

2. Give the correct symbol ($<$, $=$, or $>$) for each ◯.
   A $3.732 \bigcirc 3.722$ **B** $0.07 \bigcirc 0.01$

3. Find the sum $2.78 + 16.5 + 4.295$

4. Find the differences.
   A $8.62 - 1.95$ **B** $3.1 - 1.92$

5. Which is the best estimate for the product 38.705×19.1?
   A 550 **B** 650 **C** 750

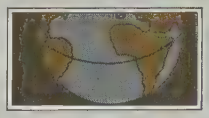
6. Find the products.
   A 0.02×1.6
   B 100×27.4
   C 0.001×37429
   D 42.3×0.29

7. Find the quotients.
   A $7\overline{)2.24}$ **B** $0.54\overline{)0.4212}$

8. Find a repeating decimal for $\frac{3}{11}$.

9. Express $\frac{9}{16}$ as a mixed decimal numeral in hundredths.

10. The equator of the Earth is about 40000 kilometres long. This is about 3.14 times as long as the diameter of the Earth.



   A What is the diameter of the Earth? Round your answer to the nearest hundred kilometres.
   B Express your rounded answer in part **A** in scientific notation.

RESEARCH PROJECTS

A A micrometer is an instrument used measuring small dimensions. Find a micrometer and see if you can figure out how to use it. Then use it to measure the following items:
   A The thickness of a pencil.
   B The thickness of a sheet of notebook paper.
   C The thickness of a hair.

B Learn how to multiply and divide with decimals using a slide rule. Most slide rules come with a pamphlet explaining their use.

C Find out some of the history behind the metric system of measurement. Prepare a display of various metric units. Sufficiently emphasize how the prefixes of the metric units are used.

D Figure out how "decimals" are used in bases other than ten and be able to explain their use to your class.

For example, $3.2_{(6)} = 3\frac{1}{3}_{(\text{ten})}$

$1.23_{(5)} = 1.52_{(\text{ten})}$

TEST YOURSELF

This self-evaluation test should give students an adequate assessment of their understanding and skill in use of decimals as presented in this module.

Objective and test item correlations are shown by annotations on the student book page.

RESEARCH PROJECTS

For Research Project A, shop micrometers and instructional materials might be obtainable from the industrial arts department of your school. Micrometer manufacturers are often glad to supply informative materials concerning the use of the instruments.

Research Project B would be an excellent project to carry out in conjunction with the lessons concerning estimation with decimals. Most products and quotients found by means of a slide rule are approximations, and facility in use of a slide rule requires placement of the decimal point by mentally estimating the answer.

Information for Project C can be found in encyclopedias, government publications, and other reference materials.

Note that in Research Project D the use of the word "decimals" in reference to bases other than ten is a misnomer because "decimal" refers to ten. However, we can use this "point" notation in other bases. For example:

$4.32_{(6)}$ means $4 + \frac{3}{6} + \frac{2}{6^2}$, or

in base ten $4 + \frac{20}{36} = 4\frac{5}{9} = 4.\overline{5}$.

And $1.23_{(5)} = 1.52_{(\text{ten})}$ because

$1.23_{(5)} = (\text{base ten})\ 1 + \frac{2}{5} + \frac{3}{25} = 1 + \frac{13}{25}$

$= 1 + \frac{52}{100} = 1.52.$

MATHEMATICAL RECREATION

This imaginary operation, $\overset{2}{+}$, is not difficult to understand. To find a "squm," simply square each addend and then find their sum, as in the examples below.

$2 \overset{2}{+} 1 = 2^2 + 1^2 = 4 + 1 = 5$

$2 \overset{2}{+} 4 = 2^2 + 4^2 = 4 + 16 = 20$

$10 \overset{2}{+} 10 = 10^2 + 10^2 = 100 + 100 = 200$

Answers

1. A. 10 C. 98 E. 2 G. 40
   B. 41 D. 850 F. 481 H. 149
2. A. Yes
   B. No, $(2 \overset{2}{+} 1) \overset{2}{+} 3 = 34$ and $2 \overset{2}{+} (1 \overset{2}{+} 3) = 104$
   C. No, $8 \overset{2}{+} 0 = 64$
   D. Commutativity principle holds true. Associativity and zero do not.
3. A. 60
   B. 180
   C. No, $3 \cdot (2 \overset{2}{+} 4) = 60$ and $(3 \cdot 2) \overset{2}{+} (3 \cdot 4) = 180$
   D. No
4. A. $n = 5$ B. $x = 1$ C. $y = 5$
5. Answers will vary.

MATHEMATICAL RECREATIONS

THE $\overset{2}{+}$ OPERATION

You are familiar with the four operations, **addition, multiplication, subtraction,** and **division** as well as the **basic principles** for addition and multiplication. Just for fun, let's invent a new operation called "squareaddition." Study the examples to figure out how to "squareadd." We might call the answer to a squareaddition problem a "squm."

Examples
$2 \overset{2}{+} 1 = 5$
$2 \overset{2}{+} 4 = 20$
$10 \overset{2}{+} 10 = 200$

1. Find the "squms."

A $3 \overset{2}{+} 1$ C $7 \overset{2}{+} 7$ E $1 \overset{2}{+} 1$ G $2 \overset{2}{+} 6$

B $4 \overset{2}{+} 5$ D $15 \overset{2}{+} 25$ F $9 \overset{2}{+} 20$ H $7 \overset{2}{+} 10$

2. A Does $2 \overset{2}{+} 5 = 5 \overset{2}{+} 2$?

B Does $(2 \overset{2}{+} 1) \overset{2}{+} 3 = 2 \overset{2}{+} (1 \overset{2}{+} 3)$?

C Does $8 \overset{2}{+} 0 = 8$?

D Do you think that the commutative, associative, and zero principles hold true for "squareaddition"?

3. A Find $3 \cdot (2 \overset{2}{+} 4)$.

B Find $(3 \cdot 2) \overset{2}{+} (3 \cdot 4)$.

C Does $3 \cdot (2 \overset{2}{+} 4) = (3 \cdot 2) \overset{2}{+} (3 \cdot 4)$?

D Does the distributive principle seem to hold true with respect to $\overset{2}{+}$?

4. Solve the equations.

A $3 \overset{2}{+} n = 34$ B $x \overset{2}{+} 4 = 17$ C $2 \cdot y \overset{2}{+} 3 = 109$

5. Invent a new operation of your own. See if a classmate can figure it out. Check to see if the basic principles hold true.

MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. A $\frac{217}{1000}$; B 5.37 2. A >; B < 3. 23.575 4. A 6.67; B 1.18
5. C 6. A 0.032; B 2740; C 37.429; D 12.267 6. A 0.032 7. A 0.3
   B 0.78 8. 0.27 9. $0.56\frac{1}{4}$ 10. A 12700 km; B 1.27×10^4

UNIT B: THE ARITHMETIC OF FRACTIONAL NUMBERS

Module 5: Ratio, Proportion, and Percent

General Objectives

To introduce the concept of ratio.
To use ratios and proportions to solve problems.
To introduce the idea of percent and percent notation.
To relate percent notation to fraction and decimal notation.
To provide experience with practical percentage problems such as sales taxes and interest.
To provide estimation experiences with percents.

	Performance Objectives	Pupil Text	Reteach-Reinforce	Related Activities	
71	**RED** The student can use the simplest-terms ratio to compare two numbers.	B-100, 101	**WB** 65 **DM** 45		
72	**RED** The student can solve a problem by writing and solving a proportion.	B-102, 103 (B-98, 99)	**WB** 66 **DM** 46	**DS** b-2	**SWM 2** 200, 201
73	**RED** The student can write a percent as a decimal or as a fraction.	B-108, 109 B-112, 113	**WB** 68, 70 **DM** 48, 50	**ASC** D-11, 12 **PD** h-2	
74	**RED** Given a fraction or a decimal, the student can find the corresponding percent.	B-108, 109	**WB** 68 **DM** 48	**ASC** D-13, 14 **PD** h-2	
75	**RED** The student can find the percent of a number.	B-110, 111	**WB** 69 **DM** 49	**PD** h-3 **DS** g-4	
76	**YELLOW** Given word problems involving percent such as sales tax or interest, the student can solve them.	B-116, 117 B-118, 119 B-111, 113	**WB** 70, 71 **DM** 50, 51	**ASC** AP-18, 19, 20, 21 **PD** h-3	
77	**YELLOW** Given two numbers, the student can find what percent one number is of the other.	B-120, 121	**WB** 72 **DM** 52		
78	***GREEN*** *The student can use ratio in measurement and scale drawing problems.*	B-104, 105 B-106, 107	**WB** 67 **DM** 47	**SP** f-1	
79	***GREEN*** *The student can estimate using percent.*	B-122, 123	**WB** 73		**SWM 1** 302
	Reviewing the Ideas	B-124	**WB** 74		

MATHEMATICS

Ratio, proportion, and percent constitute the main ideas of the module.

The *ratio* of a number x to a number y, $y \neq 0$, written $x:y$, is the fractional number $\frac{x}{y}$. Two ratios are equal if each names the same fractional number. The statement that two ratios are equal is called a *proportion*. In the proportion $\frac{a}{b} = \frac{c}{d}$, or $a:b = c:d$, it is customary to call a and d the *extremes* of the proportion and b and c the *means*. Using the property of equality of fractional numbers, we can show that

$$\frac{a}{b} = \frac{c}{d} \text{ if and only if } a \cdot d = b \cdot c.$$

This is often stated verbally as, "In a proportion, the product of the means is equal to the

product of the extremes." This idea often forms the basis for solving problems that involve proportions.

The language of percent is used to show that we are thinking of hundredths. Thus, if n is any number, $n\%$ means $n \cdot \frac{1}{100}$ or $\frac{n}{100}$ or $0.01 \cdot n$. Using this basic definition, we can write fractions or decimals for percents or vice versa.

One of the main objectives of this module is for the students to gain a clear understanding of the relation between percents and ratio. Thus, when students read statements such as "The sales tax rate is 5%," they should be able to state that this means "5¢ out of each 100¢ is sales tax" or "$5 out of each $100 is sales tax." If such simple relations are clear, work with more difficult ideas pertaining to percent will be made easier.

TEACHING THE MODULE

Materials

Cardboard or posterboard, map pins, paper clips, centimetre rulers, graph paper, geoboards, geoboard paper or dot paper, newspapers (for advertisements)

Vocabulary

extremes	proportion
interest	ratio
interest rate	sales tax
means	scale
percent	simplest terms
principal	

The first lesson of the module introduces the student to the idea of a ratio as a way of comparing two numbers. This is followed by work with proportions applied to problems concerning measurement and scale drawings.

Although some students may have had an introduction to percent in their previous work in mathematics, we assume little prior knowledge of percent and develop the concept as an outgrowth of the work on ratio. The relation between fractions, decimals, and percents is given careful consideration. Two kinds of percent problems are presented in this module: (1) finding a percent of a number; (2) finding what percent one number is of another.

Practical applications of percent to sales tax problems and simple interest are also covered. The module concludes with estimation problems using percents.

The amount of emphasis you will want to give to the topics of this module will depend, of course, on the ability of your class. Since much of the material is likely to be new to your students, your pace through the material should be geared to the ease with which the students can master it.

Lesson Schedule

Although generally each lesson is designed to constitute one day's work, you may find it necessary to spend more than one day on some lessons. Furthermore, the bulk of the material is likely to be new to most of your students, so a pace slower than usual may be desirable.

Depending upon the material covered, the amount of remediation, extension, and testing, and the ability of your students, you probably should allow a minimum of 12 days and a maximum of 15 days to complete your coverage of the module.

Evaluation

If you design your own achievement test or evaluation measures, you should find the performance objectives stated at the beginning of the module extremely helpful. By comparing your test items with the objectives, you can readily decide whether your are overemphasizing certain topics and neglecting other topics. It may be that you have designed some test items for which no specific objective was listed. Analyzing your own tests in this manner will lead to tests that are close to meeting the objectives of the module. (Module and Unit Achievement Tests are also available from the publisher as part of the test package for this series.)

Utilize your day-to-day contact with your students as an aid in evaluating them. Short quizzes and oral questioning are also useful as evaluation devices.

RESOURCES FOR ACTIVE LEARNING

Activities

Developmental Math Cards, K12, L2, 4, Addison-Wesley

Experiences in Mathematical Ideas, Vol. 2, "Ratio," pp. 69–95, NCTM

Fractional Numbers, Geocards 24–26, Creative Publications

Let's Play Games in Mathematics, Vol. 7, "Per Cent In Between," pp. 50–51, National Textbook Co.

Manipulative Devices

Decimal and Percentage Board (Ideal)

Geoboards (Addison-Wesley)

Pantograph (Geyer)

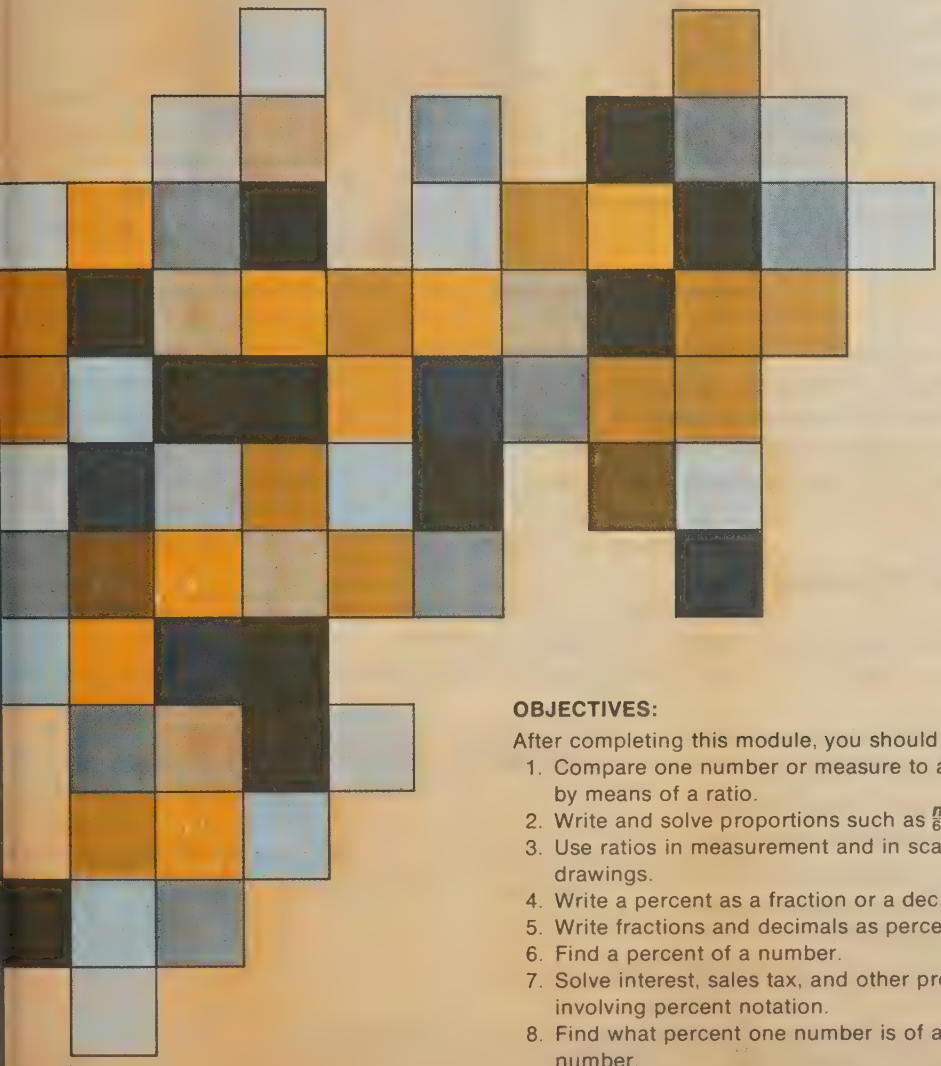
Tangrams (Addison-Wesley; Creative Publications)

Games and Puzzles

In-order Game Kit (Midwest Publications)

UNIT B: *The Arithmetic of Fractional Numbers.*

MODULE 5: *Ratio, Proportion, and Percent*



OBJECTIVES:

After completing this module, you should be able to:

1. Compare one number or measure to another by means of a ratio.
2. Write and solve proportions such as $\frac{n}{6} = \frac{20}{24}$.
3. Use ratios in measurement and in scale drawings.
4. Write a percent as a fraction or a decimal.
5. Write fractions and decimals as percents.
6. Find a percent of a number.
7. Solve interest, sales tax, and other problems involving percent notation.
8. Find what percent one number is of another number.
9. Make estimates using percent.

Objective

The student can compare two numbers by means of a ratio.

PREPARATION

Materials: Heavy cardboard, map pins or tacks, paper clips.

The illustration in the text suggests a model for a spinner. This spinner consists of a pin stuck in a piece of heavy cardboard, with a paper clip pointer. An alternate type of spinner can be made from a plastic margarine tub, with the circular top serving as the dial. If you prefer, use wooden cubes with the letter A marked on 3 faces, B marked on 2 faces, and C on the third face. Tossing such a cube will give results equivalent to those of the spinner.

INVESTIGATION

The purpose of the Investigation is to introduce ratios through a physical activity. Although the Investigation deals with a probability experiment, the emphasis is on recording the results of the experiment by means of a ratio. Encourage students to work in pairs or in small groups to carry out the Investigation. A variety of results for the Investigation will enhance the discussion and provide numerous examples of different ratios.

DISCUSSION

To capitalize on the results of the Investigation, you may want to discuss the results of the experiments as compared to expected results. The pointer should be expected to stop in region A for $\frac{1}{2}$ of the tosses, in region B for $\frac{1}{3}$ of the tosses, and in region C for $\frac{1}{6}$ of the tosses.

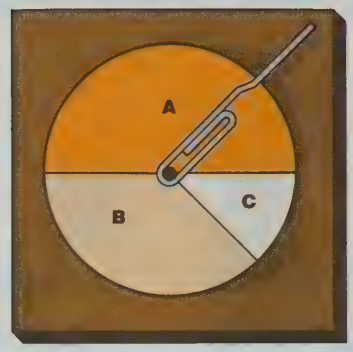
The basic idea of a ratio is introduced in Exercise 3. Two kinds of symbols for ratios are also introduced. Point out that the language of ratio is essentially just another way of talking and thinking about fractional numbers.

What are Ratios?

Investigating the Ideas

Make a spinner like the one shown at the right. Region A covers $\frac{1}{2}$ of the dial, region B covers $\frac{1}{3}$, and region C covers $\frac{1}{6}$.

Spin the pointer 60 times and make a tally of the results.



Can you write fractions that compare the number of times the pointer stopped in each region to the total number of spins?

Region	Tally
A	
B	
C	

Discussing the Ideas

1. Did the pointer stop in region A for about $\frac{1}{2}$ of the number of trials?

2. Linda made this tally sheet for the experiment.

Region	Tally
A	
B	
C	

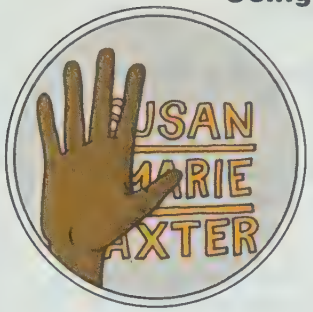
   a. How many times did the pointer stop in region A?
   b. What **fractional** part of the trials did the pointer stop in region A?
   c. Repeat the question for regions B and C.

3. In order to compare two numbers, you can often use a **ratio**. For example, Linda could have said, "The ratio of the number of times the pointer stopped in region A to the total number of trials was **32 to 60**." You can write the ratio "32 to 60" as 32:60 or $\frac{32}{60}$.

   What other ratios could Linda have written for her results in the Investigation?

Using the Ideas

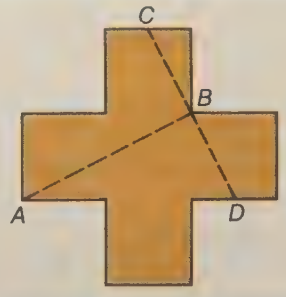
Find the following ratios.



1. The number of boys to the number of girls in your class today.
2. The number of people in your class who wear glasses to those who do not wear glasses.
3. The number of windows in your mathematics classroom to the number of doors.
4. The number of letters in your first name to the number of letters in your last name.
5. The number of letters in your middle name to the number of letters in your first name.
6. The number of eyes you have to the numbers of fingers you have.
7. The length of a millimetre to the length of a centimetre.
8. The length of a centimetre to the length of a metre.
9. The number of vowels in the alphabet to all the letters in the alphabet.
10. The number of prime numbers less than 50 to the number of odd numbers less than 50.
11. The number of provinces in Canada whose name ends in the letter "a" to the number of provinces.

think

Trace the cross and the dotted lines. Cut out your tracing and then cut along the dotted lines. Now fit four pieces together to make a square.



UTILIZATION

Note that the answers to Exercises 1 through 5 will vary from class to class and, in some instances, from student to student. Since these questions may be highly motivating to your students, you might find it worthwhile to have several different students write ratios that they find for these exercises on the chalkboard for comparison and discussion.

Exercises 6 through 11 may be assigned as written work.

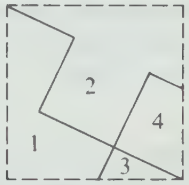
EXTENSION

Remedial: Ask students to write each ratio in the forms $\frac{a}{b}$ and $a:b$.

A. 3 to 4	E. 6 to 10	I. 1 to 3
B. 7 to 3	F. 3 to 2	J. 3 to 1
C. 1 to 10	G. 12 to 7	K. 5 to 9
D. 8 to 9	H. 1 to 100	L. 99 to 100

Enrichment: Have the students invent two problems like the ones in this lesson. Students should be able to write their own problems.

Think Solution



Assignments

Minimum 1–5, oral. Average 1–5, oral; 6–9. Maximum 1–5, oral; 6–11.

Objectives

Given two ratios, the student can express them in simplest terms.

Given two ratios, the student can determine if they are equal.

Given a proportion, the student can show that the product of the means is equal to the product of the extremes.

PREPARATION

Review the basic ideas concerning the language of ratio and the various symbols used to denote ratios.

INVESTIGATION

This Investigation uses team standings to introduce the idea of equal ratios. Usually the team standings given in newspapers are printed as decimals and incorrectly referred to as "percentages." In this Investigation, students are expected to use their knowledge of fractional numbers and ratios to decide which team has the best winning record.

Most students will probably give the fractions denoting the win ratios in lowest terms. A few students will probably find a decimal for each fraction and compare the decimals. Either method should be accepted.

DISCUSSION

Encourage various students to describe the methods they used to compare the win ratios in the Investigation. Exercises 1 through 4 utilize the ratios given in the Investigation to introduce *equal ratios*, *simplest terms* for a ratio, and *proportion*.

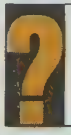
The words "proportion," "extremes," and "means" may be new to most students. It is important to emphasize that in any proportion the product of the means is equal to the product of the extremes, because this gives the student a method of solving for one of the parts of a proportion when the other three are given. You may want to remind your students that this idea was studied earlier, when pairs of fractions were tested for equivalence by finding "cross products." That is,

$\frac{a}{b} = \frac{c}{d}$ if and only if $a \cdot d = b \cdot c$.

Ratios and Proportions

Investigating the Ideas

Study the team records in the chart.



Can you decide which team has the best record?

Team	Games won	Games played	Win ratio
EXPOS	18	24	$\frac{18}{24}$
CUBS	15	20	$\frac{15}{20}$
PIRATES	10	16	$\frac{10}{16}$
CARDINALS	16	20	$\frac{16}{20}$

Discussing the Ideas

1. Explain how you decided which team had the best record.

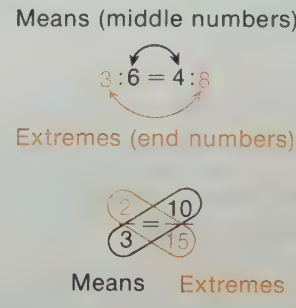
2. The Expos won 18 out of 24 games or $\frac{18}{24}$ of their games.
   - **A** What is the lowest-terms fraction for $\frac{18}{24}$?
   - **B** Do you think it is correct to say, "The Expos won 3 out of every 4 games that they played?"

3. The ratio $\frac{18}{24}$ in **simplest terms** is $\frac{3}{4}$. What are the win ratios for the Cubs and the Pirates in simplest terms?

4. The ratios 18:24 and 3:4 are **equal ratios** because $\frac{18}{24} = \frac{3}{4}$.

   A statement that two ratios are equal such as $\frac{18}{24} = \frac{3}{4}$ is called a **proportion**.
   - **A** Could the ratios $\frac{18}{24}$ and $\frac{15}{20}$ form a proportion?
   - **B** What does this tell you about the Expos and the Cubs?
   - **C** Could the ratios $\frac{10}{16}$ and $\frac{15}{20}$ form a proportion?

5. In each proportion, we call two of the numbers **means** and the other two **extremes**
   - **A** Which numbers are the means and which numbers are the extremes in the two proportions at the right?
   - **B** Complete this statement:
     In a proportion, the product of the means is equal to the product of the ___?___.



Using the Ideas

1. Express each ratio in simplest terms.

A $\frac{6}{18}$ **B** 8 to 10 **C** 25:15 **D** $\frac{15}{25}$ **E** 36 to 20

2. In each proportion, tell which pair of numbers is the means and which is the extremes.

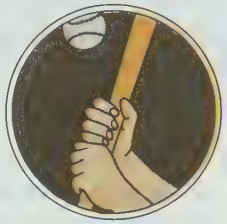
A $\frac{2}{5} = \frac{6}{15}$ **B** $1:4 = 7:28$ **C** $\frac{5}{3} = \frac{20}{12}$ **D** $5:8 = 200:320$

3. For each proportion, show that the product of the means is equal to the product of the extremes.

A $2:3 = 4:6$ **B** $1:2 = 12:24$ **C** 5 to 3 = 15 to 9 **D** 10 to 8 = 5 to 4

4. Write a proportion using the four numbers in the equation $9 \cdot 4 = 3 \cdot 12$. Can you find more than one proportion?

5. A ballplayer's ratio of the number of hits to the number of times at the bat is 4 to 14.



A Give two different symbols for the ratio which compares the number of hits with the number of times at bat.

B Write a proportion using the ratios in part **A**.

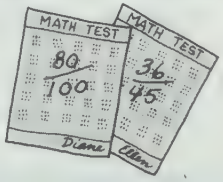
6. Out of a total 42 cars in a parking lot, 15 were sports cars.

A What was the ratio of the number of sports cars to all the other cars?

B What is this ratio in simplest terms?

C Write a proportion using the ratios of parts **A** and **B**.

7. On a test, Diane said that she got $\frac{36}{45}$ of the problems correct. Ellen said that she got $\frac{80}{100}$ of the problems correct.



A Express each ratio in simplest terms.

B Can the two ratios form a proportion?

think

A date such as February 12, 1924, can be abbreviated as 2/12/24. This is a **product date** because $2 \cdot 12 = 24$. March 24, 1972, was also a product date because $3 \cdot 24 = 72$.

1. Which dates in the year 1980 are product dates?

2. Are the dates of any days of the present year product dates?

UTILIZATION

The exercises present proportions using different notations. Thus the proportion $\frac{3}{4} = \frac{6}{8}$ may be also written $3:4 = 6:8$ or "3 to 4 = 6 to 8." You may want to encourage students to read these proportions as "three is to four as six is to eight."

Many students may feel more comfortable working with proportions using fractional notation, but they should become familiar with the other forms as well.

EXTENSION

Workbook page 65 and *Duplicator Masters* page 45 offer additional practice exercises.

Think Solution

The solution for 1980 can be found by listing pairs of factors of 80. From this listing, those pairs meeting the condition that the month date is a number from 1 to 12 and the day is a number from 1 to 31 can be easily found.

April 20, 1980	(4/20/80)
May 16, 1980	(5/16/80)
August 10, 1980	(8/10/80)
October 8, 1980	(10/8/80)

The solution for the second question will of course depend upon the year of solving.

Assignments

Minimum 1–5. Average 1–6. Maximum 1–7.

Objectives

Given a proportion, such as $\frac{n}{4} = \frac{12}{16}$, *the student can solve the proportion.*

Given an appropriate word problem, the student can solve the problem by means of a proportion.

PREPARATION

Write several ratios on the chalkboard, including several that are equal ratios. Ask two students to name which pairs could form a proportion and then check this by determining whether the product of the means equals the product of the extremes.

INVESTIGATION

Although the problem in the Investigation can be solved by means of a proportion, it is not likely that many students will use this method. Some students may use an argument such as this: "There are $\frac{3}{5}$ as many girls as boys. Since there are 80 boys, there must be $\frac{3}{5} \cdot 80 = 48$ girls."

A more intuitive argument would be: "There are 3 girls for every 5 boys. There are 16 sets of 5 boys. Therefore, there must be 16 sets of 3 girls or a total of 48 girls."

DISCUSSION

The greatest emphasis of the discussion should be on writing and solving proportions. Go through the steps in the Example in Exercise 2 carefully. The proportion $\frac{3}{5} = \frac{n}{80}$ can be written because if n represents the number of girls, the ratio $\frac{n}{80}$ must equal $\frac{3}{5}$. The second step states that the product of the means equals the product of the extremes. To complete the solution, the student must solve the equation $5 \cdot n = 240$. This solution is found by dividing to determine the missing factor.

Present and discuss other examples similar to Exercises 4 and 5, if necessary, before moving on to the Utilization stage.

Solving Problems with Proportions

Investigating the Ideas

Can you find the solution to this problem?

In Jamestown Junior High School the ratio of girls to boys in the seventh grade is 3 to 5. If there are 80 boys in the seventh grade, how many girls are in the seventh grade?

Discussing the Ideas

1. Explain your solution to the problem in the Investigation.

2. You can write a proportion to help you solve the problem. Explain each step in the example, then give the number for n.

Let n represent the number of girls in the seventh grade.

$$\frac{3}{5} = \frac{n}{80}$$

$$5 \cdot n = 3 \cdot 80$$

$$5 \cdot n = 240$$

$$n =$$

3. What is the total number of students in the seventh grade?

4. Explain how you would solve this proportion to find the number for n. $\frac{n}{4} = \frac{20}{16}$

5. Read the problem below.

   At a picnic, the ratio of adults to children was 3 to 5. If there were 24 adults, how many children were there?

   A If you let n = number of children what ratio compares the number of adults to the number of children?

   B Which proportion is correct for the problem? **A** $\frac{3}{5} = \frac{n}{24}$ **B** $\frac{3}{5} = \frac{24}{n}$

   C Solve the correct proportion for n.

Using the Ideas

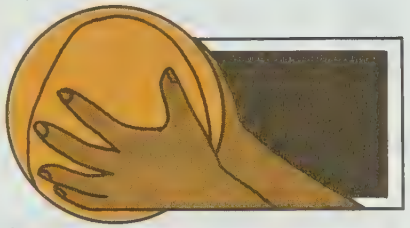
1. Solve the following proportions for n.

A $\frac{n}{8} = \frac{12}{16}$ **C** $2:3 = n:15$ **E** $\frac{1}{5} = \frac{n}{100}$ **G** $\frac{2}{n} = \frac{50}{75}$

B $\frac{n}{25} = \frac{9}{15}$ **D** $5:3 = 10:n$ **F** $\frac{3}{n} = \frac{75}{100}$ **H** $\frac{3}{1} = \frac{n}{47}$

In Exercises 2 through 9, write a proportion for each problem and then solve it.

2. For a class party, Jeanne bought 10 doughnuts for every 3 students. How many doughnuts did she buy if there are 24 students in the class?



3. The ratio of field goals scored in a basketball game to field goals attempted was 2:5. Eighty field goals were attempted. How many field goals were scored?

4. A car can travel 7 kilometres on one litre of gasoline. How many litres would be needed to travel 266 kilometres?

5. The ratio of kilometres driven to litres of gasoline used was 27 to 4. How far was it possible to drive on 72 litres of gasoline?

6. The ratio of distance in centimetres shown on a map to the actual distance in kilometres is 1 to 50. If the map distance between two cities is 9 centimetres, how many kilometres is it from one city to the other?

7. A baseball player has made 2 hits for every 7 times at bat. If the player has been at bat 210 times, how many hits has he made?

8. In a group of students, there are 7 girls for every 5 boys. Determine **x** if

A there are 14 girls for every **x** boys
B there are 35 girls for every **x** boys
C there are **x** girls for every 15 boys
D the ratio of girls to boys is **x** to 50
E the ratio of girls to boys is 42 to **x**
F the ratio of boys to girls is **x** to 119

9. The ratio 5:3 compares the population of City A with the population of city B. If a total of 8000 persons live in the two cities, how many live in each city?

City *A*

City *B*

More practice, page S-22, Set 38

UTILIZATION

Assign the exercises according to the needs and abilities of your students. After the students have completed the exercises, allow sufficient time for selected students to show the proportions they have written and their method of solution.

EXTENSION

To provide additional practice, make selective assignments from Supplementary Exercise Set 38 on page S-22, *Workbook* page 66, and *Duplicator Masters* page 46.

Enrichment: Some students will be interested in learning how proportions can be used to find the heights of objects by measurement of their shadows. At the same time of day, the ratios of the heights of any two objects to their shadows are equal.

Suppose a vertical stick 2 metres long casts a shadow 1 metre long at the same time that the shadow of a flagpole casts a shadow 9 metres long. Then, letting x denote the height of the flagpole, we have

$$\frac{x}{9} = \frac{2}{1}, \text{ or } x = 18 \text{ metres.}$$

Although the proof of this rests upon the idea of similar triangles, students can usually use the method without being concerned with the geometric ideas involved. This activity provides an opportunity for some field work and diversion from the usual classroom routine.

Assignments

Minimum 1–5. Average 1–8. Maximum 1–9.

Objective

Given two measures with different units, the student can express the measures with a common unit and find the ratio of the measures.

PREPARATION

Materials: Rulers

Give several ratios orally and have students decide whether the ratio is in simplest terms.

INVESTIGATION

The important concept to develop in this lesson is that, when we are discussing the ratio of two measures, it is desirable to express both measures in the same unit.

Allow the students to choose for themselves the unit they wish to use in carrying out the Investigation. If the centimetre is used as the unit, the ratio is easily found to be 6:8 or 3:4. Students who elect to use the inch as the unit may find it difficult to give a simple ratio and report it as $2\frac{3}{8} : 3\frac{3}{16}$, which is 38:51. Note that $\frac{3}{4} \approx \frac{38}{51}$.

DISCUSSION

When discussing Exercise 1, you might suggest that students use a 2-cm segment as their unit of measure for $\overline{EF}$ and $\overline{GH}$. They will find that the ratio of the length is still 3:4. If a 3-cm unit is used, the ratio will be $2 : 2\frac{2}{3}$ or

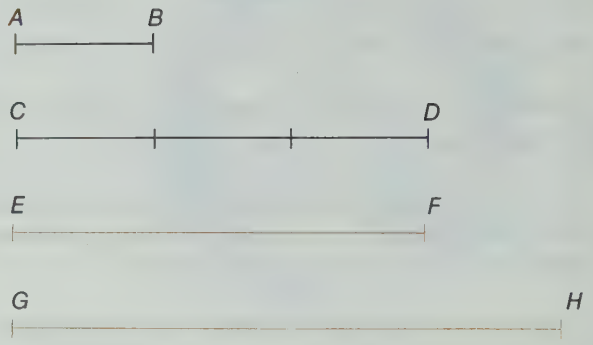
$$\frac{2}{2\frac{2}{3}} = \frac{\frac{6}{3}}{\frac{8}{3}} = \frac{6}{8} = \frac{3}{4}.$$

This should help to convince students that the ratio of the lengths is independent of the unit of length chosen.

Go through the parts of Exercises 2 and 3 carefully, discussing ways of finding the ratios in simplest terms. In order to avoid working with complex fractional numbers, the measures being compared should be expressed using the smaller of the two units. Thus the ratio $\frac{40 \text{ s}}{3 \text{ min}}$ should be rewritten so that the minutes are expressed in seconds: $\frac{40 \text{ s}}{3 \text{ min}} = \frac{40 \text{ s}}{180 \text{ s}} = \frac{2}{9}$.

Investigating the Ideas

The ratio of the length of $\overline{AB}$ to the length of $\overline{CD}$ is 1:3.



Can you find the ratio of the length of $\overline{EF}$ to the length of $\overline{GH}$?

Discussing the Ideas

1. What unit did you choose to measure the lengths $\overline{EF}$ and $\overline{GH}$? Do you think that the ratio of the lengths of the segments depends upon the unit that you chose?

2. When you use a ratio to compare two measures, it is helpful to express both measures with the same unit. Study part **A**.

   Use the relationships between the different units to give the following ratios in the simplest-terms

   A 1 m = 100 cm → $\frac{1 \text{ m}}{1 \text{ cm}} = \frac{100}{1}$

   B 1000 g = 1 kg → $\frac{1 \text{ kg}}{1 \text{ g}} =$ [illegible]

   C [illegible] mm = [illegible] cm → $\frac{1 \text{ mm}}{1 \text{ cm}} =$ [illegible]

   D 1 min = 60 s → $\frac{1 \text{ s}}{1 \text{ min}} =$ [illegible]

   E [illegible] h = 60 min → $\frac{1 \text{ h}}{1 \text{ min}} =$ [illegible]

   F 1 day = [illegible] h → $\frac{1 \text{ h}}{1 \text{ day}} =$ [illegible]

3. To find the simplest-terms ratio for "30 minutes to 2 hours." you can write

$$\frac{30 \text{ min}}{2 \text{ h}} = \frac{30 \text{ min}}{2 \cdot 60 \text{ min}} = \frac{30}{120} = \frac{1}{4}$$

   Explain how to find the simplest-terms ratio for each of the following:

   A 5 cm to 20 mm

   B 24 mm to 4 cm

   C 700 g to 1.2 kg

   D 40 s to 3 min

   E 250 mg to 1 g

   F 12 days to 2 weeks

Using the Ideas

1. Give the appropriate value for x. Then give the ratio in simplest terms.

A $\frac{3\text{ m}}{50\text{ cm}} = \frac{3 \cdot x}{50} =$ ▒▒

B $\frac{3\text{ cm}}{7\text{ mm}} = \frac{3 \cdot x}{7} =$ ▒▒

C $\frac{7\text{ km}}{850\text{ m}} = \frac{7 \cdot x}{850} =$ ▒▒

D $\frac{200\text{ m}}{4\text{ km}} = \frac{200}{4 \cdot x} =$ ▒▒

E $\frac{85\text{ mm}}{15\text{ cm}} = \frac{85}{15 \cdot x} =$ ▒▒

F $\frac{420\text{ cm}}{8\text{ m}} = \frac{420}{8 \cdot x} =$ ▒▒

2. In each part, give the indicated ratios in simplest terms.

A 5 cm to 1 m

B 2 m to 10 cm

C 1 mm to 1 cm

D 5 mm to 5 cm

E 6 cm to 4 m

F 20 cm to 4 m

G 2 m to 50 cm

H 2 m 45 cm to 50 cm

I 1 km 420 m to 200 m

3. Express each measure with the same unit, then give the ratio in simplest terms.

A $\frac{3\text{ min}}{40\text{ s}}$

B $\frac{3\text{ h}}{20\text{ s}}$

C $\frac{3\text{ day}}{75\text{ min}}$

D $\frac{8\text{ mm}}{1\text{ cm}}$

E $\frac{400\text{ m}}{2\text{ km}}$

F $\frac{500\text{ ml}}{4\ \ell}$

G $\frac{1\text{ mg}}{1\text{ g}}$

H $\frac{100\text{ g}}{1\text{ kg}}$

I $\frac{2500\text{ g}}{5\text{ kg}}$

J $\frac{3\text{ day}}{8\text{ wk}}$

K $\frac{2\text{ cm}}{11\text{ mm}}$

L $\frac{44\text{ min}}{3\text{ h}}$

4. Give the indicated ratios in simplest terms.

A $\frac{4\text{ cm }8\text{ mm}}{7\text{ cm }2\text{ mm}}$

B $\frac{1\text{ h }20\text{ min}}{3\text{ h }40\text{ min}}$

C $\frac{3\text{ m }20\text{ cm}}{5\text{ m}}$

D $\frac{1\text{ wk }3\text{ days}}{4\text{ wk }2\text{ days}}$

E $\frac{1\text{ m }23\text{ cm }4\text{ mm}}{4\text{ m }32\text{ cm }1\text{ mm}}$

F $\frac{1\text{ km }234\text{ m }56\text{ cm }7\text{ mm}}{7\text{ km }654\text{ m }32\text{ cm }1\text{ mm}}$

think

There is a shortcut for squaring numbers which are multiples of 5. First find the number for n in each example at the right. Then, without using pencil or paper, determine each product below.

$15 \cdot 15 = 10 \cdot 20 + n$

$25 \cdot 25 = 20 \cdot 30 + n$

$55 \cdot 55 = 50 \cdot 60 + n$

1. $65 \cdot 65$ **2.** $75 \cdot 75$ **3.** $85 \cdot 85$ **4.** $95 \cdot 95$

UTILIZATION

The importance of these exercises will be apparent in the following lesson, which covers scale drawings where ratios involving different units of measurement are used.

In Exercise 3, the students should express the ratio in simplest terms using the unit shown in color in each part.

Think Solution

Any number whose units digit is 5 may be squared by first finding the product of the two nearest multiples of ten of the number and then adding 25 to that product.

The shortcut used in this problem is derived from the use of several basic principles. Note especially the multiple use of the distributive principle, such that the problem becomes one of multiplying multiples of 10 and adding 25. This is illustrated for the product $15 \cdot 15$.

$15 \cdot 15 = (10 + 5)(10 + 5)$ expanded notation

$= (10 + 5)10 + (10 + 5)5$ distributive principle for multiplication over addition (dpma)

$= (10 \cdot 10 + 5 \cdot 10) + (10 \cdot 5 + 5 \cdot 5)$ dpma(2)

$= (10 \cdot 10 + 10 \cdot 5) + (10 \cdot 5 + 5 \cdot 5)$ commutative principle for multiplication (cpm)

$= (10 \cdot 10 + 10 \cdot 5 + 10 \cdot 5) + 5 \cdot 5$ associative principle for addition (apa)

$= 10(10 + 5 + 5) + 5 \cdot 5$ dpma

$= 10(20) + 25$ computation

Assignments

Minimum 1, 2. Average 1–3.
Maximum 1–4.

Objective

Given a scale for a drawing, the student can determine actual measurements of the object represented.

PREPARATION

Materials: Centimetre ruler.

If possible, secure several different kinds of scale drawings for display and discussion in conjunction with this lesson. Maps, blueprints, house plans, or other scale drawings will be useful.

INVESTIGATION

Students who realize that, because the drawing is $\frac{1}{10}$ actual size, the object it represents will be 10 times as large as the drawing will quickly find that the drawing represents a basketball. The diameter of the circle in the scale drawing is about 2.4 cm, so the object it represents has a diameter of about 24 cm.

If you wish to extend the Investigation, ask the students to show a scale drawing of each of the other balls using the same scale.

DISCUSSION

In discussing Exercise 3, you may want to point out that when we write, "Scale: 1 cm = 5 km" we do not mean literally that 1 cm *equals* 5 km but that 1 cm on the map represents an actual distance of 5 km. Since

$$\frac{1\text{ cm}}{5\text{ km}} = \frac{1\text{ cm}}{5 \cdot 1000\text{ m}} = \frac{1\text{ cm}}{5 \cdot 1000 \cdot 100\text{ cm}} = \frac{1}{500\,000},$$

we see that the map is $\frac{1}{500\,000}$ actual size.

In Exercise 4 the scale of 1 cm = $\frac{1}{10}$ mm means a ratio of

$$\frac{1\text{ cm}}{\frac{1}{10}\text{ mm}} = \frac{10\text{ mm}}{\frac{1}{10}\text{ mm}} = \frac{100}{1}.$$

Thus, the picture of the amoeba is 100 times as large as the actual size of the amoeba.

Scale Drawings

Investigating the Ideas

The figure below shows a **scale drawing** of a certain kind of ball.

Scale: $\frac{1}{10}$ actual size

Kind of Ball	Approximate diameter
Golf ball	4.3 cm
Baseball	7.4 cm
Softball	10.2 cm
Volley ball	21.0 cm
Basketball	24.3 cm

Can you use your centimetre ruler and the information in the table to decide which kind of ball the scale drawing represents?

Discussing the Ideas

1. Explain how you decided which ball the scale drawing represented?

2. The **scale** for the drawing of the ball gives the ratio of the diameter of the drawing to the actual diameter of the ball.
   - A What is this ratio?
   - B The diameter of the actual ball is how many times that in the drawing?

3. A scale on a map reads 1 cm = 5 km. If the distance between two points on the map measured 3.8 centimetres. how can you find the actual distance between the two points? What is the distance in kilometres?

4. This is a drawing of a microscopic creature called an amoeba.
   - A How many times larger is the drawing than an actual amoeba?
   - B The drawing of the amoeba is about 2.5 cm long. What is the actual size of an amoeba?

Scale: 1 cm = $\frac{1}{10}$ mm

Using the Ideas

1. The table gives distances from some of the planets to the sun.

Planet	Distance (km)
Mercury	58 000 000
Venus	108 000 000
Earth	150 000 000
Mars	228 000 000
Jupiter	777 000 000
Saturn	1 426 000 000

Scale: 1 cm = 100 000 000 km

Tell which planet in the table is represented by

A Planet *A* B Planet *B* C Planet *C* D Planet *D*

2. If the planet Jupiter were shown on the scale drawing in Exercise 1, about how many centimetres away from the point for the sun would it be?

3. The map at the right is a scale drawing of France.

A What is the distance from Paris to Nantes?

B What is the distance from Nantes to Marseilles?

C What is the distance from Paris to Marseilles?

Scale: 1 cm = 320 km

4. A scale drawing of a whale as it might appear in a dictionary is shown at the right.

A How many millimetres long is the drawing?

B Estimate the length of the whale in metres.

$\frac{1}{1000}$

5. The drawing at the right shows a microscopic creature called a paramecium.

A How many millimetres long is the drawing?

B Find the actual length of the paramecium to the nearest tenth of a millimetre.

Scale: 250 times actual size

★ **6.** Measure the length and width of your classroom. Choose a scale and make a scale drawing of the room.

UTILIZATION

Students will need centimetre rulers for the exercises. In Exercise 4 the ratio $\frac{1}{1000}$ means the drawing is $\frac{1}{1000}$ actual size. Large dictionaries often show scale drawings in this manner. The scale is such that 1 unit on the drawing represents 1000 actual units.

Although the problems can be solved without using proportions, some students may set up and solve proportions for some problems. For example, in Exercise 5:

Let x = paramecium's actual length.

$$\frac{250}{1} = \frac{74}{x} \rightarrow 250 \cdot x = 74 \rightarrow x \approx 0.3 \text{ mm}$$

EXTENSION

To provide appropriate additional practice, make assignments from *Workbook* page 67 and *Duplicator Masters* page 47.

You may find it helpful to provide some additional map activities to augment this lesson. The students can benefit from working with the scale on a given map, whether it is a community, state, or national map.

You may also suggest that some students make a scale drawing of football fields, tennis courts, or the floor plan of their homes, as suggested by Research Project A on page B-125.

Assignments

Minimum 1–3. Average 1–4. Maximum 1–6.

Objective

Given a simple percent such as 17%, the student can express it as a fraction and as a decimal, and conversely.

PREPARATION

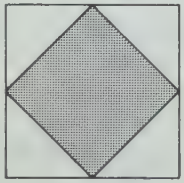
Materials: Graph paper (*Duplicator Masters*, page 83).

Review lowest-terms fractions and the relation between fractions and decimals. As examples, use several fractions with denominators of 2, 4, 5, 10, 100.

INVESTIGATION

The purpose of the Investigation is to provide semiconcrete experiences with fractions whose denominators are 100, as a prelude to introducing the idea of percent.

The square region that is $\frac{49}{100}$ of the 10 by 10 square is a square with 7 units on each side. However, to show a square that is $\frac{50}{100}$ of the 10 by 10 square is more difficult. The student must find a way of showing a square region that covers one half of the large square. The only way possible is to form the square by connecting the midpoints of the sides of the large square. If some students have difficulty seeing this, show that the 4 triangles on each corner will exactly cover the square, so it must be $\frac{1}{2}$ or $\frac{50}{100}$ as large as the 10 by 10 square.



DISCUSSION

In discussing Exercise 1, ask students to give a fraction that compares the size of each square to the large square. The term "percent" is introduced in Exercise 2. This word comes from the Latin phrase *per centum*, which means *in the hundred*. Exercises 3, 4, and 5 relate percent notation to function and decimal notation.

The basic definition of percent is:

If n is a whole number, then $\frac{n}{100}$ can be expressed as $n\%$.

The Meaning of Percent

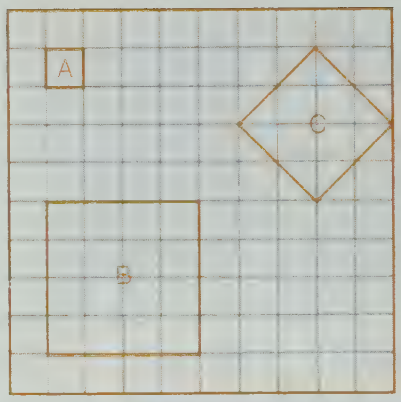
Investigating the Ideas

Square region A covers $\frac{1}{100}$ of the large square region.

Square region B covers $\frac{16}{100}$ of the large square region.

What part of the large square region does region C cover?

Outline a 10 by 10-unit square on graph paper.



Can you draw square regions that will cover $\frac{49}{100}$ and $\frac{50}{100}$ of the large square region?

Discussing the Ideas

1. What is the size of some other square regions that you could draw inside the 10 by 10 square region?

2. Fractions with denominators of 100 are easily expressed in terms of **percents**. For example, instead of saying

   "Square region A covers $\frac{1}{100}$ of the large square region."

   You could say:

   "Square region A covers 1 percent of the large square region."

   You can write: "1 percent" as "1%". 1% means $\frac{1}{100}$ or 0.01.

   What percent of the large square do square regions B and C cover?

3. You can think of 6% as meaning "6 out of 100" or $\frac{6}{100}$ or 0.06.
   - **A** How can you think of 41%?
   - **B** What fraction means the same as 41%?

4. Explain how to find a fraction for each percent.
   A 8% **B** 27% **C** 2% **D** 90% **E** 50% **F** 40%

5. Since $0.23 = \frac{23}{100}$, what percent can you write for 0.23?

Using the Ideas

1. Write a percent for each fraction. EXAMPLE: $\frac{50}{100} = 50\%$

A $\frac{17}{100}$	**C** $\frac{14}{100}$	**E** $\frac{10}{100}$	**G** $\frac{1}{100}$	**I** $\frac{66}{100}$	**K** $\frac{100}{100}$
B $\frac{29}{100}$	**D** $\frac{11}{100}$	**F** $\frac{9}{100}$	**H** $\frac{97}{100}$	**J** $\frac{75}{100}$	**L** $\frac{125}{100}$

2. Give a decimal for each percent. EXAMPLE: 35% 0.35

A 23%	**C** 25%	**E** 12%	**G** 9%	**I** 2%	**K** 95%
B 94%	**D** 20%	**F** 10%	**H** 3%	**J** 1%	**L** 102%

3. Give a percent for each decimal. EXAMPLE: [illegible] $= \frac{18}{100} = 18\%$

A 0.27	**C** 0.66	**E** 0.11	**G** 0.04	**I** 0.40	**K** 1.00
B 0.35	**D** 0.60	**F** 0.08	**H** 0.01	**J** 0.3	**L** 1.16

4. Write an equivalent fraction with a denominator of 100. Then give the percent for the fraction.

EXAMPLE: $\frac{1}{4} = \frac{25}{100} = 25\%$

A $\frac{1}{2}$	**C** $\frac{1}{5}$	**E** $\frac{1}{25}$	**G** $\frac{3}{10}$	**I** $\frac{7}{20}$	**K** $\frac{9}{10}$
B $\frac{1}{10}$	**D** $\frac{1}{20}$	**F** $\frac{3}{4}$	**H** $\frac{4}{25}$	**J** $\frac{2}{5}$	**L** $\frac{11}{50}$

5. Give the number for each ▒ in Examples B, C, and D in the table.

	Statement	Ratio	Fraction (hundredths)	Decimal	Percent
A	3 of 25 boys are on the football team.	3:25	$\frac{12}{100}$	0.12	12%
B	There are 3 boys for every 5 students.	3:5	$\frac{60}{100}$	0.60	▒%
C	I saved $7 out of $20.	7:20	$\frac{▒▒}{100}$	0.▒	▒%
D	Mike made 5 free throws in 5 tries.	5:5	$\frac{▒▒}{100}$	0.▒	▒%

think

The ratio of boys to girls in a group of children was 3 to 5. Then 24 girls left the group and 24 more boys joined it. The ratio of boys to girls became 5 to 3. How many boys and girls were in the original group?

More practice, page S-23, Set 39

UTILIZATION

The exercises on this page give students varied experiences with percent notation. Only whole percents are presented in this exercise set. Fractional and decimal percents will be treated in later lessons.

EXTENSION

Students who would benefit from further practice with the ideas presented in this lesson should be referred to Supplementary Exercise Set 39 on page S-23. You might also make selective assignments from *Workbook* page 68, *Duplicator Masters* page 48, and *Arithmetic Skill Cards* D-11 through D-14.

In addition, you might want to provide some practice in finding lowest-terms fractions for given percents, such as:

1. $25\% = \frac{25}{100} = \frac{1}{4}$
2. 90% $(\frac{9}{10})$
3. 40% $(\frac{2}{5})$
4. 75% $(\frac{3}{4})$
5. 20% $(\frac{1}{5})$
6. 50% $(\frac{1}{2})$
7. 60% $(\frac{3}{5})$
8. 35% $(\frac{7}{20})$
9. 44% $(\frac{11}{25})$
10. 80% $(\frac{4}{5})$

Think Solution

An algebraic solution follows. Let $3n$ equal the number of boys and $5n$ equal the number of girls. Then:

$$\frac{3n+24}{5n-24} = \frac{5}{3}$$
$$(3n+24)3 = (5n-24)5$$
$$9n+72 = 25n-120$$
$$192 = 16n$$
$$12 = n$$
$$36 = 3n, \text{ boys}$$
$$60 = 5n, \text{ girls}$$

Although most of your students will use trial and error to arrive at a solution, they should find clues in the problems as to possible numbers to try. Two such hints might be: the original number of girls is greater than 24; the numbers are probably multiples of 12, since 24 leave one group and 24 join the other group and the ratio still involves the same whole numbers in reversed order.

Assignments

Minimum 1–3. Average 1–4. Maximum 1–5.

Objective

The student can find a percent of a number.

PREPARATION

Materials: Geoboards and rubber bands, or geoboard paper or dot paper (*Duplicator Masters*, page 82).

If geoboards are not available, duplicated geoboard paper or dot paper can be substituted, or students can be instructed to draw a 5 by 5 array of dots on plain paper.

INVESTIGATION

Students may use various methods to find the number of nails on the geoboard that represents 16% of the total. One intuitive argument might be similar to this: "If there were 100 nails, then 16% of the nails would be $\frac{16}{100}$ of 100, or 16. Since there are only 25, or $\frac{1}{4}$ as many nails, the answer must be $\frac{1}{4}$ of 16 or 4." Some students may recall methods of computing with percent and use the method shown in Discussion Exercise 2. The success of your students in the Investigation may help you to determine the amount of instruction needed.

DISCUSSION

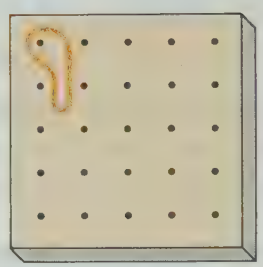
Allow sufficient time for students to discuss the various methods they used in the Investigation. In Exercise 2, 16% is written as a decimal to find 16% of 25. Of course one may use $\frac{16}{100}$ or $\frac{4}{25}$ as well as the decimal; however, most students find it easier to compute with decimals. For percents such as 25%, 50%, and 75%, the fractional notation may be the easiest to use.

Exercises 4 and 5 apply percent to problems involving money. If you wish, you may give as additional examples some problems which involve rounding answers to the nearest cent, such as those in Exercise 3 on the next page.

Finding a Percent of a Number

Investigating the Ideas

Use a geoboard like this one for the Investigation.



Can you place a rubber band around 16% of the number of nails on the geoboard?

Discussing the Ideas

1. How did you find the number of nails to enclose on the geoboard?

2. Study the method that Jim used to find 16% of the number of nails on the geoboard.

   Use Jim's method to find 24% of the number of nails on the geoboard.

JIM

16% of 25 means 16% × 25.

16% = 0.16

0.16 × 25 ⟶

$$\begin{array}{r} 25 \\ \times 0.16 \\ \hline 150 \\ 25 \\ \hline 4.00 \end{array}$$

3. Could you enclose 10% of the nails on the geoboard with a rubber band? Explain your answer.

4. Joan had $3.00. She spent 40% of her money.
   - A How can you find how much money she spent?
   - B How can you find how much money she had left?

5. A sports coat was originally priced at $60. During a sale, it was marked "25% off."
   - A What does "25% off" mean?
   - B How would you find how much you would have to pay for the coat during the sale?

Using the Ideas

1. Find the percent of each number.

A 20% of 100	**D** 50% of 124	**G** 1% of 100
B 35% of 100	**E** 10% of 850	**H** 3% of 100
C 25% of 40	**F** 23% of 200	**I** 5% of 80

2. Find the products. Replace each percent by a decimal before multiplying.

A 25% × 18	**D** 35% × 244	**G** 12% × 82.3
B 52% × 278	**E** 78% × 549	**H** 45% × 2.36
C 80% × 16	**F** 1% × 1569	**I** 2% × 139

3. Find the percent of each amount of money. Round your answer to the nearest cent.

EXAMPLE:
Find 15% of $27.98

$$\begin{array}{r} \$27.98 \\ \times 0.15 \\ \hline 13990 \\ 2798 \\ \hline \$4.1970 \text{ or } \$4.20 \end{array}$$

A 74% of $1000	**F** 24% of $1250
B 33% of $75	**G** 25% of $36.84
C 40% of $89.95	**H** 11% of $1.95
D 10% of $96.47	**I** 8% of $320.49
E 1% of $234.19	**J** 99% of $67.95

Solve each problem.

4. A winter coat was originally priced at $75. During a sale it was marked "30% off."

A How much less is the sale price?

B For what amount will it be sold?

5. Margaret earned $785 one month. 22% of her earnings were deducted for taxes. How much money was deducted for taxes?

6. Between 1960 and 1970, the population of a certain city increased 20%. If the 1960 population was 60 000, what was the 1970 population?

7. A salesman sold a car for $3400. He earned a commission of 8%. How much money did he earn for this sale?

8. A suit of clothes originally priced at $100 was reduced by 30%. Later the suit was reduced by 20% of the reduced price. What was the final price of the suit?

9. After spending 60% of her money for a sweater, Irene had $8.00 left. How much did she have originally?

UTILIZATION

Assign the exercises according to the needs and abilities of your students. Allow time for discussion of answers, especially of the word problems in this set.

Solution, Exercise 8

The suit was *not* reduced by 50%. The second reduction was on the sale price, not the original price. The first reduction was 30% of 100 = $30. The second reduction was 20% of $70 or $14. Thus the sale price is $56.

Solution, Exercise 9

Since Irene spent 60% of her money, she must have 40% of her original amount left. If x = Irene's original amount of money, then $0.40 \cdot x = \$8.00$, and $x = \$8.00 \div 0.40 = \$20.00$.

EXTENSION

Use *Workbook* page 69 and *Duplicator Masters* page 49 to provide more practice with these concepts.

Enrichment: Make a collection of newspaper advertisements which show sale prices in terms of percent reduction. Post the advertisements on the bulletin board with one problem based on each advertisement below it. Have the students solve the problems. Challenge them to make up their own problems about the advertisements.

Some students might like to follow up this activity by making charts like those described in Research Project B on page B-125.

Assignments

Minimum 1A–F, 2A–F, 3A–D, 4, 5.
Average 1–7. Maximum 1–9.

Objective

Given a percent such as $3\frac{1}{2}\%$, 12.3%, or 300%, the student can express the percent as a decimal and solve problems involving such percents.

PREPARATION

Review the meaning of percent notation and give examples in which students write a given percent as a fraction or a decimal.

INVESTIGATION

The Investigation introduces the student to a method of finding a decimal for a given percent by means of a flow chart. The ability to use this method is especially important in problems involving such percents as $5\frac{1}{4}\%$ or 18.5%. For example, if a problem asks for the amount of interest at the rate of 7% to be paid on a loan of $400, students can think: "$7.00 for each of the four 100's, or $28 interest." However this type of reasoning is of little help in finding the interest, for example, on $784.93 at a rate of 5.25%. In this case, the interpretation $n\% = n \times 0.01$ is much more practical.

DISCUSSION

Since $n\%$ means $n \times 0.01$, in Exercise 1 students should suggest that a percent can be written as a decimal by dropping the percent symbol and shifting the decimal in n two places to the left.

Stress the fact that 100% of a number means 1 times the number, 150% means $1\frac{1}{2}$ times the number, 200% means 2 times the number, and so on.

When discussing Exercise 3, you may find it helpful to give a set demonstration of $\frac{1}{2}\%$. Using a 10 by 10 square, ask the students how much of the square should be shaded to show 1% of the square. When they have observed that 1 square represents 1%, ask how much of the square should be shaded to show $\frac{1}{2}\%$. They should then be able to see that $\frac{1}{2}$ of one square represents $\frac{1}{2}\%$ of the whole square.

Using Percents and Decimals

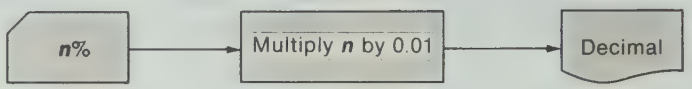
Investigating the Ideas

In problems using percents, you may often need to write a decimal for a percent.

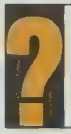
Since percent means "per hundred," you can think of

$n\%$ as meaning $n \cdot \frac{1}{100} = \frac{n}{100}$ or n .01

The flow chart below shows the steps for writing a decimal for a percent.



27% 27 0.01 0.27



Can you use the flow chart to help you write a decimal for each percent below?

A 38% **B** 3% **C** 12.5% **D** 175% **E** 0.8%

Discussing the Ideas

1. Can you describe a rule for finding a decimal for a percent that is suggested by the flow chart?

2. **A** Use the flow chart to find the decimal for 200%.
   B Jan said, "I earned 200% of my last week's earnings." What do you think Jan meant?
   - **A** 200 times her last week's earnings.
   - **B** 20 times her last week's earnings.
   - **C** 2 times her last week's earnings.

3. Explain each step for finding a decimal for $\frac{1}{2}\%$. Give the decimal for the screen.

   $\frac{1}{2}\% = 0.5\% = 0.5 \cdot 0.01 =$ ▮▮▮

4. Study Maria's method for finding $3\frac{1}{2}\%$ of 40. Explain how you would find $1\frac{1}{2}\%$ of 64.

$3\frac{1}{2}\% = 3.5\% = 0.035$

$$\begin{array}{r} 40 \\ \times 0.035 \\ \hline 200 \\ 120 \\ \hline 1.400 \end{array}$$

maria

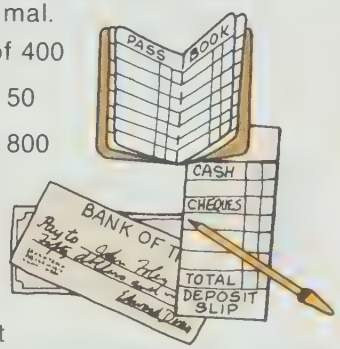
Using the Ideas

1. Write each percent as a decimal.

A 11%	**E** 123%	**I** 150%	**M** .03%	**Q** 34.2%	**U** $87\frac{1}{2}\%$
B 24%	**F** 1%	**J** 400%	**N** 12.35%	**R** 3.42%	**V** $2\frac{1}{2}\%$
C 81%	**G** 7%	**K** 250%	**O** 42.6%	**S** 0.5%	**W** $12\frac{1}{2}\%$
D 101%	**H** 4%	**L** 1000%	**P** 12.5%	**T** 0.1%	**X** $3\frac{1}{4}\%$

2. Find the products. First replace each percent by a decimal.

A 35% of 84
B 300% of 55
C 125% of 48
D 1000% of 25
E 240% of 6.5
F 6% of 45.3
G $37\frac{1}{2}\%$ of 400
H $8\frac{1}{2}\%$ of 50
I $6\frac{1}{2}\%$ of 800



3. Alvin had 64 customers on his paper route. He got several new customers and now has 125% as many as before. How many customers does he have now?

4. Marge earned $12 one week from babysitting. The next week she only earned $62\frac{1}{2}\%$ as much. How much money did she earn the second week?

5. Mrs. Johnson owed a bill of $54. She was charged $1\frac{1}{2}\%$ of this amount for a service charge.

A How much did she have to pay for the service charge?

B What was the total amount she had to pay?

CHARGE-IT-ALL Bank card statement	
Charges to date	$54.00
Service charge $1\frac{1}{2}\%$	
TOTAL	

6. Mr. Allen earned $240 one week. 12.5% of earnings were held for taxes. How much was withheld for taxes?

7. Lindbergh's flight from New York to Paris in 1927 took about 33 hours 30 minutes. In 1961, Major W. R. Payne flew from New York to Paris in a Hustler jet bomber in about 9.9% of this time. What was Major Payne's time?

think

There are 12 dentists in a town. In one class of 26 students, each student goes to one of these dentists.

1. Is it possible that some dentist has none of these students as a patient?
2. Is it possible that each student goes to a different dentist?
3. Show that at least three students go to the same dentist.

More practice, page S-24, Set 40

UTILIZATION

Exercises 1 and 2 provide practice in writing percents as decimals, and Exercise 2 also involves finding products. The remaining exercises involve word problems using percents like those in the first two exercises.

When the students have completed the exercises have some of them show their word-problem solutions on the chalkboard and explain their methods.

EXTENSION

Further practice exercises are provided by Supplementary Exercise Set 40 on page S-24, *Workbook* page 70, *Duplicator Masters* page 50, and *Arithmetic Skill Card* D-11.

Think Solution

1. Yes. Each student must go to one of the 12 dentists, but all 26 students could go to one of the dentists and still satisfy this condition.
2. No. There are not enough different dentists.
3. If each of the 12 dentists had two patients, 24 students would be accounted for. The two students left over would have to go to one of these dentists. Thus, at least three students must go to the same dentist.

Assignments

Minimum 1–4. Average 1–6. Maximum 1–7.

Objective

The student can solve percent problems based on the pieces of the tangram puzzle.

PREPARATION

This is an optional lesson and should be considered a change of pace or enrichment rather than a part of the basic sequence. (See the Long-Range Planning Chart in the Introduction to this Teachers' Edition.)

Considerable time will be saved if duplicated copies of the tangram puzzle square are available for students' use in this lesson. The pieces should be the same size as shown in the text.

INVESTIGATION

Have the students trace and cut out the tangram puzzle (unless duplicated copies are available). In order to determine what percent of the complete square the small square represents, some comparisons are necessary. It can be readily seen that each of the two large triangles is $\frac{1}{4}$ or 25% of the large square. If the middle-sized triangle is placed on the large triangle, it can be seen that the former is only $\frac{1}{2}$ as large, so it must be $12\frac{1}{2}\%$ of the large square. Each of the two small triangles is $\frac{1}{2}$ the size of the middle-sized triangle, or $6\frac{1}{4}\%$ of the large square. Since the two small triangles can cover the small square or the parallelogram, each of the latter is $12\frac{1}{2}\%$ of the large square.

DISCUSSION

As the various percents are determined for each piece of the tangram puzzle, have students record the percent on the piece. This will aid them in doing the Utilization exercises. After all the percents have been determined, have a student list them on the chalkboard and show that the total is 100%.

Large Triangle	25%
Large Triangle	25%
Middle Triangle	$12\frac{1}{2}\%$
Small Triangle	$6\frac{1}{4}\%$
Small Triangle	$6\frac{1}{4}\%$
Square	$12\frac{1}{2}\%$
Parallelogram	$12\frac{1}{2}\%$
Total	100%

Combining Percents

Investigating the Ideas

The seven colored regions at the right are the pieces of the tangram puzzle.

What percent of the complete square region do you estimate the small square to be?

Can you trace the puzzle, cut out the pieces, and find a way to check your estimate?

Discussing the Ideas

1. The two large triangular regions cover half of the large square of the tangram puzzle.
   - A What percent of the complete square region do they cover?
   - B What percent of the entire square region does one of the large triangular regions cover?

2. A What percent of one of the large triangular regions does the middle sized triangular region cover?
   - B What percent of the complete square region does the middle sized triangular region cover?

3. What percent of the complete square region does one of the small triangular regions cover? How can you decide?

4. Explain how to find the percent of the complete square region covered by the small square.

5. Find what percent of the large square region is covered by the parallelogram.

6. Can you use some of the pieces and form a square that is 25% as large as the complete square region?

Using the Ideas

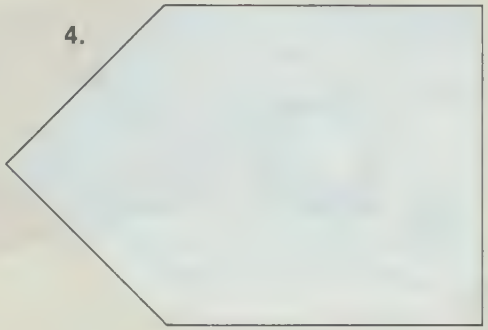
Cover each region with your tangram pieces. Then give the percent of the entire puzzle that the pieces would cover.

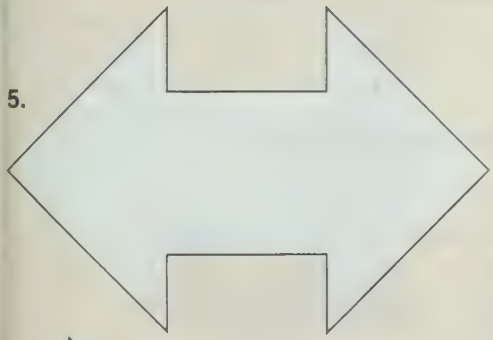
1.

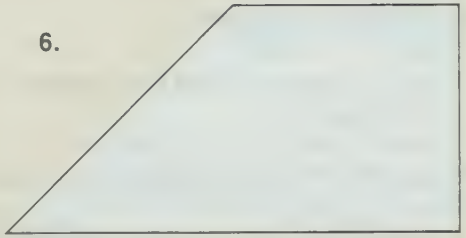
Answer : 25%

2.

3.

4. 

5. 

6. 

7.

8.

UTILIZATION

Many of the regions in the exercises can be covered in more than one way. In Exercise 1, for example, the region can be covered by the two small triangles and the square, by the middle-sized triangle and the two small triangles, or by the parallelogram and the two small triangles.

If you wish, you may have the students record the pieces they use to cover each shape so that afterwards they can discuss various ways to determine the solution.

EXTENSION

Enrichment: Have the students create additional problems like the ones on this page. Select the best designs and prepare duplicated copies of the problems for students to solve.

Some students might be encouraged to use posterboard or thin pieces of wood to make more durable copies of the tangram pieces.

Assignments

Minimum 1–4. Average 1–6. Maximum 1–8.

Objective

Given the price of an article and the sales tax rate on the article, the student can determine the amount of sales tax.

PREPARATION

Materials: Newspaper advertisements, sales tax table (optional).

Discuss the local sales tax rate with your students. Almost everyone will have had some experience with sales tax. Discuss what kinds of items are not subject to sales tax. Many local and provincial governments do not levy a sales tax on food, except for restaurant meals. Sales taxes are not ordinarily charged on gasoline since provincial and federal taxes are already included in the price of the gasoline.

INVESTIGATION

Students should be supplied with a wide variety of old newspapers from which they may select their item. If this is not possible, use the items pictured in the Investigation section. If there is no sales tax in your locality, ask the students to pretend they are making the purchase in a locality in which a given sales tax applies.

If possible, secure a sales tax table like those used in retail stores, and suggest that students use it to check their findings for the Investigation.

DISCUSSION

Allow sufficient time for students to tell what article they selected and how they computed the amount of sales tax.

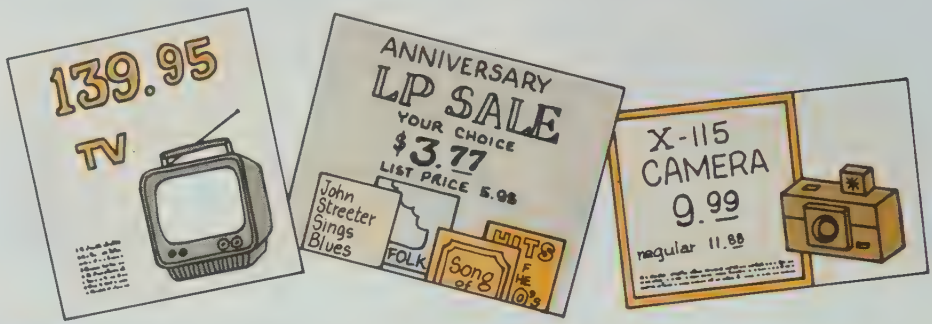
Since computing sales tax is simply one of the more familiar applications of finding a percent of a number, there is little that is mathematically new in the lesson. You should devote some time to discussing Exercises 2 and 3, which concern finding the total cost of an item including sales tax.

Sales Tax

Investigating the Ideas

In most provinces you must pay a sales tax on certain items that you buy. The sales tax rate is usually expressed as a percent of the amount purchased and may vary from 5% to 10%.

Do you know the sales tax rate in your community?

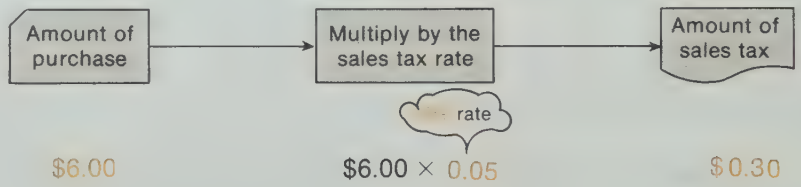


Can you choose an item from an advertisement like those above and find out how much sales tax you would have to pay on it?

Discussing the Ideas

1. Explain how you found the amount of sales tax on the item that you choose.

2. Study the flow chart below for computing the amount of sales tax.



How can you find the **total amount** that you must pay for the item costing $6.00 when the sales tax is 5%?

3. Explain how you would find the amount of sales tax and the total amount of the purchase if the item costs $18.75 and the local sales tax rate is 7%.

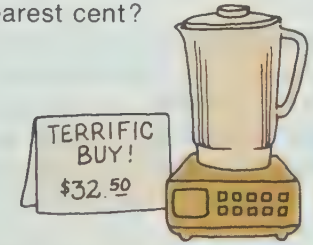
Using the Ideas

1. A new automobile costs $4200. The sales tax rate is 7%. How much sales tax must be paid on the car?

2. A suit of clothes was priced at $80. The sales tax rate was 5%.
   - A How much tax must be paid on the suit?
   - B What is the total cost of the suit including the sales tax?

3. What is the total cost of a pair of boots priced at $18.50 if the sales tax rate is 7%?

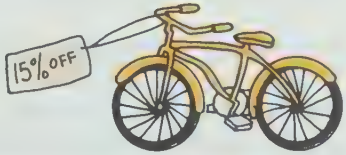
4. An electric blender costs $32.50. The sales tax rate is 6%. What is the total cost of the blender to the nearest cent?



5. Mrs. Wright had a grocery bill of $23.40. She had to pay a 5% sales tax on all the non-food items. These items totalled $4.80
   - A What was the amount of sales tax?
   - B What was the total cost of the groceries?

6. In a certain city, the sales tax rate was $5\frac{1}{2}$%. How much tax must be paid on a purchase of $100? On a purchase of $50?

7. Pamela bought a $30-dress that was marked "20% off." The sales tax rate was 6%. What was the total amount she had to pay for the dress?



8. Roger bought a new 10-speed bicycle. Its price was $120 but he got 15% off on a sale. The sales tax rate was 8%. What was the total cost of the bicycle?

Think

A used car dealer bought a car from a wholesaler for $600 and marked up the price 50% from what he had paid for it. When the car did not sell in two months' time, he had a 40%-off sale. How much money did he make or lose on the car?

UTILIZATION

The exercises provide experience in determining the amount of sales tax and the total cost of purchase for various items.

When the students have completed the exercises, have them display their solutions on the chalkboard and explain them to the other students.

EXTENSION

Arithmetic Skill Card AP-19 is recommended for students who would benefit from additional practice.

Enrichment: Ask students to cut out a newspaper advertisement and paste or tape it to a piece of paper. Beneath it they should write a story problem involving the local sales tax rate for one of the articles advertised. If each student makes up one or two such problems, an interesting booklet of sales tax problems can be compiled.

Now would be a good time to encourage some students to undertake Research Project C on page B-125.

Think Solution

$600 · 150% = $900 (Dealer's original marked price)

40% of $900 = $360 (amount of discount)

$900 − 360 = $540 (sale price)

The dealer bought the car for $600 and sold it for $540, so he lost $60.

Assignments

Minimum 1–4. Average 1–6. Maximum 1–8.

Objective

Given an amount of principal, the interest rate, and the time, the student can determine the amount of interest.

PREPARATION

Materials: Advertisements of bank or savings and loan institution rates.

The lesson will be more interesting if you use local interest rates in the Investigation. A field trip to a bank could be planned, or a representative from a bank might be invited to speak to your class briefly on the basic functions of banks. Many banks have brochures that advertise interest rates. Spend some time discussing how simple interest is determined for a given amount of money.

INVESTIGATION

If you want an open-ended Investigation, have different students choose different interest rates and compute the amount $200 would earn in a year at each different rate. Otherwise, choose a standard rate and have everyone use that rate.

DISCUSSION

The terms *principal*, *interest*, and *interest rate* are introduced in the first two discussion exercises. Exercises 3 and 4 are designed to help the student understand the interest formula, which is given in Exercise 5 with the steps shown by means of a flow chart. Only simple interest is treated in this lesson.

Interest

Investigating the Ideas

Suppose that you had this much money.

Can you find how much your money would earn if you put it into a savings account in a local bank for one year?

Discussing the Ideas

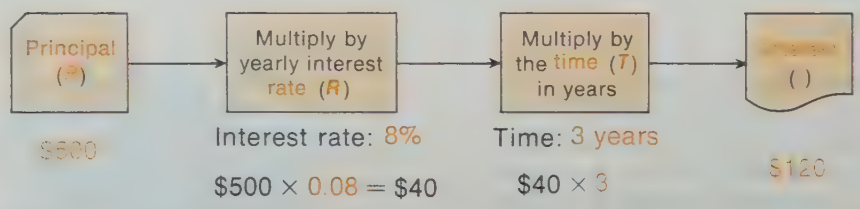
1. The money that a bank pays you for the use of your money (**principal**) is called **interest**.
   How much interest would you earn on the amount of money in the Investigation? How did you find it?

2. The **interest rate** is usually expressed as a percent and is often a **yearly** rate.
   What are some yearly interest rates at some local banks?

3. It is helpful to think of interest as so many dollars per $100 saved or borrowed.
   For example, if you borrow $500 for one year at a yearly rate of 8%, the interest charge is $8.00 per $100
   How much interest must you pay if you borrowed $500?

4. If you borrowed $500 at 8% yearly interest rate for **3 years**, you would have to pay 3 times as much interest as for one year. How much interest would you have to pay for 3 years?

5. The flow chart will help you understand the interest formula $I = P \cdot R \cdot T$.



Can you use the flow chart to find the interest on $1000 at a yearly rate of 6% for 6 months $\left(\frac{1}{2}\text{ year}\right)$?

Using the Ideas

1. Find the interest for one year given the following amounts and rates.

A \$200 at 6% per year
B \$600 at 7% per year
C \$500 at 5% per year
D \$800 at 4% per year
E \$300 at 4% per year
F \$200 at 8% per year
G \$700 at 2% per year
H \$600 at 9% per year
I \$400 at $6\frac{1}{4}$% per year
J \$300 at $5\frac{1}{2}$% per year
K \$1000 at $8\frac{3}{4}$% per year
L \$1200 at $\frac{3}{4}$% per year

2. Find the interest using the formula $I = P \cdot R \cdot T$.

A \$355 at 5% per year for 2 years
B \$800 at 7% per year for 4 years
C \$4200 at 8% per year for 3 years
D \$21,500 at 4% per year for 10 years
E \$450 at 6% per year for $1\frac{1}{2}$ years
F \$620 at 8% per year for 4 mon

3. Sometimes interest rates are expressed as **monthly rates** rather than yearly rates. For example, if the monthly interest rate is 2%, this is equivalent to a yearly interest rate of $12 \times 2\%$ or 24% per year.

Express each monthly rate as a yearly rate.

A 1% **B** $1\frac{1}{2}$% **C** 3% **D** $2\frac{1}{2}$% **E** $1\frac{1}{4}$% **F** $\frac{3}{4}$%

4. Give the monthly rates equivalent to the following yearly rates.

A 12% **B** 18% **C** 6% **D** 8% **E** 9% **F** 30%

5. Mr. Nickols borrowed \$300 for one year from his bank. The interest rate was 1% per month. How much interest did Mr. Nickols have to pay at the end of one year?

6. Suppose that you wanted to borrow \$1000 for one year. Bank A has a 10% yearly interest rate. Bank B has a $\frac{3}{4}$% monthly interest rate.

A Which bank would charge you less interest?
B How much less interest would you have to pay that bank?

7. Carol had \$150 in her savings account. The yearly interest rate was $4\frac{1}{2}$%.

A How much interest did her money earn in one year?
B How much money would Carol have in her account after the interest was added to her savings?

8. Mr. Marsh borrowed \$22500 from a bank to buy a house. He paid a $7\frac{1}{4}$% interest rate on the amount he borrowed during the first year. How much interest did he pay the first year?

UTILIZATION

Exercise 3 introduces the idea of a monthly interest rate. This might lead into a discussion of how borrowers can be misled by advertising which quotes the interest rate on a monthly basis. For example, a company may advertise that they lend money at the rate of $1\frac{1}{2}$% per month. Although this seems a small amount, it actually corresponds to 18% interest on a yearly basis.

When your students have completed the assignment, allow some time for discussion of these exercises.

EXTENSION

For reinforcement, use *Workbook* page 71, *Duplicator Masters* page 51, and *Arithmetic Skill Card* AP-18.

Enrichment: A problem that involves the idea of *compound interest* can be used to challenge your more able students:

Suppose you place \$100 in a savings account at 5% yearly interest rate. At the end of 1 year you would have \$100 + \$5.00 interest in your savings. If you allowed the interest to accumulate in this manner each year how much money would you have in your savings after 5 years? after 10 years?

Answer: After 5 years: \$127.63
After 10 years: \$162.89

This would be an opportune time to refer interested students to Research Project D on page B-125, which suggests a compound interest activity.

Assignments

Minimum 1A–D, 2A–C, 3–5.
Average 1E–L, 2–6.
Maximum 1–8.

Finding What Percent One Number is of Another

Objective

Given two numbers, the student can find what percent the first number is of the second.

PREPARATION

Plan a brief review of writing decimals for fractions and writing a fraction for some decimals.

INVESTIGATION

Have the students read the Investigation and use their own method of determining which school has won the greater part of their games. Some students may use ratios that have the same denominator, while others may find a decimal or a percent for the ratio. The Investigation provides the basis for the Discussion, which introduces the idea of finding what percent one number is of the second.

DISCUSSION

Notice the sequence of exercises. In Exercise 1 the student describes his method of comparing the number pairs in the Investigation. Exercise 2 suggests comparing the ratios by means of decimals. Exercise 4 introduces the language "x is what percent of y?" Parts A, B, and C relate this to the numbers used in the Investigation. You might want to summarize the ideas brought out in Exercises 2–4 as shown below.

School	Ratio	Decimal	Percent
Rockville	$\frac{9}{20} = \frac{108}{240}$	0.4500	45%
Jamestown	$\frac{6}{15} = \frac{96}{240}$	0.4000	40%
Brownsburg	$\frac{7}{16} = \frac{105}{240}$	0.4375	$43\frac{3}{4}\%$

Give additional examples of the type in Exercise 4, if needed, before moving on to the Utilization stage.

Investigating the Ideas

Study the score board below.

SCHOOL	GAMES WON	GAMES PLAYED
Rockville Junior High	9	20
Jamestown Junior High	6	15
Brownsburg Junior High	7	16

Can you find which school has won the greater part of their games?

Discussing the Ideas

1. How did you find which school had won the greater part of their games?

2. To compare the school's records, you could compare the fractions $\frac{9}{20}$, $\frac{6}{15}$, and $\frac{7}{16}$. This is often easier to do if you use decimals. Find a decimal in hundredths for each fraction.

   A $\frac{9}{20} \rightarrow 20\overline{)9.00}$ B $\frac{6}{15} \rightarrow 15\overline{)6.00}$ C $\frac{7}{16} \rightarrow 16\overline{)7.00}$

3. Express each decimal in Exercise 2 as a percent.

4. In order to find what percent one number **x** is of another number **y**, you can write $\frac{x}{y}$ and find a percent for the fraction.

   EXAMPLE: 3 is what percent of 8?

   SOLUTION: $\frac{3}{8} = 0.37\frac{1}{2} = 37\frac{1}{2}\%$

   3 is $37\frac{1}{2}\%$ of 8.

   Use your answers to Exercise 2 to complete each statement.

   A 9 is ____% of 20. B 6 is ____% of 15. C 7 is ____% of 16.

Using the Ideas

1. Find the number for each ▒.

A 4 is ▒% of 20
B 3 is ▒% of 10
C 8 is ▒% of 16
D 4 is ▒% of 5
E 1 is ▒% of 100
F 54 is ▒% of 200
G 27 is ▒% of 300
H 19 is ▒% of 76
I 12 is ▒% of 160

Solve each problem.

2. If 1 of every 2 persons has at least one cold a year, what percent of all people have at least one yearly cold?

3. If 4 of every 10 trees in an orchard are apple trees, what percent of all trees are apple trees?

4. If 1 litre of fruit juice is used in making 5 litres of punch, what percent of the punch is fruit juice?

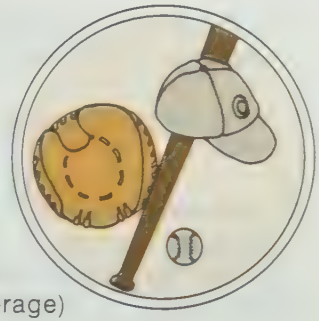
5. The areas of two pieces of property are approximately in the ratio 3:5. Express the area of lot A as a percent of the area of lot B.

6. In a group of 500 people, 35 are more than seventy years old. What percent of the group is more than seventy years old?

7. A coat was priced at $60 but was later sold for $48. The sale price was what percent of the original price?

8. Find what percent the first number is of the second. Round your answers to the nearest tenth of a percent.

A 11, 46
ANSWER: 23.9%
B 28, 55
C 34, 175
D 87, 266
E 28, 21
F 3.2, 12.4
G 136,541
H 725, 8088



9. In baseball, a "300 hitter" is a person who makes a hit in at least 0.300 or 30% of his times at bat. In 1973, Pete Rose made 230 hits in 680 times at bat.

A Was Pete Rose a "300 hitter" that year?
B Express the percent of times he got hits (batting average) as a percent to the nearest tenth of a percent.

10. Don Kessinger has been at bat 311 times and made 90 hits.

A What is his average?
B How many consecutive hits does he need in order to be a "300 hitter"?
C What will his average be if he is hitless for 7 times at bat?

More practice, page S-24, Set 41

UTILIZATION

Assign the exercises for independent work by the student. Most students will benefit by a discussion of the exercises after completing them. Solutions to the word problems can be placed on the chalkboard for student analysis and discussion.

EXTENSION

You may want to refer students to Supplementary Exercises Set 41 on page S-24, *Workbook* page 72, and *Duplicator Masters* page 52 for additional practice.

Assignments

Minimum 1–5. Average 1–8. Maximum 1–10.

Objectives

The student can estimate what percent the area of one region is of another.

The student can use rounding to estimate a given percent of a number.

PREPARATION

Plan an oral review of simple fractions and percents. Students should know these relations:

$\frac{1}{4} = 25\%$, $\frac{1}{2} = 50\%$, $\frac{3}{4} = 75\%$

$\frac{1}{10} = 10\%$, $\frac{1}{5} = 20\%$, $\frac{3}{5} = 60\%$, $\frac{4}{5} = 80\%$

Some practical estimation of percents would also be helpful. Use questions like these:

1. What percent of the wall area is occupied by chalkboards?
2. What percent of the class are boys?
3. What percent of the chairs or desks are occupied?
4. What percent of your desk top does your book cover?

INVESTIGATION

After the students have read the Investigation and studied the graph, ask them to write their estimates on a piece of paper. If possible, have reference books available so that they can check their estimates.

Pacific Ocean	166 000 000 km²
Atlantic Ocean	85 000 000 km²
Indian Ocean	73 000 000 km²
Arctic Ocean	10 000 000 km²

DISCUSSION

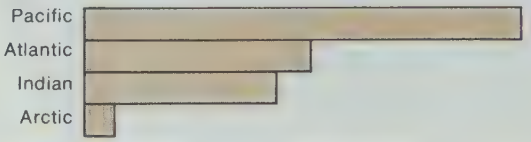
The students will probably use ratio ideas in comparing the areas of the oceans and then express the ratios in terms of percents. Thus the Atlantic Ocean is about $\frac{1}{2}$ or 50% as large as the Pacific. The Indian Ocean is about 45% as large as the Pacific. The Arctic Ocean is less than 10% of the size of the Pacific. Encourage comparison of other pairs of oceans during the discussion.

Exercises 5 and 6 introduce estimation of a percent of a number by means of rounding. This practical kind of estimation is valuable in everyday affairs and also will aid students in checking their work for gross errors in their answers.

Estimating with Percents

Investigating the Ideas

The bar graph compares areas of the oceans of the world. The area of the Pacific Ocean is about 166 000 000 square kilometres.



Can you estimate the area of each of the other oceans?	Use a reference book to check your estimates.

Discussing the Ideas

1. How did you make each of your estimates?

2. The area of the Atlantic Ocean is about what percent of the area of the Pacific Ocean?

3. Is the area of the Arctic Ocean more or less than 10% of the area of the Pacific Ocean?

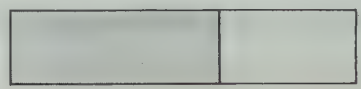
4. How did you make your estimate of the area of the Indian Ocean?

5. To estimate 31.7% of 496.4, you can round the numbers and think "30% of 500 is 150." Therefore, 31.7% of 496 is about 150. Explain how you would make estimates for these products.

   A 48% of 81.42 **B** $6\frac{1}{2}\%$ of 7295 **C** 19.8% of 698.2

6. If the Arctic Ocean is about 12.1% as large as the Atlantic Ocean and the Atlantic Ocean is 86 550 000 square kilometres, what would you estimate the area of the Arctic Ocean to be?

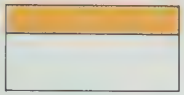
7. What is your estimate for the percent of the rectangle that is shaded?



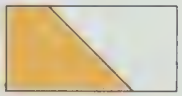
Using the Ideas

1. Estimate what percent of each rectangular region is shaded.

A

B 

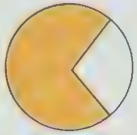
C

D 

2. Estimate what percent of each circular region is shaded.

A

B

C 

D

3. The bar graph compares the areas of the seven continents.

- **A** Which continent is about 50% as large as Asia?
- **B** Which continent is about 50% as large as South America?
- **C** Antarctica is about what percent of the size of Asia?

4. Mr. Freed took his family to a restaurant for dinner. The check for the dinner totalled $23.85. Mr. Freed figured approximately 15% of this amount for a tip. Which amount is the best approximation of the tip?

A $3.00 **B** $3.50 **C** $3.75 **D** $4.00

5. Estimate the sales tax, to the nearest dollar, on a used car which is priced at $2499 if the provincial sales tax rate is

A 5% **B** 6% **C** 7% **D** 8% **E** 9%

6. Choose the best estimate for each exercise.

A	3% of 369	(100, 10, 1000, .1, 0.01)
B	27% of 46	(12, 1.2, 120, 0.12, 0.012)
C	98% of 3.7	(0.37, 0.037, 37, 3.7, 370)
D	16% of 284	(0.45, 0.045, 450, 4.5, 45)
E	49% of 68.4	(34, 3.4, 0.34, 340, 0.034)
F	33% of 200	(660, 6.6, 66, 0.66, 0.066)
G	.9% of 167	(167, 1.67, 16.7, .167, 1670)

think

Six similar bricks balance 3 large blocks that are also similar. Each block weighs 2 kg more than a brick. What is the mass of 1 block?

UTILIZATION

Exercises 1, 2, and 3 concern percentage estimates for various regions. Exercises 4, 5, and 6 provide practice in estimating a percent of a number by rounding the numbers given in the exercises. Exercises 1 and 2 might be done orally.

EXTENSION

Page 73 of the *Workbook* contains exercises that would be appropriate for additional practice.

Remedial: Encourage students to estimate 1% or 10% of numbers and then use the results to make final estimates.

Give problems like these:

1. If $x = 56$, find each amount.
   - A. 10% of x
   - B. 1% of x
   - C. 0.01% of x
   - D. 1000% of x
2. If $y = 28.3$, find each amount.
   - A. 10% of y
   - B. 1% of y
   - C. 100% of y
   - D. 0.01% of y

Think Solution

An algebraic solution to this problem follows: Let x equal the weight of a small block. Thus, $2x + 3$ equals the weight of a large block.

$$(2x + 3) + (2x + 3) + 3x = 41$$
$$7x + 6 = 41$$
$$7x = 35$$
$$x = 5$$

A large block weighs $(2 \cdot 5) + 3 = 13$.

Your students might reason like this: If s = small block and L = large block, then

$$s + s + s + L + L = 41$$
$$s + s + s + (s + s + 3) + (s + s + 3) = 41$$

If you deduct the extra 3 kg in each large block, you would have

$$s + s + s + (s + s) + (s + s) = 35,$$

or 7 small blocks weighing 35 kg. Each small block must weigh 5 kg; thus a large block weighs $2 \cdot 5 + 3$ or 13 kg.

Assignments

Minimum 1, 2, oral; 3; 4. Average 1–5. Maximum 1–6.

Objective

The student can demonstrate the ability to work with the concepts covered in this module.

UTILIZATION

The exercises can be worked independently by the students and then discussed, or they might form the basis for a class discussion of the main topics covered in the module.

Some students should put their work for selected exercises on the chalkboard and discuss their solution methods with their classmates.

Additional appropriate review exercises are provided on page 74 of the *Workbook*.

REVIEWING THE IDEAS

1. Express each ratio in simplest terms.

A 8:12 B 20:50 C 35:25 D 16:4

2. In the proportion $3:8 = 15:40$,

A which numbers are the means?

B which numbers are the extremes?

3. Solve the proportions for n.

A $\frac{2}{5} = \frac{n}{20}$ C $\frac{3}{n} = \frac{4}{12}$

B $4:15 = 12:n$ D $n:6 = 20:8$

4. Express each ratio in simplest terms.

A 300 cm to 5 m C 240 mm to 12 cm

B 40 dm to 3 m D 600 mm to 1 m

5. The scale for a drawing of a house is 1:100. The living room is pictured on the drawing as a 6.2 by 9.5-cm rectangle. What is the actual length and width of the living room?

6. Express each percent as a decimal.

A 23% C 250% E 0.01%

B 6% D 1.5% F 0.003%

7. Express each decimal as a percent.

A 0.35 C 1.25 E 0.7

B 0.04 D 5 F 0.009

8. Express each percent as a lowest-terms fraction.

A 50% C 75% E 2%

B 30% D 80% F $\frac{1}{2}$%

9. Express each fraction as a percent.

A $\frac{1}{4}$ C $\frac{7}{10}$ E $\frac{3}{8}$

B $\frac{3}{5}$ D $\frac{3}{10}$ F $\frac{1}{500}$

10. Find the percent of each number.

A 42% of 87 C 60% of \$72.95

B 9% of 283 D $37\frac{1}{2}$% of \$800

11. A sweater costs \$14. The provincial sales tax rate is 7%.

A How much tax on the sweater?

B What is the total cost of the sweater?

12. Find the interest on each amount.

A \$200 at 7% per year for 1 year

B \$450 at 6% per year for 3 years

C \$225 at 8% per year for $1\frac{1}{2}$ years

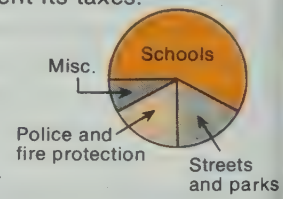
D \$600 at 5% per year for 4 months

13. What is the yearly interest rate of the monthly rate is $1\frac{1}{4}$%?

14. A baseball team won 9 out of 21 games. What percent of the games, to the nearest tenth, were won?

15. A television set priced at \$300 was reduced in price by \$24. What percent was it reduced?

16. The circle graph pictures how a certain city spent its taxes. Estimate the percent of all tax money spent for each category.



17. Choose the number closest to 2% of 79 534.12.

A 160 000 C 1600 E 16

B 16 000 D 160 F 1.6

★ **18.** The French franc is worth 20% as much as the dollar.

A What is the value in dollars of 500 French francs?

B How many francs could you get in exchange for \$50?

TEST YOURSELF

1. What is the simplest terms ratio for 16:20?

2. Solve the proportion: $\frac{n}{8} = \frac{12}{32}$

3. The ratio of 18 mm to 3 cm is

   A $\frac{18}{3}$ B $\frac{3}{5}$ C $\frac{3}{18}$ D $\frac{5}{3}$

4. A scale on a map is 1 cm = 48 km. What is the actual distance between two cities if the distance on the map is 3.5 cm?

5. Express as a decimal.

   A 17% B 6.2% C 0.4%

6. Express as a percent.

   A $\frac{4}{5}$ B $\frac{1}{8}$ C $0.08\frac{3}{4}$ D 0.237

7. What is 38% of 420?

8. A How much sales tax must be paid on a purchase of $26.95 if the sales tax rate is 5%?

   B How much interest on $500 at a rate of 8% per year for $2\frac{1}{2}$ years?

9. In a class of 20 children, 4 of them wear glasses. What percent of the children wear glasses?

10. Which is the best estimate for $8\frac{3}{4}$% of 319?

    A 27 B 2.7 C 0.27 D 270

RESEARCH PROJECTS

A Make a scale drawing of the floor plan of your home or of some building. Choose a suitable scale to show the dimensions of the building as well as some details such as windows and doors.

B Select some newspaper advertisements that show different percents of discounts for articles that are on sale. Make a chart that shows the regular price, the percent of reduction (discount), the sale price, and the amount saved by purchasing the article during a sale.

C Obtain a copy of a sales tax table for your province. Show how it is used by retail sales people. Local business firms or provincial tax offices can provide tax tables.

D Visit a local bank to find out about different interest rates that are charged for borrowing money or for depositing money in savings accounts. Try to determine how long it would take for $100 of savings to double in value if it were left in a savings account.

TEST YOURSELF

This self-evaluation test should help students determine their strengths and weaknesses in terms of the concepts and skills developed in the module. Have the students reread the objectives for the module and discuss with you those objectives that they failed to reach. Adequate review or remedial instruction can then be initiated.

RESEARCH PROJECTS

Project A might provide valuable motivation for many students. Practical experience in measurement, rounding, estimation, geometry, and ratio are all involved in this project. Some creative students might be interested in drawing floor plans for houses they design themselves.

Students who undertake Research Project B should be provided with posterboard on which to display their advertisements and charts. Such posters can make an interesting and visually effective bulletin board display.

Students may find a study of a sales tax table an enlightening experience (Project C). The tables are usually constructed to show how much sales tax must be charged on each purchase amount within specified ranges. For example, in a state with a sales tax rate of 5%, all sales from 90¢ to $1.09 are subject to a 5¢ sales tax; from $1.10 to $1.29, the sales tax is 6¢; and so on. Thus, the amount of tax is rounded to the nearest cent.

Students who try Project D will learn that the amount of time it takes for a given amount of principal to double depends on the interest rate and how often the interest is compounded. At a rate of 4% the amount will double in 18 years if compounded annually. At 6% the amount will double in 12 years; and at 8%, in about 9 years.

Objective

The student can demonstrate the ability to work with the concepts presented in this unit.

UTILIZATION

Because of the length of the Cumulative Review, you may find it necessary to allow more than one day for the completion of all the exercises. A more effective plan might be to assign problems selectively and then devote some time to a discussion of the main ideas of the learning unit before administering any achievement test on it. Thus, the following correlation between review items and the modules of the unit may be useful:

Module number	1	2	3	4	5
Related exercises	1–7	8–15	16–22	23–30	31–40

Encourage students to work through the review independently, but afterwards to seek aid with those topics that have given them difficulty.

CUMULATIVE REVIEW

1. What fractional part of the figure is shaded?

A

B

2. Which fraction is equivalent to $\frac{6}{16}$?

A $\frac{3}{4}$ B $\frac{7}{8}$ C $\frac{1}{3}$ D $\frac{3}{8}$

3. Give three fractions that are equivalent to:

A $\frac{1}{3}$ B $\frac{3}{10}$ C $\frac{2}{9}$

4. In each set, which fraction is not equivalent to any of the other fractions in the set?

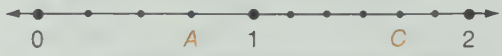
A $\{\frac{1}{3}, \frac{3}{12}, \frac{5}{15}, \frac{8}{24}, \frac{6}{18}\}$

B $\{\frac{6}{8}, \frac{15}{20}, \frac{3}{4}, \frac{9}{16}, \frac{18}{24}\}$

5. Give the lowest-terms fraction for each fraction.

A $\frac{15}{25}$ B $\frac{16}{40}$ C $\frac{14}{35}$ D $\frac{25}{150}$

6. Give a fractional number for each lettered point on the number line.



7. Give the correct symbol ($<$, $=$, or $>$) for each ▨.

A $\frac{6}{7}$ ▨ $\frac{5}{6}$ C $\frac{1}{2}$ ▨ $\frac{5}{9}$

B $\frac{5}{8}$ ▨ $\frac{7}{12}$ D $\frac{11}{5}$ ▨ $\frac{13}{6}$

8. Give an improper fraction for each mixed numeral.

A $1\frac{5}{8}$ B $3\frac{2}{5}$ C $4\frac{9}{10}$

9. Give a mixed numeral for each improper fraction.

A $\frac{11}{7}$ B $\frac{43}{10}$ C $\frac{42}{8}$

10. What is the least common denominator for $\frac{3}{5}$ and $\frac{1}{4}$?

11. Find the sums.

A $\frac{5}{12} + \frac{3}{4}$ C $\frac{1}{2} + \frac{1}{3} + \frac{1}{4}$

B $1\frac{2}{3} + \frac{5}{6}$ D $2\frac{3}{4} + 4\frac{3}{5}$

12. Find the differences.

A $\frac{11}{12} - \frac{3}{4}$ C $7 - 3\frac{2}{5}$

B $5\frac{1}{4} - \frac{7}{8}$ D $2\frac{2}{3} - 1\frac{1}{4}$

13. Give the missing numbers in each function table.

Function Rule $n + \frac{1}{6}$

	n	$f(n)$
	2	$2\frac{1}{6}$
A	$\frac{1}{2}$	▨
B	0	▨
C	$\frac{2}{5}$	▨
D	▨	1

Function Rule $n - \frac{3}{4}$

	n	$f(n)$
	2	$1\frac{1}{4}$
E	$1\frac{1}{2}$	▨
F	$4\frac{1}{8}$	▨
G	$3\frac{5}{6}$	▨
H	10	▨

14. Round each number to the nearest whole number.

A $3\frac{7}{8}$ B $\frac{19}{3}$ C $\frac{56}{8}$

15. It took $46\frac{9}{10}$ litres of gasoline to fill Mr. Wagner's car. The next time he filled his car, it took $43\frac{1}{2}$ litres. How many litres did it take for both fill ups?

16. Solve the equations.

A $\frac{1}{3} \cdot \frac{1}{2} = n$ C $3 \cdot \frac{1}{4} = n$

B $\frac{1}{4} \cdot \frac{1}{6} = n$ D $\frac{1}{8} \cdot 5 = n$

17. Find the products.

A $\frac{1}{4}$ of 12 **C** $\frac{1}{3}$ of 36

B $\frac{1}{8}$ of 72 **D** $\frac{1}{10}$ of 250

18. Find the products.

A $\frac{3}{5} \cdot \frac{1}{3}$ **C** $1\frac{2}{3} \cdot \frac{3}{4}$

B $\frac{9}{5} \cdot \frac{1}{3}$ **D** $2\frac{1}{2} \cdot 7\frac{3}{5}$

19. Give the reciprocals for each.

A $\frac{2}{7}$ **B** 5 **C** $2\frac{1}{3}$

20. Find the quotients.

A $\frac{3}{4} \div \frac{1}{2}$ **C** $1 \div \frac{9}{10}$

B $\frac{3}{5} \div \frac{2}{3}$ **D** $4\frac{1}{4} \div 2\frac{1}{8}$

21. Simplify each complex fraction.

A $\dfrac{\frac{3}{4}}{\frac{5}{6}}$ **B** $\dfrac{\frac{6}{7}}{\frac{9}{14}}$

22. If the system is in equilibrium, what is the distance **x**?

23. Write a fraction for each decimal.

A 1.7 **B** 0.318 **C** 0.04

24. Write a decimal for each fraction.

A $\frac{37}{100}$ **B** $1\frac{356}{1000}$ **C** $6\frac{9}{100}$

25. Find the sums.

A $2.8 + 3.47 + 19.068$

B $167.3 + 84.9 + 72.006$

26. Find the differences.

A $12.47 - 9.6$

B $83.49 - 17.864$

27. Find the products.

A 6.2×0.12 **B** 43.09×7.6

28. Find the quotients.

A $1.9\overline{)7.22}$ **B** $0.008\overline{)31.2}$

29. Find the repeating decimal for $\frac{4}{9}$.

30. Express $\frac{5}{6}$ as a mixed decimal numeral.

31. Solve the proportion: $\frac{n}{9} = \frac{12}{36}$

32. Express the ratio "300 cm to 5 m" in simplest terms.

33. Express each percent as a decimal.

A 27% **B** 115% **C** 8%

34. Express each decimal as a percent.

A 0.89 **B** 0.9 **C** 0.03

35. Express each percent as a lowest-terms fraction.

A 20% **B** 35% **C** 4%

36. Express each fraction as percent.

A $\frac{2}{5}$ **B** $\frac{3}{4}$ **C** $\frac{5}{8}$

37. What is 15% of 642?

38. A pair of shoes cost \$18.95 plus 5% sales tax. What was the total cost of the shoes?

39. How much interest on \$600 at a rate of $5\frac{1}{2}$% per year for 3 years?

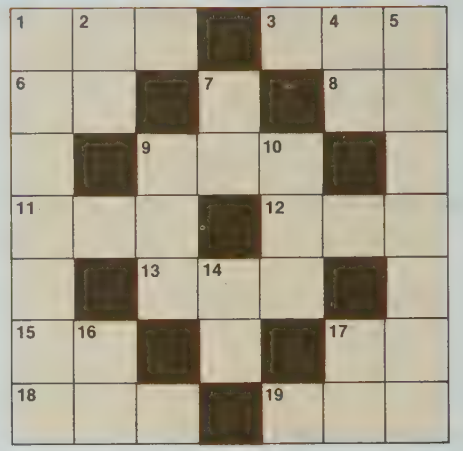
40. In a school survey, 12 students out of 40 preferred a shorter lunch hour. What percent of the students preferred a shorter lunch hour?

MATHEMATICAL RECREATION

Many students can enjoy this kind of puzzle. Encourage some of your more inventive students to create their own cross number puzzle. These can be most easily constructed on 1-cm graph paper by tracing over the grid lines and shading squares as appropriate. If the students use light-blue colored pencils to write in the solutions to the puzzles they create, the puzzles can then be duplicated by any of several processes without the solutions being reproduced, and given to other students to solve.

MATHEMATICAL RECREATIONS

A Cross Number Puzzle



Make a copy of the cross number puzzle grid on graph paper. Then find the numbers for the squares.

ACROSS

1. 2^8

3. The number of a large jet airplane.

6. XLVII

8. 3^4

9. The number of centimetres in 1.78 metres.

11. Base two numeral for 5.

12. 2381 − 1979

13. $19 \cdot 9$

15. $15\frac{3}{4} \times 4$

17. $0.002\overline{)0.07}$

18. The square of a number greater than 15 that ends with a five.

19. CMVII

DOWN

1. 2154×1129

2. $19.38 \div 0.34$

4. $\frac{12}{25} = \blacksquare\blacksquare\%$

5. The digits in the first seven decimal places for $\frac{5}{7}$.

7. The largest prime number less than 100.

9. $[(3 \cdot 3 \cdot 2 \cdot 2) + 1] \cdot 3$

10. 29^2

14. $2^2 \cdot 3^2$

16. Two to the fifth power.

17. A number whose prime factors are 2, 3, and 5.

MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. 4:5 **2.** $n = 3$ **3.** B **4.** 168 km **5.** A 0.17 B 0.062 C 0.004 **6.** A 80%; B $12\frac{1}{2}\%$; C $8\frac{3}{4}\%$; D 23.7% **7.** 159.6 **8.** A \$1.08; B \$100 **9.** 20% **10.** A

B–128

UNIT C: GEOMETRY AND MEASUREMENT

Module 1: Basic Concepts of Geometry

General Objectives

To review and extend the basic concepts of points, lines, and planes.
To study parallel and intersecting lines.
To provide experience in comparing pairs of segments.
To introduce half-lines and half-planes and to review rays and angles.
To provide interesting geometric experiences by means of reflections.
To review various types of triangles.
To introduce convex and concave figures.
To study some properties of quadrilaterals and other polygons.

	Performance Objectives	Pupil Text	Reteach-Reinforce	Related Activities
80	**RED** The student can demonstrate an understanding of basic properties of points, lines, and planes.	C-2 through 7	**WB** 75, 76 **DM** 53	
81	**RED** Given a pair of line segments, the student can decide if they are congruent or if one is greater than the other.	C-8, 9 C-10, 11	**WB** 77 **DM** 54	**SWM 2** 50, 51 **SWM 3** 154, 155
82	**RED** The student can identify rays and angles.	C-14, 15	**WB** 78 **DM** 55	
83	**RED** Given a geometric figure and a line, the student can find the reflection image of the figure in the line.	C-16, 17	**WB** 79	
84	**RED** The student can identify parallel and perpendicular lines.	C-4, 5 C-16, 17 C-18, 19	**WB** 80	**SWM 1** 126
85	**RED** The student can recognize and name various special angles.	C-20,21	**WB** 81 **DM** 56	
86	**RED** Given a triangle, the student can decide if it is scalene, isosceles, or equilateral.	C-24, 25	**WB** 82 **DM** 57	**DS** f-3
87	**YELLOW** The student can identify half-lines and half-planes.	C-12, 13		
88	**YELLOW** The student can identify any of the special quadrilaterals.	C-28, 29	**WB** 83	**SWM 1** 122, 123
89	***GREEN*** *Given a polygon, the student can determine if it is convex or concave.*	C-26, 27		
90	***GREEN*** *The student can name a polygon according to the number of its sides.*	C-30, 31		
	Reviewing the Ideas	C-32	**WB** 84	

MATHEMATICS

This module contains a large number of geometric ideas. Many of the geometric figures will be familiar to the students, but the development includes many fresh approaches to this familiar content. Basically, the module is concerned with nonmetric geometry.

In this module, students should gain a feeling for the fact that points, lines, and planes (indeed, all geometric figures) are simply ideas. They are abstractions which exist only in the mind. The marks that we put on paper or the models that we can hold in our hands are simply imperfect representations of the ideas we think about in an abstract sense. Of course, we often

speak of the physical models as if they are the geometric objects themselves; this is simply a matter of convenience in communication and should not detract from the mathematical concepts involved.

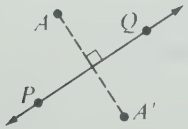
Parallel lines are introduced early in the module. The student is provided with intuitive experiences that will lead him to accept the Parallel Postulate:

There is one and only one line through a point parallel to a given line.

In formal treatments of geometry, many theorems such as the ones about the sum of the angles of a triangle and even the Pythagorean Theorem cannot be proved unless the Parallel Postulate is assumed.

The idea of reflection in a line is introduced in this module, and will be extended in the next module. A reflection is one of several transformations of the points in a plane that is a *rigid motion*. A rigid motion is a motion under which all distances are preserved.

If $\overleftrightarrow{PQ}$ is a line in a plane and A is any point not on $\overleftrightarrow{PQ}$, then the *reflection* image of A in $\overleftrightarrow{PQ}$ is the point A' such that $\overleftrightarrow{PQ}$ is the perpendicular bisector of $\overleftrightarrow{AA'}$.



Point A is also the reflection image of A'. Each point on the line is its own image and thus each is *fixed point* under the reflection. Because of the way reflections are defined, we can think of the reflecting line as a mirror.

TEACHING THE MODULE

Materials

Geoboards and rubber bands, dot paper, graph paper, rulers, scissors, cardboard or posterboard, pins or thumbtacks, plexiglass or acetate sheets.

Vocabulary

adjacent
acute angle
angle
concave
congruent segments
convex
diagonal
equilateral
exterior
greater than
half-line
half-plane
height
hypotenuse
image
interior
intersection
isosceles
less than
line
midpoint
obtuse angle
parallel
parallelogram
perpendicular
plane
point
polygon
quadrilateral
ray
rectangle
reflection
reflection image
regular polygon
rhombus
right angle
rigid motion
scalene
segment
skew
square
straight angle
supplementary angles
symmetry
trapezoid
triangle
vertex
vertical angles

The geometry of this module provides the necessary background for the material in the modules that follow. Since your students' backgrounds in geometry may vary considerably, it is important for the basic concepts of geometry to be reviewed thoroughly.

The main purpose of the module is to provide students with a wide variety of geometrical experiences. Terminology is kept to a minimum; we are more concerned with ideas than with names of objects.

Lesson Schedule

You should allow at least 12 days to cover the material in this module. Tests, projects, and optional material might extend the time requirement to at least 15 days.

Evaluation

In making evaluation plans, you should remember that the main purpose of this module is to provide students with interesting geometrical experiences rather than simply to expose them to a list of definitions and geometric facts.

RESOURCES FOR ACTIVE LEARNING

Activities

Boxes, Squares, and Other Things, NCTM
Developmental Math Cards, L15, 18, Addison-Wesley
Experiments in Mathematics, Stage 1, pp. 46–47, 52–55; Stage 3, pp. 20–27, Houghton Mifflin
Experiments With the Geo Strips, Midwest Publications
Geoboard Activity Card Kit, "Triangles," Cards 61–79; "Kites," Cards 117–119, Cuisenaire Co.
Geosquare Activity Cards, "Map Problems, Networks, Junctions," Cards 80–97, Scott Resources
Mira Activities, Activities 1–80, Creative Publications
Nuffield Project; *Angles, Courses and Bearings; Problems*—Red Set, Nos. 9B, 10, 24, 26, Wiley
Relationshapes Activities Cards, Nos. 66–68, Addison-Wesley
Sourcebook for Substitutes, "Four-color problem," p. 81; "Jordan Curve Theorem," pp. 85–86; "Paths, routes, and circuits," pp. 92–93, Addison-Wesley
Triangles, Cuisenaire Co.

Manipulative Devices

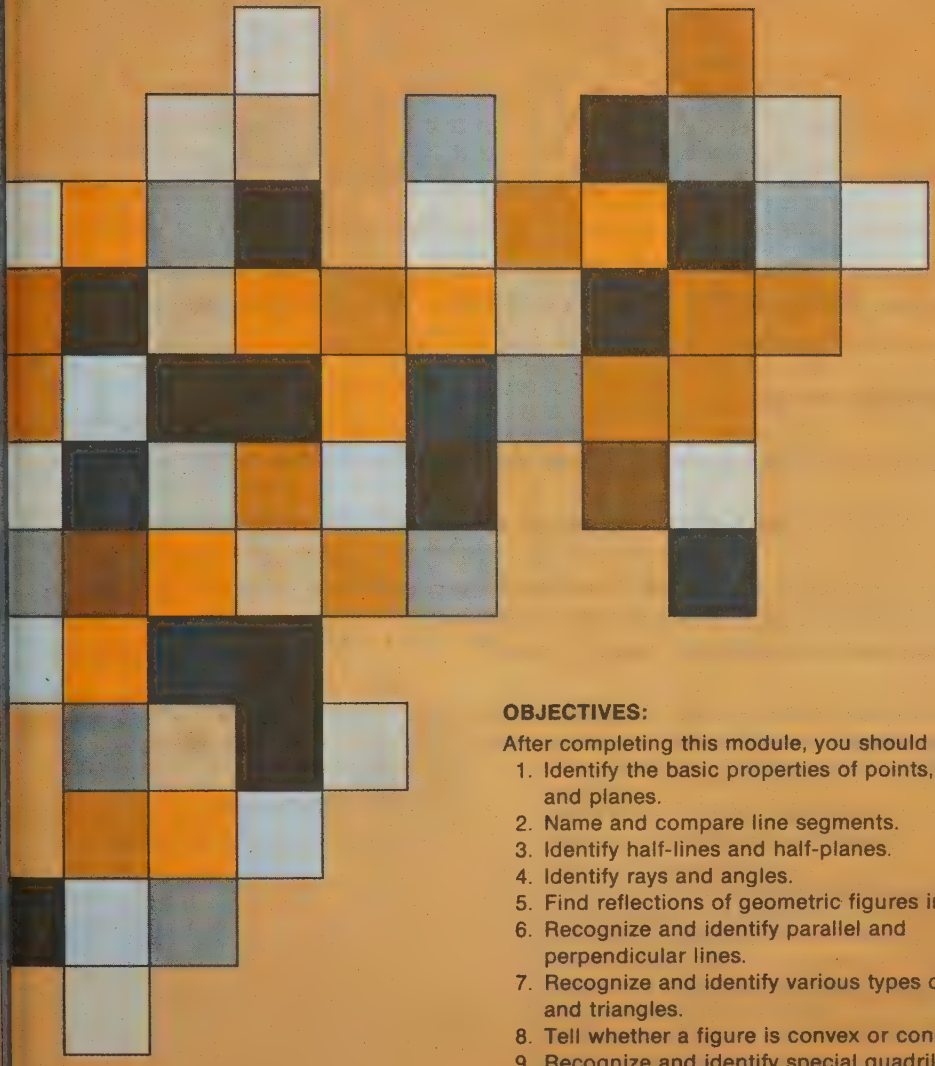
Circle Master Compass (Creative Publications)
Clinometer (Selective Educational Equipment; Sigma)
Geoboards (Addison-Wesley)
Geo Strips (Mind/Matter; Selective Educational Equipment)
Mira (Creative Publications; Cuisenaire Co.)
Protractors (Geyer; Selective Educational Equipment)
Transparent Geosquare (Scott Resources)

Games and Puzzles

Euclid (Midwest Publications)
Net Results (Cuisenaire Co.)
Perception Games (Edmund Scientific; Selective Educational Equipment)
Psyche-Paths (Cuisenaire Co.)
Twixt (Sigma)

UNIT C: *Geometry and Measurement*

MODULE 1: *Basic Concepts of Geometry*



OBJECTIVES:

After completing this module, you should be able to:

1. Identify the basic properties of points, lines, and planes.
2. Name and compare line segments.
3. Identify half-lines and half-planes.
4. Identify rays and angles.
5. Find reflections of geometric figures in a line.
6. Recognize and identify parallel and perpendicular lines.
7. Recognize and identify various types of angles and triangles.
8. Tell whether a figure is convex or concave.
9. Recognize and identify special quadrilaterals.

Objective

The student can name points and lines and can determine points of intersection of lines.

PREPARATION

Materials: Geoboards and rubber bands, dot or graph paper (*Duplicator Masters,* pages 82, 83), rulers.

Discuss points and lines with your students, pointing out that a line contains infinitely many points and has no endpoints. However, the drawings we make to represent lines are finite, so we put arrow tips on the "ends" to indicate that the lines continue endlessly. When we say "line," we always mean "*straight* line."

INVESTIGATION

The Investigation demonstrates that two points determine a unique line and that a given point can be on many different lines. When students record their lines, they should draw lines completely through all points in a line, thus:

DISCUSSION

Even though some students may not have found *all* the lines through point P in the Investigation, it is important that they understand the implications of their findings: *Many lines can contain the same point, but given two distinct points, there is a unique line containing the points.*

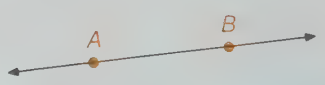
Exercise 4 brings out the fact that, although points and lines are abstract geometric concepts, we can use pictures to represent them.

Exercise 5 develops the idea that the symbols $\overleftrightarrow{RS}$, $\overleftrightarrow{RT}$, ℓ, and $\overleftrightarrow{ST}$ all represent the same set of points, that is, the line. Thus, it would be correct to say that $\overleftrightarrow{RS} = \overleftrightarrow{RT} = \overleftrightarrow{ST} = \ell$.

Points and Lines

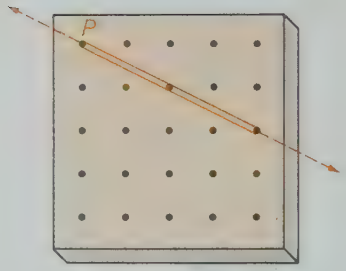
Investigating the Ideas

There is exactly one line that contains any two points.



"line AB"

SYMBOL: $\overleftrightarrow{AB}$ or $\overleftrightarrow{BA}$



How many different "lines" can you find on your geoboard that contain point P?

Record your findings on dot paper or graph paper.

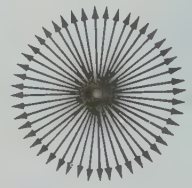
Discussing the Ideas

1. How many lines did you find that contain point P?

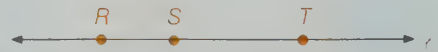
2. What are some objects that remind you of the idea of a point?

3. Lines, in geometry, never end. However, we often see physical objects which remind us of lines. What are some objects that suggest lines?

4. When we say "draw a line," we mean **draw a picture to represent a line**. What does the picture at the right suggest about the number of lines that can contain one point?



5. A line can have many names. Three names for this line are $\overleftrightarrow{RS}$, $\overleftrightarrow{TR}$, and ℓ. What are some other names for this same line?



6. The figure shows lines ℓ and m intersecting at point P. Can two lines intersect in more then one point?

$\ell \cap m = P$

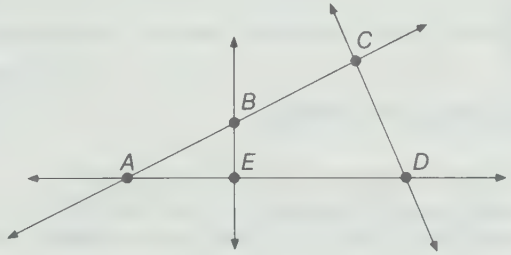
Using the Ideas

1. Mark two points on your paper and label them P and Q.
   Use your ruler to draw the line through P and Q.
   Mark a point A not on $\overleftrightarrow{PQ}$. Draw $\overleftrightarrow{AP}$ and $\overleftrightarrow{AQ}$.

2. If three points are chosen not on the same line, how many lines can be drawn that contain pairs of these points?

3. Mark four points on your paper and label them A, B, C, and D.
   Make sure no three of them are on the same line.
   Draw all the lines that contain pairs of these points.
   Give names for the lines such as $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, and so on.
   You should have six lines in all.

4. A How many lines are shown in the figure at the right?
   B Give two other names for $\overleftrightarrow{AE}$.
   C Give two other names for $\overleftrightarrow{AC}$.
   D What is $\overleftrightarrow{AB} \cap \overleftrightarrow{CD}$?
   E What is $\overleftrightarrow{BC} \cap \overleftrightarrow{ED}$?



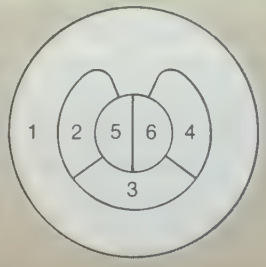
5. Draw four lines and label them m, n, o, and p.
   Make sure that no three of them go through the same point and that each one intersects the other three.
   How many points are determined by the intersections of pairs of these lines?

6. How many lines are determined by five points if no three of them were on the same line?

7. How many lines are determined by six points if no three of them were on the same line?

Think

Copy the figure on your paper.
If you had six different crayons, you could easily color the six regions of this "map" so that no two bordering regions had the same color.
Try to do it using only five colors.
Then try to color the map using only four colors.
Can you color the regions using only three colors?
Remember, bordering regions may not have the same color.



UTILIZATION

The exercises on this page provide practice in representing points and lines, naming points and lines, and determining the number of lines through finite sets of points not on the same line (noncollinear points).

Exercise 2, 3, 6, and 7 form a sequence of exercises that could be conveniently arranged in a table so that students can see a pattern.

Number of points not on one line	Number of possible lines
3	3
4	6
5	10
6	15

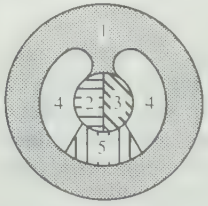
EXTENSION

Enrichment: Challenge students to try to develop a formula for the numbers in the table above. The sequence consists of "triangular numbers." The general formula is

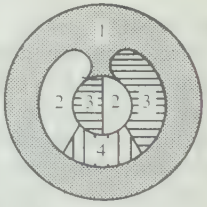
$$N = \frac{p\,(p-1)}{2},$$

where p is the number of points not on the same line and N is the number of lines.

Think Solution



Five colors



Four colors

You cannot color the regions as specified using only three colors.

Assignments

Minimum 1–4. Average 1–5. Maximum 1–7.

Objective

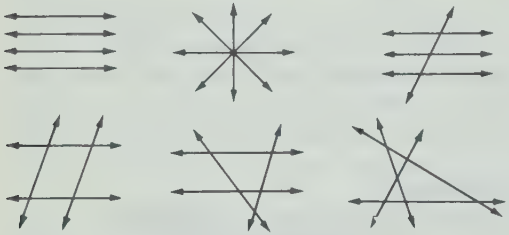
The student can identify parallel lines and intersecting lines.

PREPARATION

If you prefer, you might discuss the definition of parallel lines *before* beginning the Investigation (see Discussion Exercise 2). However, you could begin immediately with the Investigation.

INVESTIGATION

The Investigation should prove interesting because four lines in one plane can intersect in 0, 1, 3, 4, 5, or 6 points, but not in just 2 points.



If we do not restrict ourselves to lines in a plane, four lines in space can intersect in exactly 2 points as shown.

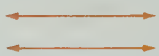
DISCUSSION

In discussing the definition of parallel lines, emphasize that two parallel lines are always coplanar; that is, they are in the same plane. Illustrate both parallel lines and skew lines, and discuss the fact that although skew lines do not intersect, they are not parallel.

Exercise 4 introduces, without naming it, one of the most famous postulates of geometry—The Parallel Postulate. This postulate asserts that in a plane there is no more than one line through a point not on a given line parallel to the line.

Intersecting Lines and Parallel Lines

Investigating the Ideas

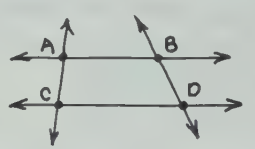
Two lines in a plane can have  no point of intersection or exactly one point of intersection.

Three lines in a plane can have zero, one, two, or three points of intersection.

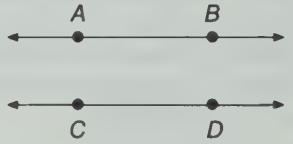
How many different points of intersection can you find for four lines?

Discussing the Ideas

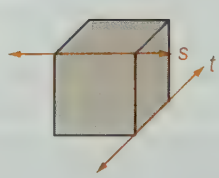
1. Andrea drew this picture to illustrate four lines intersecting in four points. Was she correct? Explain your answer.



2. Two lines that are in the same plane and do not intersect are called **parallel lines**. To show that $\overleftrightarrow{AB}$ **is parallel to** $\overleftrightarrow{CD}$, we write $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. What are some physical objects that remind you of parallel lines?



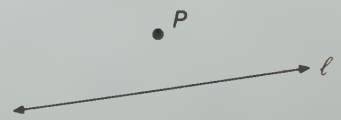
3. **Skew lines** are two lines that are not in the same plane. In the figure lines s and t illustrate skew lines. They do not intersect, yet they are not parallel lines.



Use two pencils to demonstrate

A intersecting lines **B** parallel lines **C** skew lines

4. Suppose point P is not on a line ℓ.

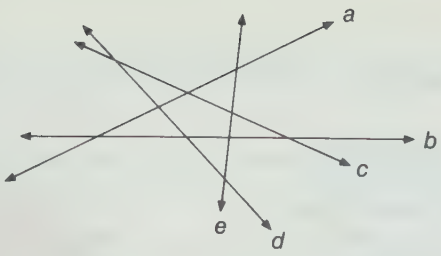


A Is there a line in the plane of P and ℓ that contains point P and is parallel to line ℓ?

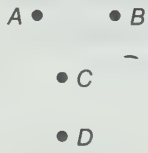
B Is there more than one line through P that is parallel to line ℓ?

Using the Ideas

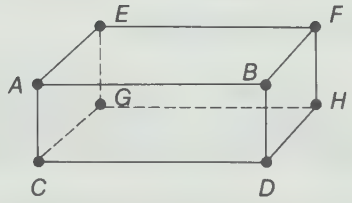
1. How many points of intersection are shown in the figure for five lines?



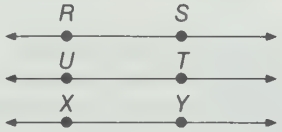
2. **A** Mark four points on your paper in the positions of points *A*, *B*, *C*, and *D*.
   B Draw all the lines that you can through pairs of points.
   C How many new points of intersection, other than *A*, *B*, *C*, and *D*, do the lines form?



3. Use the points at the corners of the box in the figure to name some parallel lines. How many pairs can you find?
   EXAMPLE: $\overleftrightarrow{AC} \parallel \overleftrightarrow{BD}$
4. Name some skew lines using the points at the corners of the box in Exercise 3.



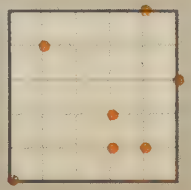
5. Suppose $\overleftrightarrow{RS} \parallel \overleftrightarrow{UT}$ and $\overleftrightarrow{UT} \parallel \overleftrightarrow{XY}$. Is $\overleftrightarrow{RS} \parallel \overleftrightarrow{XY}$?



6. Draw a picture to illustrate the following:
   Five lines *a*, *b*, *c*, *d*, and *e* in a plane such that $a \parallel b$ and $c \parallel d$, *a* is not parallel to *c*, *e* intersects the other four lines, and the five lines intersect in four points.

think

Trace the square and the seven points. Draw three straight line segments joining the sides of the square so that each of the seven points will be separated from each other.



UTILIZATION

The exercises provide a variety of practice with parallel, intersecting, and skew lines.

In naming parallel lines for the figures in Exercise 3, a few students may name pairs such as $\overleftrightarrow{AF}$ and $\overleftrightarrow{CH}$, or $\overleftrightarrow{AG}$ and $\overleftrightarrow{BH}$. Of course such diagonal segments are parallel even though the segments are not pictured in the text.

EXTENSION

Additional practice exercises are provided on page 75 of the *Workbook*.

Enrichment: Ask students to try to find the maximum number of points of intersection for any given number of coplanar lines. (Solution: $\frac{n(n-1)}{2}$, where n is the number of coplanar lines.)

Think Solution

This Think problem illustrates the fact that the greatest number of regions into which three intersecting lines can divide the plane is 7.

Assignments

Minimum 1–4. Average 1–5. Maximum 1–6.

Objective

Given pictures or models of planes, the student can identify certain properties of planes.

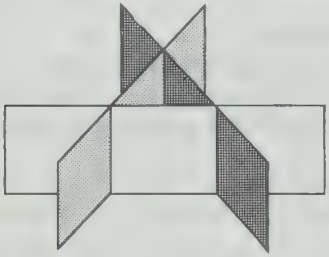
PREPARATION

Materials: Pieces of thin cardboard, scissors.

Distribute the materials needed for the Investigation. Students can work in pairs or in small groups on this Investigation.

INVESTIGATION

The purpose of the Investigation is to provide a concrete representation of intersecting planes. The model when put together will be used in discovering and discussing properties of planes. The assembled pieces should look like this:



DISCUSSION

The discussion questions should bring out the following ideas about planes.

A. The intersection of two planes is a line (Exercise 1).

B. Three noncollinear points determine a unique plane (Exercise 6).

C. A line and a point not on that line determine a unique plane (Exercise 5).

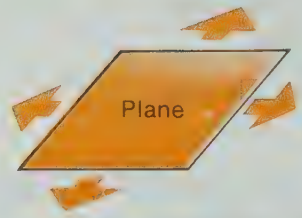
D. A plane separates space into disjoint regions (Exercises 3 and 4).

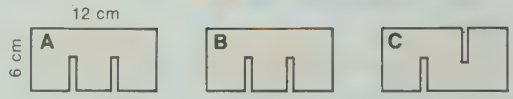
Point out that, given 4 noncollinear points, any three of them are in the same plane but the fourth point may not be. This accounts for the fact that a 3-legged stool always sits steady on all three legs, but stools, tables, and chairs having four legs, may "rock" because the bottom of one of the four legs is not in the same plane as the other three.

Planes

Investigating the Ideas

A **plane** is a flat surface that extends indefinitely. You can make some models of planes using pieces of cardboard or tag board. Cut out three pieces of cardboard like A, B, and C. Then cut the slots so that they go at least halfway through each piece.





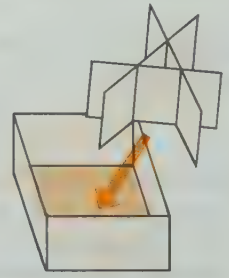
Can you place the three pieces together to make a model of three intersecting planes?

Discussing the Ideas

1. The intersection of two planes is the set of all points that are common to both planes. Look at your model. What geometric figure is formed by the intersection of two planes?

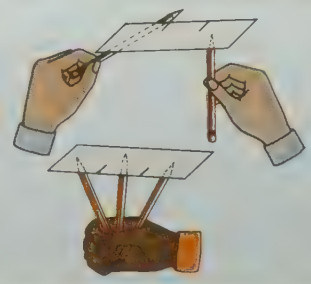
2. How many lines of intersection are formed by the planes in your model?

3. Imagine that your model will just fit into a rectangular box. Into how many separate compartments would the model partition the box?



4. Can you arrange the pieces of your model so that it would partition a box into six compartments?

5. There is only one plane that will contain a line and a point not on the line. Hold two pencils to balance one of your planes in order to illustrate this.



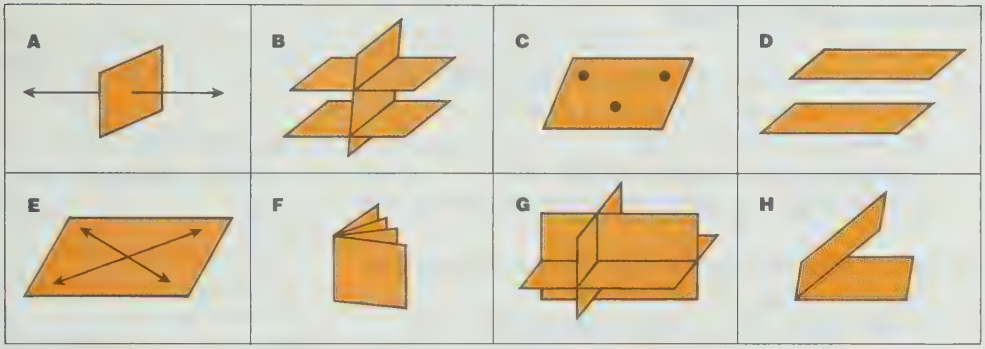
6. If three points are not in a line, then there is just one plane that contains all three points. Try to balance your model of a plane on three pencil points.

Using the Ideas

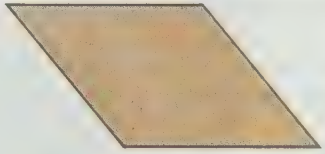
1. Mark three points on your paper so that only one plane contains these points.

2. Mark three points on your paper so that each of many planes contains these points.

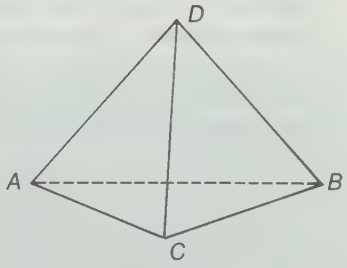
3. Name the figure suggested by the phrase.
   - **A** Two planes that do not intersect.
   - **B** Four planes intersecting in one line.
   - **C** A line intersecting a plane in a point.
   - **D** Two intersecting lines in a plane.
   - **E** Three planes intersecting.
   - **F** Two planes intersecting in a line.
   - **G** Three points in a plane.
   - **H** Three planes intersecting in two lines.



4. **A** Draw a picture of a plane.
   - **B** Mark two points X and Y in the plane.
   - **C** Draw $\overleftrightarrow{XY}$.
   - **D** Mark a point P on $\overleftrightarrow{XY}$ between X and Y.
   - **E** Mark a point Q in the plane but not on $\overleftrightarrow{XY}$.
   - **F** Draw $\overleftrightarrow{PQ}$.



5. In the figure there are four points so that no three lie on one line and no plane contains all of them. Points A, C, and D determine one plane called plane ACD.
   Name three other planes determined by other sets of three points in the figure.



6. How many planes would be determined by five points? (See Exercise 5.)

UTILIZATION

The exercises emphasize the interpretation of pictures involving planes and lines. Exercise 3 involves many geometric ideas which students can see intuitively. These ideas should be discussed, but formal statements of them should not be stressed at this time. Some of these ideas are given below.

A. The intersection of a plane and a line is a point.
B. Two intersecting lines are always coplanar.
C. Many planes can contain the same line.
D. Pairs of planes which do not intersect are parallel.
E. If two planes intersect, their intersection is a line.

EXTENSION

Further practice with the concepts introduced in this lesson is provided on page 76 of the *Workbook* and page 53 of the *Duplicator Masters*.

Remedial: The faces of a cube or a rectangular prism are portions of planes. Use models of these figures to illustrate the intersection of planes. For example, the intersection of 3 of the faces of a cube is one point, a vertex of the cube. Each face of the cube intersects all but one of the other faces of the cube.

Enrichment: Models of lines and planes can be made from cardboard and string or wire. Encourage students to make some models of the figures shown in Exercise 3.

Assignments

Minimum 1–3. Average 1–5. Maximum 1–6.

Objective

The student can draw, demonstrate, or name pairs of congruent segments.

PREPARATION

Materials: Geoboards and rubber bands, dot paper or graph paper (*Duplicator Masters*, pages 82, 83).

You will probably want to have students begin the Investigation as soon as the needed materials have been distributed. If geoboards are not available, dot paper or graph paper can be used as a substitute.

INVESTIGATION

Although the Investigation does not seem to involve measurement, most students will probably think about placing rubber bands so that they go "over 4 and up 2" or "over 2 and up 4." Therefore some intuitive notions of metric geometry are involved in finding segments as long as the one shown on the geoboard.

DISCUSSION

The definition and symbol for *segment* is given in the first exercise. Note that if P is a point on $\overline{AB}$ and if P is neither A nor B, P must be between A and B.

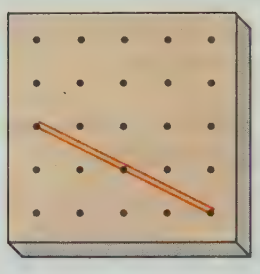
The term *congruent segment* is introduced intuitively in Exercise 2 as two segments whose endpoints are equally far apart. Although congruence of segments is a nonmetric concept, some students may observe that congruent segments have the same length (a metric concept), and this is certainly permissible since two segments are congruent if and only if they have the same length.

In connection with Exercise 3, you might want to have students display on the chalkboard or overhead projector the congruent segments they found in the Investigation. It is unlikely that many students will have found all eleven of the possible segments, but by combining their individual findings they will probably achieve the maximum total.

Congruent Segments

Investigating the Ideas

You can represent a **segment** on your geoboard by looping a rubber band around any two nails on the board.

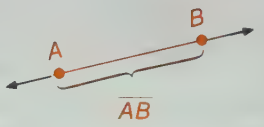


How many segments can you find on your geoboard that are just as long as the one shown above?

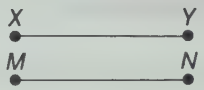
Record your findings on dot paper.

Discussing the Ideas

1. The segment AB with endpoints A and B is the set made up of points A and B and all points between A and B on $\overleftrightarrow{AB}$.
The symbol for segment AB is $\overline{AB}$.
In what way is a segment different from a line?

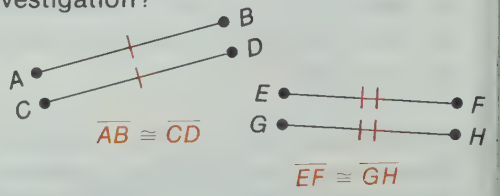


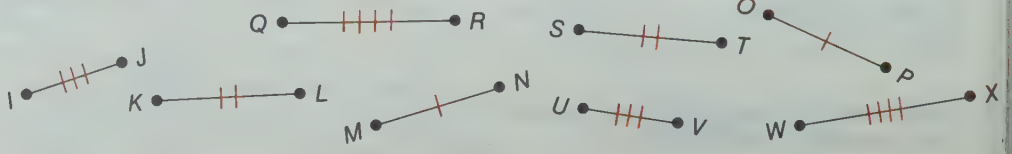
2. If two segments have their endpoints equally far apart, they are congruent.
In the figure, $\overline{XY}$ is congruent to $\overline{MN}$ and we write $\overline{XY} \cong \overline{MN}$.
Draw and name two segments that you think are congruent.



3. How many different segments did you find on your geoboard that were congruent to the segment in the Investigation?

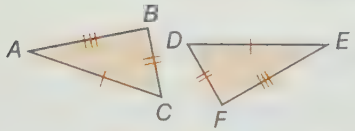
4. We often use slash marks to indicate that two segments are congruent. The same number of slashes indicates that the two segments are congruent.
Name the pairs of congruent segments as indicated by the slash marks.



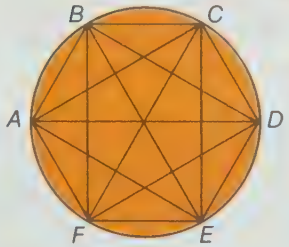


Using the Ideas

1. List the pairs of congruent segments shown in the two triangles.



2. The six points A, B, C, D, E, and F are equally spaced around the circle. List all the segments in the figure that are congruent to
   - **A** $\overline{AB}$
   - **B** $\overline{AC}$
   - **C** $\overline{AD}$

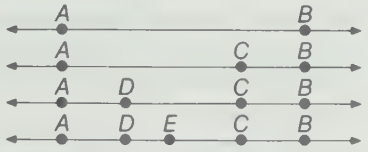


3. How many segments are shown in the figure for Exercise 2?

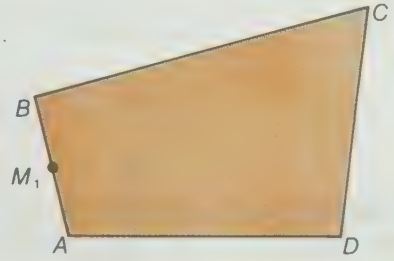
4. Complete the sentence: If $\overline{JK} \cong \overline{DE}$ and $\overline{DE} \cong \overline{HG}$, then $\overline{JK} \cong$? .

5. Mark two points X and Y on your paper. Then mark a third point Z so that $\overline{XY} \cong \overline{XZ}$ and $\overline{XY} \cong \overline{ZY}$.

★ 6. How many segments can you name using
   - **A** two points on a line?
   - **B** three points on a line?
   - **C** four points on a line?
   - **D** five points on line?

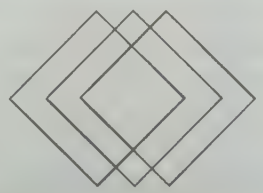


★ 7. The **midpoint** of a segment divides a segment into two congruent segments. Draw a 4-sided figure such as $ABCD$. Label the midpoints of $\overline{AB}$ as M_1, the midpoint of $\overline{BC}$ as M_2, the midpoint of $\overline{CD}$ as M_3, and the midpoint of $\overline{AD}$ as M_4. Draw the figure $M_1M_2M_3M_4$. Can you find some congruent segments in your drawing?



think

Starting at any point, draw the figure without crossing any lines, retracing any lines, or lifting your pencil from the paper.



UTILIZATION

Although circles have not yet been studied formally, students should be able to find pairs of congruent segments in the figure for Exercise 2. You might point out to the students that they are expected to list only those segments with endpoints on the circle.

Exercise 4 illustrates that congruence of segments is a *transitive* relation. If $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} \cong \overline{EF}$. Congruence of segments is also *reflexive* ($\overline{AB} \cong \overline{AB}$), and *symmetric* (if $\overline{AB} \cong \overline{CD}$, then $\overline{CD} \cong \overline{AB}$).

Unless your students are already familiar with the use of the compass in geometric constructions, you will want to point out to the students that their drawings for Exercises 5 and 7 need only be approximations.

The geometric idea underlying Exercise 7 is that it can be proved that the quadrilateral formed by connecting the midpoints of the sides of any quadrilateral is a parallelogram. Hence the students should be able to guess from these drawings that $\overline{M_1M_2} \cong \overline{M_3M_4}$ and $\overline{M_1M_4} \cong \overline{M_2M_3}$.

EXTENSION

Provide a duplicated copy of a circle with a diameter of about 15 cm and 24 points spaced evenly around the circle. Have the students draw all the possible segments joining the 24 points. If this is done carefully, a beautiful design will result. The same figure with segments constructed of fine thread on cardboard will provide an interesting decoration. Challenge the students to count the total number of segments. (There will be 276 altogether.)

Think Solution

One solution for the diagram is shown on the student page. This is not the only solution, however. A further activity might be to see how many different solutions can be found, starting at a particular point. Consider:

Assignments

Minimum 1–4. Average 1–6. Maximum 1–7.

Objective

The student can compare two given segments and can write a statement that shows the comparison.

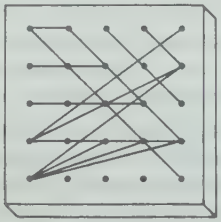
PREPARATION

Materials: Geoboards and rubber bands, dot paper or graph paper (*Duplicator Masters*, pages 82, 83).

Draw a segment $\overline{AB}$ on the chalkboard. Tell the students to *think* about a point X that is between A and B. Write AB, AX, and XB on the board. Then ask: "Which segment must be the greatest, no matter where X is located?" "Where must X be if $\overline{AX} \cong \overline{XB}$?" "If $\overline{XB}$ is the shortest segment, which segment is the next shortest?"

INVESTIGATION

Students could work effectively on this Investigation in pairs. It is not likely that pairs of students will find all the different possible segments, but if students' results are combined it is likely that all will be found. Encourage students to think about "families of lines" and to use notions of length to help them. There are 14 different-size segments, as shown below.



DISCUSSION

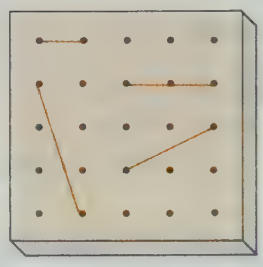
Although the symbols $>$ and $<$ have been used with numbers, the use of the same symbols with segments may be new to your students.

In discussing Exercises 3 and 4, stress the fact that there are certain inequalities that all students should agree upon regardless of the placement of the two "in between" points, C and E. Thus $\overline{CE} < \overline{AE}$, $\overline{CE} < \overline{CK}$, and $\overline{CE} < \overline{AK}$ if C and E are between A and K.

Comparing Segments

Investigating the Ideas

Use your geoboard for this Investigation.



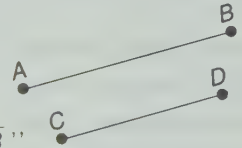
How many different segments can you find on your geoboard so that no two are congruent?

Record your findings on dot paper.

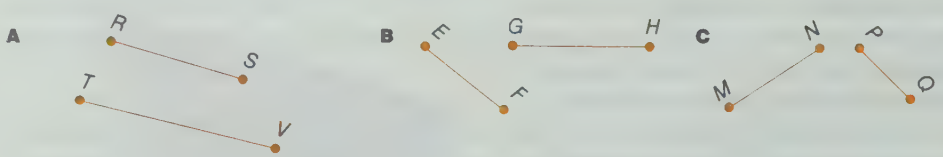
Discussing the Ideas

1. Are these two geoboard segments congruent? How can you tell?

2. The endpoints of $\overline{AB}$ are farther apart than the endpoints of $\overline{CD}$. We say that "**$\overline{AB}$ is greater than $\overline{CD}$**" or "**$\overline{CD}$ is less than $\overline{AB}$.**" We write this as $\overline{AB} > \overline{CD}$ or $\overline{CD} < \overline{AB}$



Compare the pairs of segments below.



3. Label four points A, C, E, and K in that order on a line. Name all the different segments. You should have six in all.

4. Find and name three inequalities about the six segments in Exercise 3. EXAMPLE: $\overline{AC} < \overline{AE}$

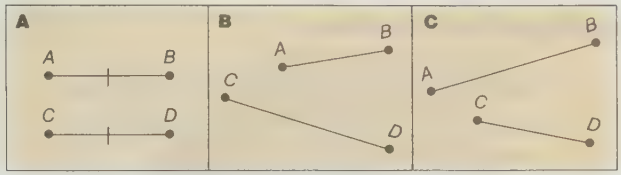
5. In the figure, point P is the midpoint of $\overline{AB}$.

What is the correct symbol for each ◍?

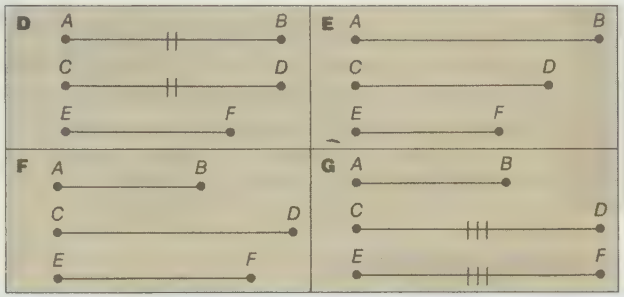
A $\overline{AB}$ ◍ $\overline{BP}$ B $\overline{AP}$ ◍ $\overline{PB}$ C PB ◍ AB

Using the Ideas

1. Select the correct figure.
   - A $\overline{AB} > \overline{CD}$
   - B $\overline{AB} \cong \overline{CD}$
   - C $\overline{CD} < \overline{AB}$
   - D $\overline{CD} > \overline{AB}$
   - E $\overline{AB} < \overline{CD}$

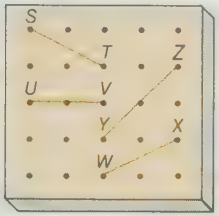


2. Select the correct figure.
   - A $\overline{AB} < \overline{CD}$ and $\overline{EF} < \overline{CD}$
   - B $\overline{AB} > \overline{CD}$ and $\overline{CD} > \overline{EF}$
   - C $\overline{AB} < \overline{CD}$ and $\overline{CD} \cong \overline{EF}$
   - D $\overline{CD} > \overline{EF}$ and $\overline{EF} > \overline{AB}$
   - E $\overline{AB} \cong \overline{CD}$ and $\overline{AB} > \overline{EF}$
   - F $\overline{AB} > \overline{EF}$ and $\overline{CD} < \overline{AB}$

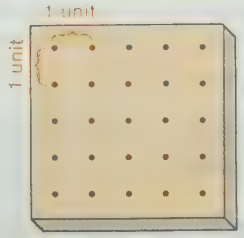


3. Select the statement that you think completes the sentence correctly. Be prepared to explain your selection.
   - A If $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$, we know that $\begin{cases} \overline{AB} \cong \overline{EF}. \\ \overline{AB} < \overline{EF}. \end{cases}$
   - B If $\overline{AB} \cong \overline{CD}$ and $\overline{CD} > \overline{EF}$, we know that $\begin{cases} \overline{AB} \cong \overline{EF}. \\ \overline{AB} > \overline{EF}. \end{cases}$
   - C If $\overline{AB} < \overline{CD}$ and $\overline{CD} < \overline{EF}$, we know that $\begin{cases} \overline{AB} < \overline{EF}. \\ \overline{AB} \cong \overline{EF}. \end{cases}$

★ 4. Write as many statements as you can that will compare different pairs of segments shown on the geoboard.



★ 5. Suppose that the distance between two adjacent nails on a geoboard either across or down is 1 unit.
   - A How many segments are 1 unit long?
   - B How many segments are 2 units long?
   - C How many segments are 3 units long?
   - D How many segments are 4 units long?



UTILIZATION

If students have difficulty with Exercise 3, you might suggest that they draw pictures to help them decide which of the two statements completes the sentence correctly.

Remind students that the same number of slash marks on two segments indicates congruent segments.

Solution, Exercise 4

$\overline{ST} \cong \overline{WX}$	$\overline{UV} < \overline{YZ}$
$\overline{ST} > \overline{UV}$	$\overline{UV} < \overline{WX}$
$\overline{ST} < \overline{YZ}$	$\overline{YZ} > \overline{WX}$

Each of these statements can also be reversed.

EXTENSION

To provide additional practice, make selective assignments from *Workbook* page 77 and *Duplicator Masters* page 54.

Assignments

Minimum 1, 2. Average 1–3.
Maximum 1–5.

Objective

The student will be able to identify or show half-planes and half-lines.

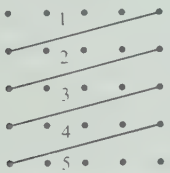
PREPARATION

Materials: Geoboards and rubber bands, dot paper or graph paper (*Duplicator Masters*, pages 82, 83).

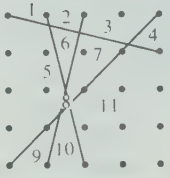
Since the Investigation may require considerable time, it would be best to proceed immediately with it.

INVESTIGATION

The least number of regions will occur when no two of the segments intersect. The greatest number will occur when each segment intersects all of the others and no three segments have the same point in common. Possible arrangements of segments that will produce the minimum and maximum number of regions are shown below.



Five regions



Eleven regions

DISCUSSION

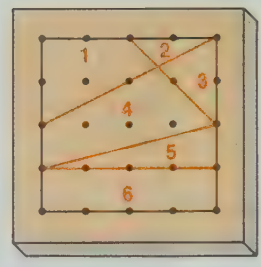
You might use the geoboard pictured in the Investigation as a model to illustrate *half-planes*. If each rubber band is thought of as a line, then it separates the geoboard (plane) into two regions or half-planes. By definition, the separating line is not included in either of the half-planes. (Similarly, a half-line does not include the separating point.)

Many of the notions of *separation* or *betweenness* that are treated intuitively in this lesson can be treated quite rigorously in formal geometry courses; some of the notions are assumed as postulates and others can be proved as theorems. However, such rigor is inappropriate at this level.

Half-lines and Half-planes

Investigating the Ideas

The four colored segments divide the square region on the geoboard into six smaller regions.



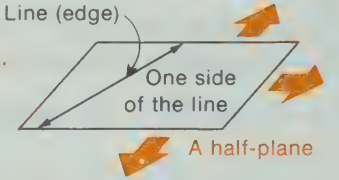
Can you find the fewest and the greatest number of regions into which a square can be divided by four segments whose endpoints are on two different sides of the square?

Record your findings on dot paper.

Discussing the Ideas

The region on each of the two sides of a line in a plane is called a **half-plane**.

We say: A line in a plane separates the plane into three sets, the line and two half-planes.

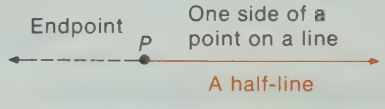


1. The red line on a hockey rink suggests a line that separates a plane. What are some other examples?

2. Do you think that a half-plane includes the separating line or edge?

Each of the two sides of a point on a line is called a **half-line**.

We say: A point on a line separates the line into three sets, the point and two half-lines.

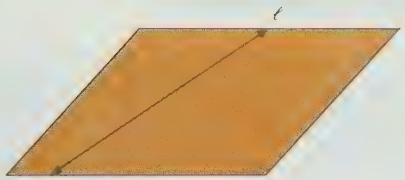


3. What are some physical objects that suggest that a point separates a line?

4. Does a half-line include the endpoint?

Using the Ideas

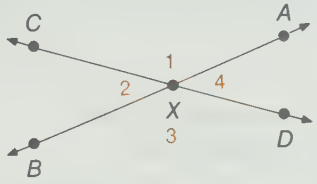
1. **A** Draw a picture of a plane with line ℓ separating the plane into two half-planes.



B Mark a point P in one of the half-planes and a point Q in the other half-plane. Draw segment PQ. Does segment PQ necessarily intersect line ℓ?

C Mark another point R in the same half-plane as point Q. Does RQ intersect line ℓ? Does PR intersect line ℓ?

2. **A** Draw a figure like the one at the right.



B Shade (▥) the half-plane that is the C-side of $\overleftrightarrow{AB}$.

C Shade (▤) the half-plane that is the B-side of $\overleftrightarrow{CD}$.

D Which region has been shaded twice, 1, 2, 3, or 4?

3. Refer to $\overleftrightarrow{WX}$ to give the correct letter for each ▥.

A X and Z are on opposite sides of ▥.

B Z and ▥ are on the same side of Y.

C W and Y are on opposite sides of ▥.

D W, Y and Z are on the same side of ▥.

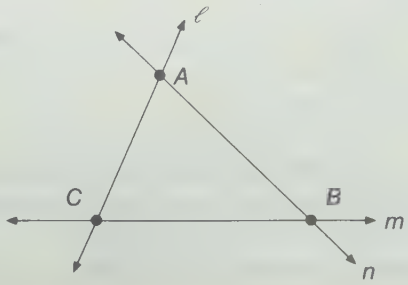
E V and X are on the same side of ▥.

★**4.** **A** Shade (▨) the set of points that are on the B-side of ℓ.

B Shade (▨) the set of points that are on the C-side of n.

C Shade (▥) the set of points that are on the A-side of m.

D Describe the set of points that are in all three half-planes.



UTILIZATION

Exercise 1 illustrates that a line in a plane separates the plane into two half-planes so that:

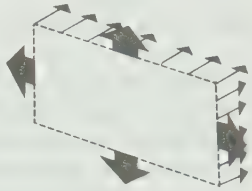
(a) If two points are not in the same half plane, the segment joining the two points intersects the separating line.

(b) If two points are in the same half-plane, the segment joining them will not intersect the separating line.

EXTENSION

Page 78 of the *Workbook* provides exercises that would be appropriate for further practice with the ideas presented in this lesson.

Enrichment: Just as each line separates the plane into half-planes, each plane in space separates the points in space into two *half-spaces*, the two regions on each side of the plane.



Half-space

Challenge students to find the maximum number of regions into which three intersecting planes will divide space. (*Answer:* 8) This could be illustrated by means of a cardboard model.

Assignments

Minimum 1, 3. Average 1–3. Maximum 1–4.

Objective

The student can name rays and angles shown by drawings and compare the sizes of angles.

PREPARATION

Since there is a considerable amount of geometry in this lesson, you may need to spend extra time in discussing the ideas and the symbols that are introduced.

INVESTIGATION

There are just 6 angles possible with the four rays from point O.

A common misconception among students is that two rays with a common endpoint determine 2 different angles. This misunderstanding

may be caused by the students thinking that the interior and the exterior of an angle *are* two angles. Another possible source of confusion is related to the way we measure angles. Some students think of the amount of rotation required to make one ray coincide with the other, and of course there are two different numbers that measure this rotation, depending on the direction of the rotation. Stress the fact that the two different rays with the common endpoint determine a unique angle, the set of points on the two rays.

DISCUSSION

Exercise 2 presents a nonmetric method of comparing two angles. To compare the size of the angles, first choose one pair of corresponding points on one side of each angle. Then follow the same procedure for the other side of each angle. This gives two pairs of points that determine the "openings" for the two angles, and these corresponding openings are used to compare the size of the two angles.

If an overhead projector is available, use it in discussing Exercise 3, to show the intersection of different half-planes. Shade the half-planes on separate overlays and show the class the different half-planes; then put the overlays together to show the intersection.

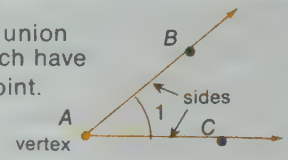
Rays and Angles

Investigating the Ideas

A ray is the union of a half-line and its endpoint.

NAME: Ray AB SYMBOL: $\overrightarrow{AB}$

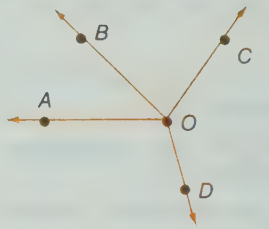
An angle is the union of two rays which have the same endpoint.



NAMES: Angle A, Angle BAC, Angle 1

SYMBOLS: $\angle A$, $\angle BAC$, $\angle 1$

How many rays and angles can you find and name in the figure at the right?



Discussing the Ideas

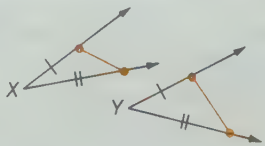
1. In naming an angle with three letters, such as $\angle BAC$, which letter denotes the vertex of the angle?

2. If the openings of two angles are the same, then the two angles are **congruent** ($\cong$). If the two angles are not congruent, then one angle is **greater than** or **less than** the other. In the following pairs of figures, the same number of slash marks on the segments tells you which pairs of segments are congruent. How do each pair of angles below compare in size?

$\angle D \cong \angle F$

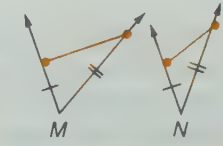
$\angle E > \angle F$ $\angle D < \angle E$

A



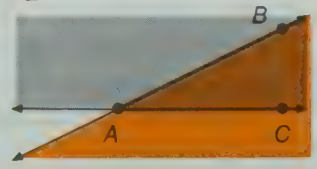
B

C



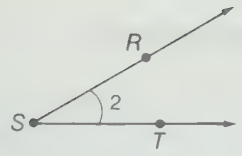
3. A The **interior** of $\angle BAC$ is the intersection of two half-planes as shown in the figure. Describe the two half-planes.

   B What is meant by the **exterior** of the angle?

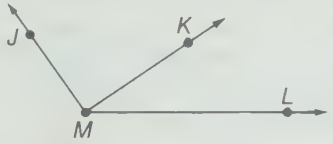


Using the Ideas

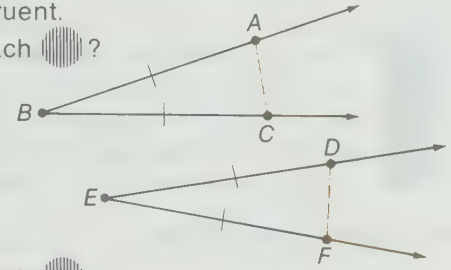
1. **A** Name the rays that form the angle.
   B Which point is the vertex of the angle?
   C Use three different symbols to name the angle.



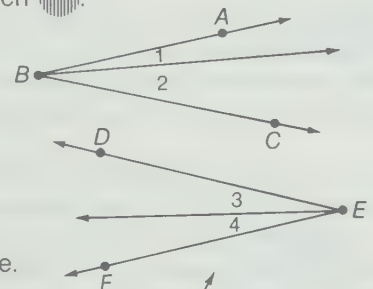
2. How many different angles are shown in the picture? Use three letters to name each angle.



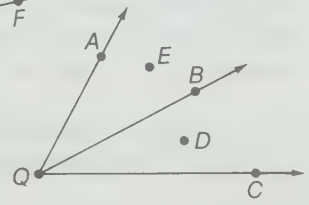
3. Suppose that $\overline{BA}$, $\overline{BC}$, $\overline{ED}$, and $\overline{EF}$ are congruent.
   Which symbol ($<$, $>$, or $\cong$) should go in each ▨?
   A If $\overline{AC} < \overline{DF}$, then $\angle B$ ▨ $\angle E$.
   B If $\overline{AC} > \overline{DF}$, then $\angle B$ ▨ $\angle E$.
   C If $\overline{AC} \cong \overline{DF}$, then $\angle B$ ▨ $\angle E$.



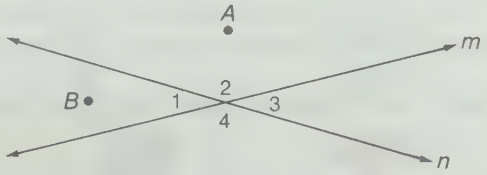
4. Give the correct symbol ($<$, $>$, or $\cong$) for each ▨.
   A If $\left\{\begin{array}{l}\angle 1 \cong \angle 4\\ \angle 2 \approx \angle 3\end{array}\right\}$ then $\angle ABC$ ▨ $\angle DEF$.
   B If $\left\{\begin{array}{l}\angle 1 < \angle 4\\ \angle 2 \cong \angle 3\end{array}\right\}$ then $\angle ABC$ ▨ $\angle DEF$.
   C If $\left\{\begin{array}{l}\angle ABC > \angle DEF\\ \angle 1 \cong \angle 4\end{array}\right\}$ then $\angle 2$ ▨ $\angle 3$.



5. Complete each sentence by naming an angle.
   A Point E is in the interior of $\angle$? .
   B Point D is in the exterior of $\angle$? .
   C Point B is in the interior of $\angle$? .
   D Point A is in the exterior of $\angle$? .



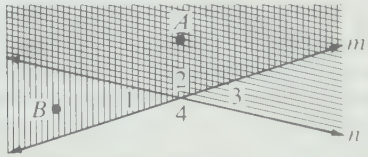
★ 6. Draw two intersecting lines m and n and choose points A and B as shown.
   Shade (▨) the half-plane that is on the B-side of m.
   Shade (▨) the half-plane that is on the A-side of n.
   A What is the intersection of the shaded regions?
   B What is their union?



UTILIZATION

Exercise 4 illustrates various relations between pairs of angles. Part A suggests a kind of "addition" for congruent angles. Parts B and C deal with combinations of inequalities and congruences. Students are expected to respond to these parts intuitively rather than through strict geometrical reasoning.

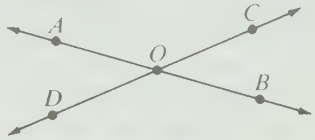
Construction, Exercise 6



EXTENSION

Additional practice exercises appropriate for this lesson are provided on page 79 of the *Workbook* and page 55 of the *Duplicator Masters*.

Remedial: Ask students to name at least 4 rays in the figure below; at least 4 angles. Name two straight angles with vertex O. What pairs of angles seem to be congruent? Shade the interior of $\angle AOC$.



Enrichment: Ask students to find a formula for the number of angles formed by n rays from the same point.

Solution: $\frac{n \cdot (n-1)}{2}$

Assignments

Minimum 1–3. Average 1–5. Maximum 1–6.

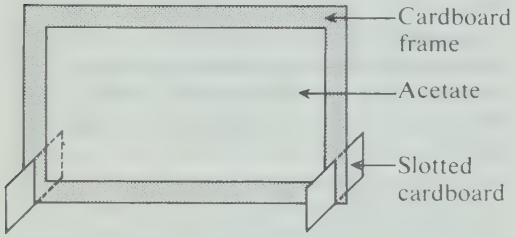
Objectives

The student can find the reflection of geometric figures in lines and can use reflections in a line to identify symmetric figures.

PREPARATION

Materials: Plexiglas or heavy acetate sheets, geoboards and rubber bands or dot paper (*Duplicator Masters*, page 82).

It is highly desirable for each student or small group of students to have a small rectangle of acetate or Plexiglas for the work on reflections in this module and in Module 3. Commercial devices are available for this purpose, but satisfactory home-made models can be made from Plexiglas or heavy acetate sheets. Used and cleaned x-ray film, usually available from x-ray labs, can also be used. Tape the acetate or other reflector to a 3-sided cardboard frame to increase the sturdiness of the mirror.



Two slotted cardboard strips can be used so that the mirror will stand upright.

INVESTIGATION

Students using geoboards can use one rubber band to show line ℓ and another band to show the triangle. After the students have shown and recorded the reflection of $\triangle XYZ$, suggest that they form some other geometric region on their geoboards and show its reflection.

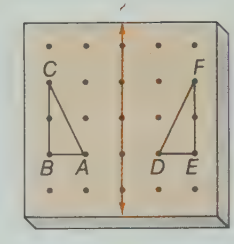
DISCUSSION

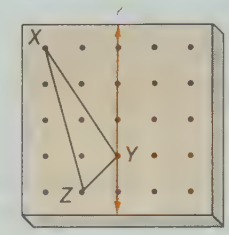
Be sure to emphasize the fact that the reflection images of geometric figures are the same size and shape as the original figures. Such ideas are more important than the accuracy of students' constructions.

Reflections

Investigating the Ideas

If line ℓ on the geoboard represented an upright mirror, you could see the reflection of $\triangle ABC$ in the mirror. The reflection is $\triangle DEF$. You can visualize this by thinking of folding the geoboard at the line. Then the two triangles would match.



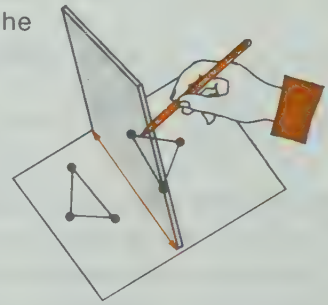


Can you show how the reflection of of $\triangle XYZ$ would look on the geoboard?

Record the reflection on dot paper.

Discussing the Ideas

1. The triangle that you found for the question above is the **reflection image** of triangle XYZ in the line ℓ. Does the reflection image that you found have the same size and shape as the original triangle?



2. You can use a rectangular piece of clear plastic to help you draw reflection images of figures on paper. Hold the plastic upright to the paper. Look through the plastic to see the reflection image. Try drawing the reflections of various figures.

3. Do you think that the reflection image of a geometric figure in a line is always the same size and shape as the original figure?

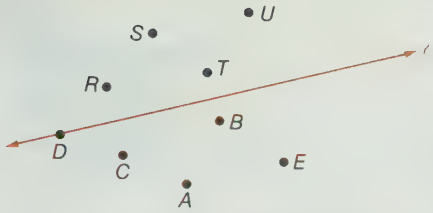
4. The region shown is a **symmetric region** with ℓ as a **line of symmetry**. Each half of the region is a reflection image of the other.

How can you use a plastic mirror to see if these regions are symmetric?

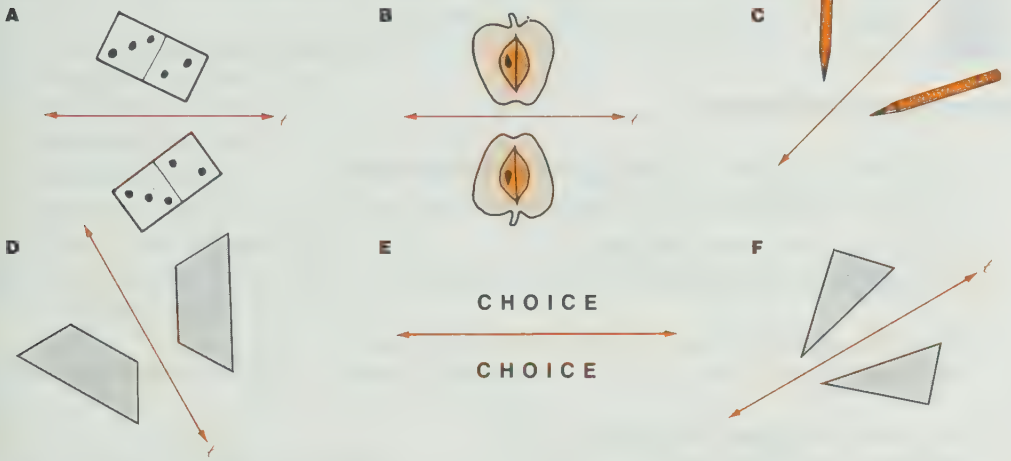
A T B D C Z D M

Using the Ideas

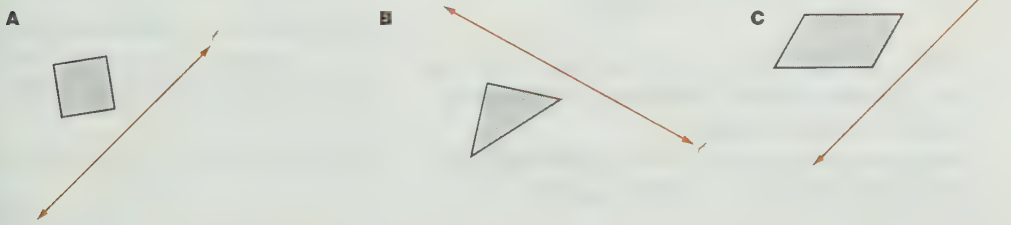
1. Which point is the reflection image of each of the points *A*, *B*, *C*, *D*, and *E* in line ℓ?



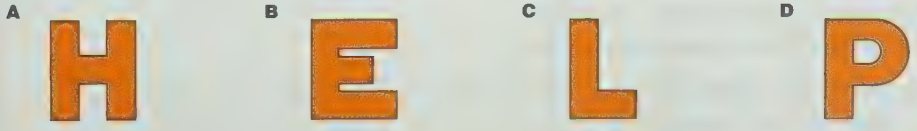
2. For each picture, tell whether the two objects are reflection images of each other in the line.



3. Copy each figure below. Then draw the reflection image of each figure in the line.



4. Which regions are symmetric?



★5. Draw four segments on your paper. Use your plastic mirror and reflections to find the **midpoint** (middle point) of each segment.

UTILIZATION

Although most of the exercises could be answered without the aid of a plastic mirror, the students' interest and understanding will be enhanced if they can use a device that enables them to see the reflections.

Reflections in a line are also sometimes called "flips" about a line. We do not use the term "flips" in this book, however.

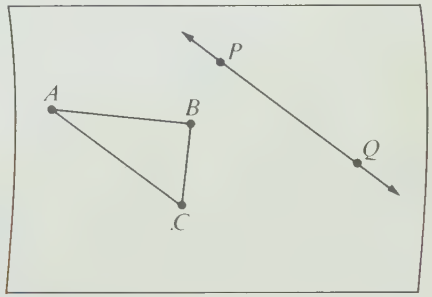
In using the plastic mirror in Exercise 5, students should discover that if they can make the reflection image of one end of the segment fall upon the other end of the segment, then the mirror must intersect the segment at its midpoint.

In general, the lesson should be kept on an intuitive level.

EXTENSION

Page 80 of the *Workbook* provides exercises designed to reinforce the ideas presented in this lesson.

Remedial: If an overhead projector is available, use it to demonstrate reflections. Draw a line PQ and a $\triangle ABC$ on one side of the line. Place a sheet of acetate over this drawing, and trace $\triangle ABC$ and points P and Q. Now turn the sheet of acetate over, aligning P and Q again. The reflection of $\triangle ABC$ will appear on the opposite side of $\overleftrightarrow{PQ}$.



Assignments

Minimum 1–4. Average 1–5. Maximum 1–5.

Objective

The student can identify right angles and construct them by means of reflections.

PREPARATION

Materials: Plastic mirrors.

Briefly review reflections. Have the students mark two points A and B on their paper. Then ask them if they can place their plastic mirrors in such a position that B is the reflection image of A. By the basic property of reflections, the line of the mirror will be the perpendicular bisector of $\overline{AB}$.

INVESTIGATION

The Investigation will not require much time, but the underlying idea is important:

If two lines intersect so that one line is the reflection image of itself when reflected in the other line, then the two lines are perpendicular.

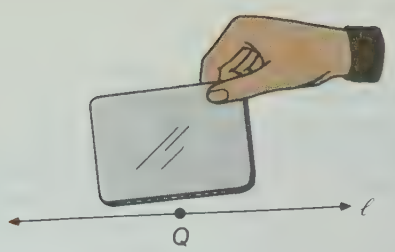
DISCUSSION

Right angles are defined in Exercise 5 as the nonstraight angles formed by pairs of perpendicular lines. Using the definition and the same construction process as in the Investigation, some right angles can be drawn for Exercise 6.

Perpendicular Lines

Investigating the Ideas

Draw a line ℓ and mark a point Q on the line.



Can you find a line m through Q so that ℓ is the reflection image of itself when m is the reflecting line?

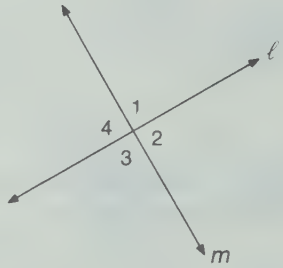
Discussing the Ideas

1. The lines ℓ and m are called **perpendicular lines**. You can write this as $\ell \perp m$.

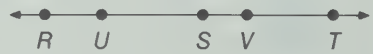
   Is each line the reflection image of itself in the other line?

2. What are some physical objects that remind you of perpendicular lines?

3. If $\ell \perp m$, the angles shown are congruent to each other. Explain how you can use a plastic mirror to check whether or not lines ℓ and m are perpendicular.

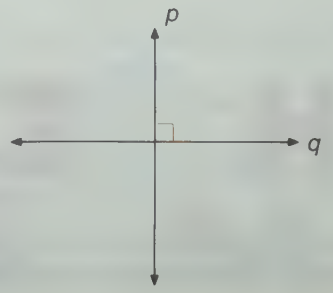


4. When the two rays of an angle lie on a line and are opposite in direction, the angle is called a **straight angle**. Give some names for the straight angle at the right with point S as the vertex.



5. The four non-straight angles formed by two perpendicular lines are called **right angles**. The symbol ⌝ is used on drawings to indicate a right angle.

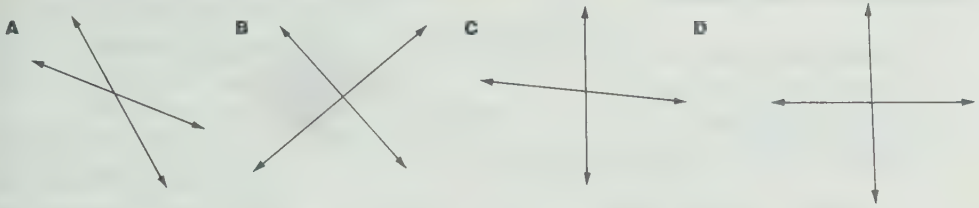
   What are some objects in your classroom that suggest right angles?



6. Can you explain how to use your plastic mirror to draw a right angle.

Using the Ideas

1. Which pairs of lines are perpendicular?



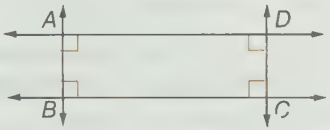
2. **A** Draw a line t and then mark a point P on line t. Use your mirror to draw a line perpendicular to line t through point P.

   B Mark another point Q on line t and then draw a perpendicular line to line t through point Q.

   C What do you think is the relationship between the two lines perpendicular to line t?

3. Use your plastic mirror to draw a right angle.

4. Use your plastic mirror to draw a figure like this one.



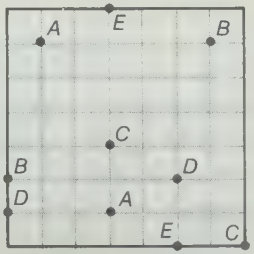
5. Use your mirror to draw a line through point Q that is perpendicular to line ℓ.

★ 6. Construct a square with the aid of your plastic mirror.

Think

Two points are labelled A, two points are labelled B, two C, two D, and two E.

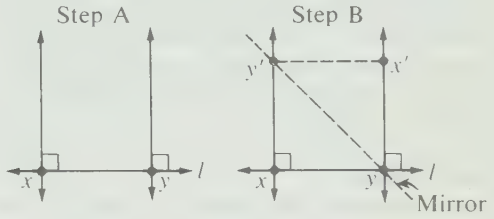
Join A to A, B to B, C to C, D to D, and E to E with paths along the lines of the grid so that no two paths cross or touch at any point.



UTILIZATION

Students will need their plastic mirrors for this set of exercises. Exercise 2 suggests the geometric theorem that, in a plane, two lines perpendicular to the same line are parallel.

In Exercise 6, allow for various methods of construction. One method is shown below.



In Step A, construct two perpendiculars to a line ℓ through arbitrary points X and Y. In Step B, place the mirror on Y and turn it until the image of X (denoted by X' above) falls on the perpendicular through Y. Mark X' and then complete the square by constructing a perpendicular to $\overleftrightarrow{X'Y}$ at X' by the method described in Step A.

EXTENSION

Additional practice exercises appropriate for this lesson are provided on page 81 of the *Workbook*.

Think Solution

This Think problem should be solved by trial and error. The solution is shown by the annotated drawing on the student page.

Objective

The student can identify adjacent angles, supplementary angles, vertical angles, and acute and obtuse angles.

PREPARATION

Materials: Plastic mirrors.

Review the various methods of denoting angles. Point out to the students that in drawings involving more than one angle it is best to use the three-letter notation or numeral notation.

DISCUSSION

Your students may already be familiar with some of the definitions given in this section. If so, you should simply review these definitions and perhaps quickly work through all the exercises as a class activity. However, if your students have not had previous experience with these definitions, you should progress more slowly and give additional examples in order to be sure that students understand them thoroughly.

The Discussion introduces three types of related angles: adjacent, supplementary, and vertical angles. Acute and obtuse angles are also discussed. The Discussion Exercises will give the students a chance to clarify these ideas, and the Utilization exercises w ll give them an opportunity to work with the definitions.

At this level, we rely mainly on pictorial definitions. It is probably more desirable for the student to be able to name pairs of related angles in a drawing than it is for him to verbalize complicated definitions.

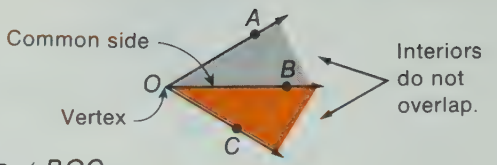
Vertical angles can be defined as two of the nonstraight and nonadjacent angles formed by two intersecting lines. In Exercise 3, students can use a plastic mirror to discover that pairs of vertical angles are the same size. The mirror must be placed along the line that bisects one pair of the vertical angles.

The definitions of obtuse and acute angles rely upon an intuitive understanding of "greater than" and "less than" for angles. At this stage, one angle is "less than" another if it has a smaller "opening" than the other.

Related and Special Angles

Discussing the Ideas

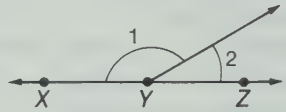
1. The figure shows that $\angle AOB$ and $\angle BOC$ are **adjacent angles**.



We say: $\angle AOB$ is **adjacent to** $\angle BOC$.

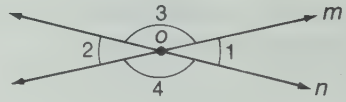
In **A** and **B**, explain why $\angle 1$ and $\angle 2$ are not adjacent angles.

2. If $\angle XYZ$ is a straight angle, then $\angle 1$ and $\angle 2$ are **supplementary angles**.



Which two pairs of angles are supplementary if $\angle BAC$ is a straight angle?

3. Lines m and n intersect at point O.



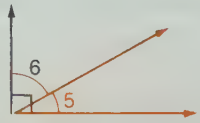
We say: $\angle 1$ and $\angle 2$ are **vertical angles**.
$\angle 3$ and $\angle 4$ are **vertical angles**.

A Do you think pairs of vertical angles are the same size?

B Can you find a way to use reflections to compare two vertical angles?

4. An angle smaller than a right angle is an **acute angle**.

Is $\angle 6$ an acute angle?



$\angle 5$ is an acute angle

5. An angle greater than a right angle but less than a straight angle is called an **obtuse angle**.

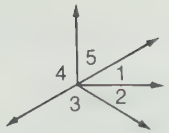
A Is $\angle 8$ obtuse or acute?

B What name is given to the pair $\angle 7$ and $\angle 8$?

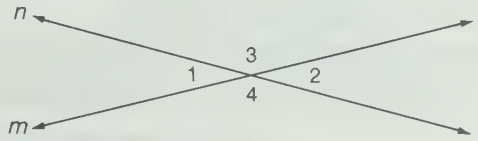
$\angle 7$ is an obtuse angle

Using the Ideas

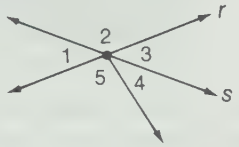
1. Using the figure at the right, name
   - A five pairs of adjacent angles.
   - B two pairs of angles that are not adjacent.



2. Complete each sentence with the correct word (**vertical**, **supplementary**).
   - A Angles 1 and 3 are ? .
   - B Angles 3 and 4 are ? .
   - C Angles 2 and 4 are ? .
   - D Angles 1 and 2 are ? .
   - E Angles 2 and 3 are ? .



3. In the figure, r and s are lines. Indicate which pairs of angles are
   - A vertical angles.
   - B supplementary angles.

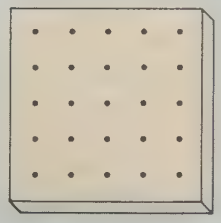


4. Tell whether each of the numbered angles in the picture at the right is
   - A right.
   - B acute.
   - C obtuse.
   - D straight.

5. Use the words **acute** or **obtuse** to complete each sentence.
   - A An angle that is congruent to an obtuse angle is an ? angle.
   - B An angle that is supplementary to an acute angle is an ? angle.
   - C If $\angle A$ and $\angle B$ are vertical angles with $\angle A$ acute, then $\angle B$ is ? .
   - D The supplement of an obtuse angle is an ? angle.
   - E An angle that is less than a right angle is an ? angle.
   - F A right angle is less than an ? angle.

think

Form an outline of a cross with a rubber band on a geoboard so that five nails are inside the cross and eight nails are outside the cross and all sides of the cross are congruent segments.



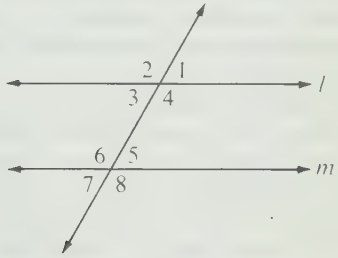
UTILIZATION

You may treat this entire exercise set as an oral activity if your students' background in geometry has been fairly broad.

EXTENSION

Workbook page 82 and *Duplicator Masters* page 56 offer exercises that may be used for extending practice with the ideas presented in this lesson.

Remedial: Ask students to study the figure below and name as many different pairs of related angles as they can. Then ask them to tell which angles are obtuse and which are acute.



Enrichment: Present students with a problem like the following.

Suppose $\angle XYZ$ and $\angle VYX$ are adjacent. What is

A. $\angle XYZ \cap \angle VYX$? ($\overrightarrow{YX}$)

B. {interior of $\angle XYZ$} $\cap$ {interior of $\angle VYX$}? (ϕ)

C. {interior of $\angle XYZ$} $\cup$ {interior of $\angle VYX$}? ({interior of $\angle VYZ$})

Think Solution

There must be 12 nails on the boundary of the cross; hence no side of the cross can contain 3 nails. The solution shown in the annotated figure is unique.

Assignments

Minimum 1–4, oral. Average 1–5. Maximum 1–5.

Objective

The student can identify the sides, vertices, interior, and exterior of a triangle.

PREPARATION

Materials: Cardboard strips, scissors, centimetre rulers, pins or thumb tacks.

For the Investigation, students will need several minutes to prepare the strips. Therefore, you will probably want to distribute the materials and begin the Investigation at once.

INVESTIGATION

The purpose of the Investigation is to provide concrete experiences in constructing a triangle. Two main ideas should evolve from this experience:

1. A triangle is a rigid figure.
2. It is not always possible to construct a triangle, given three arbitrary segments.

The student should discover that triangles can be formed from the following combinations of lengths: 3, 5, and 7 cm; 3, 7, and 9 cm; 5, 7, and 9 cm. No triangle can be formed by the combination of the 3, 5, and 9 cm strips.

DISCUSSION

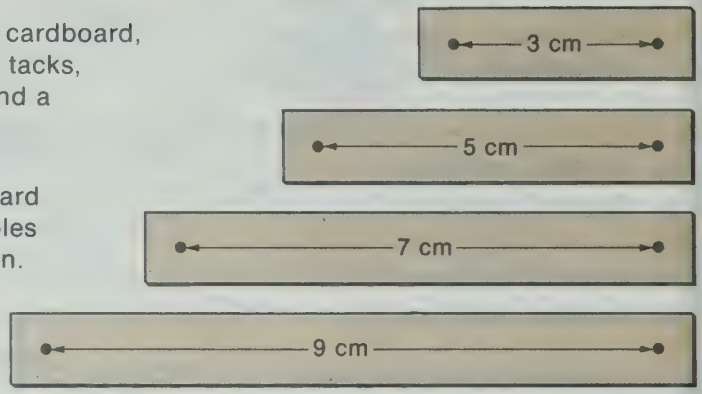
Exercise 2 introduces the idea elaborated on in Exercise 3: Any triangle formed by three of the strips can have but one shape. A triangle is *rigid* in the sense that the length of the three sides fixes its shape. The students can check the rigidity of their triangles by exerting pressure on the sides. The triangles will retain their shape unless the sides are bent or deformed. Compare the rigidity of a triangle to that of the quadrilateral suggested in Exercise 5. Note that the quadrilateral may assume many different shapes. You may want to discuss the role of triangles in providing rigidity in construction of buildings, bridges, and braces of various kinds.

Triangles

Investigating the Ideas

You will need some cardboard, three pins or thumb tacks, a pair of scissors, and a centimetre ruler.

Cut out four cardboard strips and punch holes in the ends as shown.



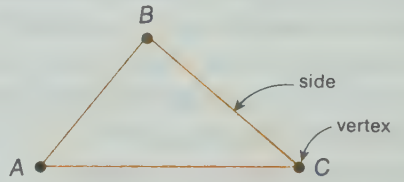
How many different shaped triangles can you make by choosing any three of the strips and putting pins through the holes?

Discussing the Ideas

1. Did you find three of the strips that would not form a triangle? Which three?

2. Can you make triangles of different shape using the same three strips?

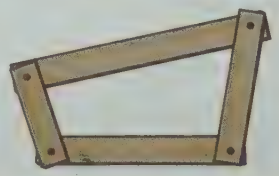
3. A **triangle** in geometry, consists of three points, not all on one line, and all the points on the segments determined by the three points.



A triangle is said to be a **rigid** geometric figure. Can you use the strips to demonstrate why a triangle is a rigid figure?

4. A How many **sides** does every triangle have?
   B How many **vertices**? (The plural of vertex is *vertices*.)

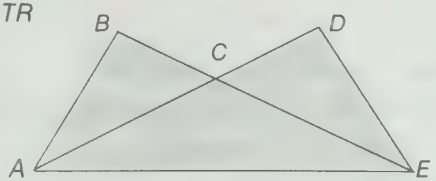
5. Join the four strips together to form a 4-sided figure. Is it a rigid figure like a triangle?



Using the Ideas

1. **A** Mark three points not in a line on your paper and label them *R*, *S*, and *T*.
   B Join the points with segments.
   C You have drawn triangle *RST* (SYMBOL: $\triangle RST$). What are some other symbols that could name the triangle? EXAMPLE: $\triangle STR$

2. Name as many triangles as you can in this figure.

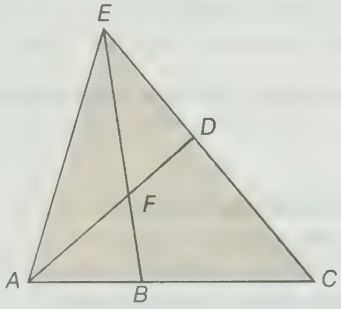


3. Draw a figure that illustrates each situation.
   A A line intersecting a triangle in two points.
   B A line intersecting a triangle in only one point.
   C A line that contains many points of a triangle.

4. A point is in the **interior of a triangle** if it is in the interior of each angle of the triangle. If a point is not in the interior or on the triangle, it is in the **exterior of the triangle.**

exterior

Draw a figure and label it like the one shown. Then mark each point described below.



   A Point *X* is in the interior of $\triangle ABE$ and in the exterior of $\triangle AFB$.
   B Point *Y* is in the interior of $\triangle ADE$ and in the exterior of $\triangle AFE$.
   C Point *Z* is in the interior of $\triangle ADC$ and in the interior of $\triangle BCE$.

think

Choose a 2-digit number such as 48.

Multiply the two digits. $4 \cdot 8 = 32$

Multiply again until you get a 1-digit number. $3 \cdot 2 = 6$

This gives a sequence of three numbers. 48, 32, 6

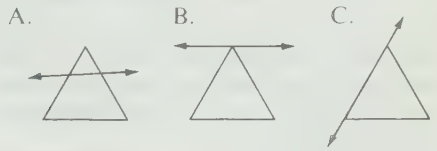
What 2-digit number will give the longest sequence?

UTILIZATION

The exercises provide practice in identifying and naming triangles and their parts.

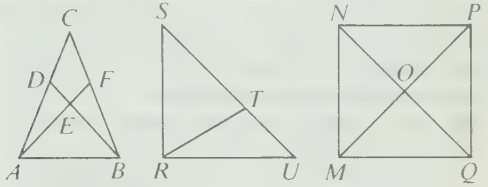
Exercise 4 defines the interior and the exterior of a triangle. If you prefer, discuss these ideas prior to assigning the problems.

Solutions, Exercise 3



EXTENSION

Remedial: To provide additional practice in naming triangles, present some figures with overlapping triangles, like those below. Ask students to name as many triangles in each figure as they can.



Enrichment: Ask students to find the total number of triangles in each figure. Extend the sequence with two more figures.

Think Solution

The students must solve the problem by trial and error. The number 77 gives a five-number sequence: 77, 49, 36, 18, 8. All other 2-digit numbers give a sequence of four or fewer numbers. Some students may be interested in seeing what 3-digit numbers will yield the longest sequence.

Assignments

Minimum 1, 2. Average 1–3. Maximum 1–4.

Objective

The student can identify the following types of triangles: scalene, isosceles, equilateral, and right.

PREPARATION

Materials: Plastic mirrors, geoboards, dot paper or graph paper (*Duplicator Masters*, pages 82, 83).

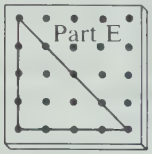
Since any preparatory discussion of the ideas is likely to preempt the Investigation, it would be best to begin the Investigation immediately.

INVESTIGATION

The Investigation consists of constructing various kinds of triangles on the geoboard or on dot paper. The related definitions will be given in the Discussion section.

The triangle described in part C of the Investigation (an equilateral triangle) cannot be formed on a square-array geoboard. If a circular geoboard of 6, 12, or 24 nails were available, such a triangle could be formed.

Sample solutions for parts E and F:



12 nails on the sides

6 interior nails

DISCUSSION

The types of triangles classified in Exercises 2, 3, and 4 in terms of their congruent elements can also be classified according to their lines of symmetry.

Scalene triangle: No line of symmetry.

Isosceles triangle: At least one line of symmetry.

Equilateral triangle: Three lines of symmetry.

You might find it interesting to point out that, if a triangle has 2 lines of symmetry, then it necessarily has 3 lines of symmetry.

In Exercise 6, you might have the students try to draw a triangle with two right angles. They will see that two of the sides will be parallel so that a triangle with 2 right angles is an impossibility.

Types of Triangles

Investigating the Ideas

How many of the triangles described below can you find on your geoboard?

Record your findings on dot paper.

A. A triangle with no two sides congruent.

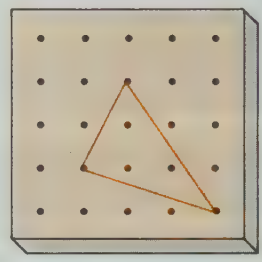
B. A triangle with a pair of congruent sides.

C. A triangle with all sides congruent.

D. A triangle that has a right angle.

E. A triangle with the greatest possible number of nails on its sides.

F. A triangle with the greatest possible number of nails in its interior.



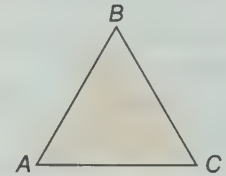
Discussing the Ideas

1. One of the triangles above is impossible to form on the geoboard. Which one do think it is?

2. A **scalene triangle** has no two of its sides congruent. Draw a scalene triangle. Does it have any lines of symmetry?

3. An **isoceles triangle** has at least two sides congruent. How many lines of symmetry does an isoceles triangle have?

4. An **equilateral triangle** has all sides congruent. How many lines of symmetry does an equilateral triangle have?

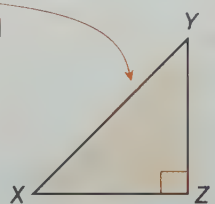


Equilateral $\triangle ABC$

5. A **right triangle** has a right angle.

   A. Show a right triangle on the geoboard that is also an isoceles triangle.

   B. The longest side of a right triangle is called the **hypotenuse**. Where is the hypotenuse of a right triangle located with respect to the right angle of the triangle?

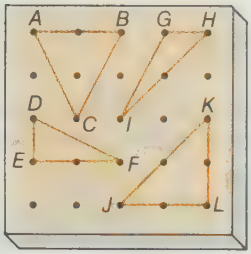


Right $\triangle XYZ$

6. Can a triangle have two right angles? Explain.

Using the Ideas

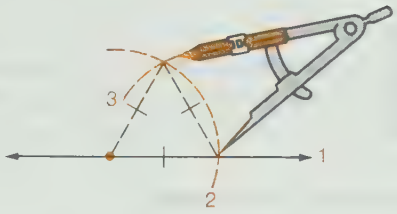
1. Each triangle on the geoboard has a special name, **scalene**, **isoceles**, **equilateral**, or **right**.
Give the name of each triangle.



A $\triangle ABC$ **C** $\triangle GHI$
B $\triangle DEF$ **D** $\triangle JKL$

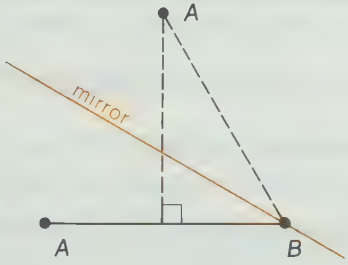
2. What properties would an **isoceles right triangle** have?

3. The drawing shows how a compass may be used to draw an equilateral triangle. Draw an equilateral triangle with sides 6 centimetres long.



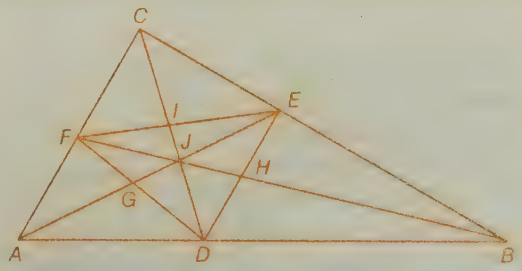
4. Draw an isoceles triangle that has a pair of sides 5 centimetres long.

★5. Use your plastic mirror and your knowledge about reflections to find each figure.
A Isoceles triangle
B Right triangle
C Equilateral triangle
(HINT: See the figure at the right.)



think

There are 47 different triangles in the figure. How many can you name?

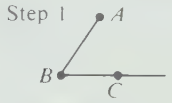


UTILIZATION

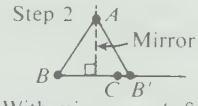
Assign these exercises according to the needs of your students.

Exercise 5 is more difficult than the others and is intended primarily for more capable students. Allow students to experiment with mirrors until they discover a method of construction.

A. Isosceles triangle



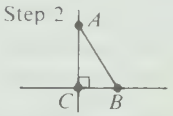
Draw $\angle ABC$.



With mirror on A, find the image of B on $\overleftrightarrow{BC}$.

B. Right triangle

Construct $\perp$ through C.



Draw any hypotenuse $\overline{AB}$.

EXTENSION

To provide additional practice, make selective assignments from *Workbook* page 83 and *Duplicator Masters* page 57.

Enrichment: Ask students to find as many isosceles right triangles of different sizes as they can on a 5 by 5 geoboard. (*Solution:* 8 triangles, as shown below.)

Think Solution

$14 + 10 + 7 + 9 + 5 + 2 = 47$

14: *CFI, CFE, CFJ, CFB, CFD, CAJ, CAE, CAD, CAB, CEI, CEJ, CED, CBJ, CBD*
10: *AFE, AFJ, AFG, AFB, AFD, ADE, ADG, ADJ, ABE, ABJ*
7: *BEH, BEF, BEJ, BDF, BDE, BDJ, BDH*
9: *FJE, FJG, FJD, FJI, FDI, FDH, FDE, FEG, FEH*
5: *EJI, EJH, EJD, EDG, EDI*
2: *DJG, DJH*

Assignments

Minimum 1–3. Average 1–4. Maximum 1–5.

Objective

The student can state whether a given plane figure is convex or concave.

PREPARATION

The lesson is of such a nature that preparatory activities are not necessary.

INVESTIGATION

The Investigation will require only a short amount of time. Emphasize that students are to think about standing *at any point* inside the room, not at one particular point. Ask students to record their choices on paper so that they can refer to them during the Discussion.

DISCUSSION

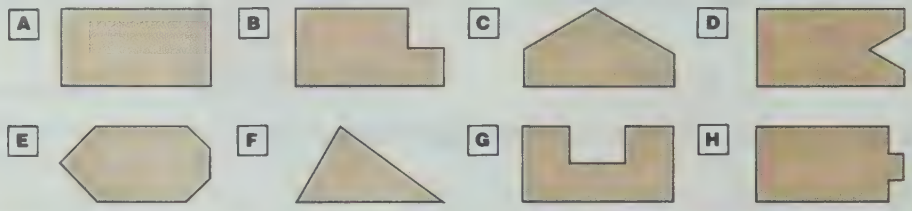
The representations of room floor plans are used to give an intuitive description of convex and concave figures. A more formal definition of convexity and concavity is given as a part of Discussion Exercise 4.

Exercises 2 and 3 bring out the idea that every triangle is a convex figure, but polygons having more than three sides can be either convex or concave.

Convex and Concave Figures

Investigating the Ideas

Suppose each drawing below represents a floor plan of a room and you are standing inside the room.



In which rooms could you see all parts of the room no matter at what point you stood in the room?

Discussing the Ideas

1. The rooms in which you can see all parts of the room from any inside point are **convex** shaped rooms. The others are **non-convex** shaped rooms. Is your classroom convex or non-convex in shape?

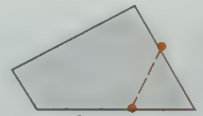
2. Can you draw a non-convex triangle?

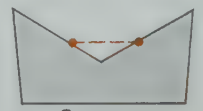
3. Is the statement true or false?

   All 4-sided geometric figures are convex.

   Draw some pictures that would illustrate your answer.

4. Another way in which you can think about convex and non-convex figures is illustrated at the right.


Convex


Concave

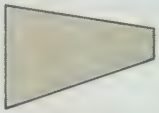
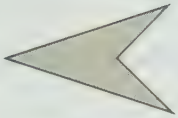
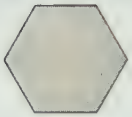
   A figure is **convex** if every segment joining any two points on the boundary of the figure contains only points in the interior of the figure or on the boundary. Non-convex figures are **concave** figures.

   A Draw a 6-sided figure that is convex.

   B Draw a 6-sided figure and show that it is concave.

Using the Ideas

1. Tell whether each figure is convex or concave.

A
B 
C 
D 
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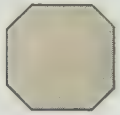
F 
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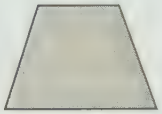
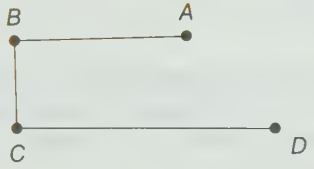
H 

2. Make a copy of each figure below. Then draw a segment that will show that each figure is concave.

A
B
C
D

3. Draw a figure like the one shown at the right.



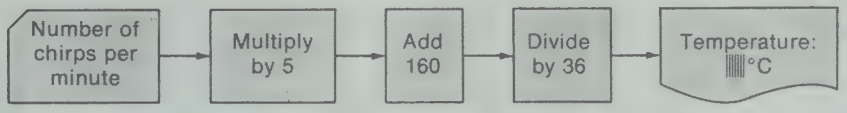
   A If *A* and *D* are connected by a segment, is the figure convex or concave?

   B Complete the figure with two segments connecting *D* with *A* to form a convex figure.

   C Complete the figure with two segments connecting *D* with *A* to form a concave figure.

think

It is said that the approximate temperature can be found by counting the number of chirps per minute of a cricket. The flow chart below gives the method.



What is the approximate temperature when the number of chirps per minute is

A 40 B 112 C 184 D 4 E 256

At what temperature would a cricket stop chirping?

UTILIZATION

Exercise 1 can be discussed orally, if you prefer. Exercises 2 and 3 involve pencil and paper activity.

EXTENSION

Enrichment: Convex and concave surfaces may provide some interesting enrichment experiences. You might want to ask some students to consult reference books to find information about practical applications of convex and concave surfaces. Reflections with convex and concave mirrors provide interesting distortions. Concave mirrors are used in telescopes, searchlights, and flashlights. Convex mirrors are often strategically placed in retail stores so that they reflect a view of large parts of the store that might otherwise not be visible to employees.

Think Solution

The temperature at which a cricket would stop chirping can be found by using 0 chirps per minute as the input in the flow chart. This gives an output of about 4.4°C.

Assignments

Minimum 1, oral; 2. Average 1–3.
Maximum 1–3.

Objective

The student can identify the following special quadrilaterals: square, rectangle, parallelogram, trapezoid, rhombus.

PREPARATION

Materials: Geoboards, plastic mirrors, dot paper or graph paper (*Duplicator Masters*, pages 82, 83).

Distribute the needed materials and have the students begin the work on the Investigation.

INVESTIGATION

The Investigation will provide several minutes of challenging activity. Do not expect all students to find all the squares of different sizes.

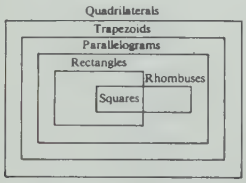
Squares with areas 1, 4, 9, and 16 are obvious. The less-obvious squares can be formed as shown below.

5 10 8 2

DISCUSSION

Exercise 2 provides pictorial and verbal descriptions of the special quadrilaterals. Observe that all squares are rectangles and all rectangles are parallelograms. A trapezoid is usually defined as a quadrilateral *with at least one pair of parallel sides*. According to this definition, a parallelogram is a special kind of trapezoid.

The various subsets of the set of all convex quadrilaterals can be pictured by means of a Venn diagram such as the one below.

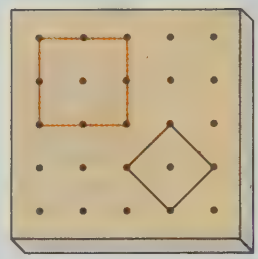


Observe that the set of squares is the intersection of the set of rectangles and the set of rhombuses. Thus, we can think of a square as either a rhombus with right angles or as a rectangle with congruent sides.

Special Quadrilaterals

Investigating the Ideas

Two squares of different size are shown on the geoboard.

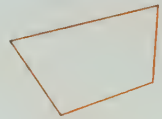


How many squares of different size can you find on your geoboard?

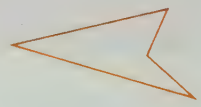
Record your findings on dot paper or graph paper.

Discussing the Ideas

1. Squares are a special kind of **quadrilateral**. Do you think that a square is a convex or a concave quadrilateral? Give your reasons.

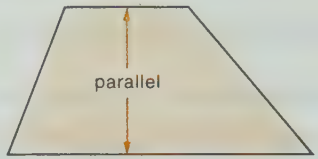


convex quadrilateral

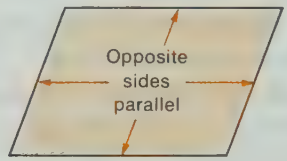


concave quadrilateral

2. Study the special quadrilaterals named below. What other properties do you think each quadrilateral might have besides those that are listed?

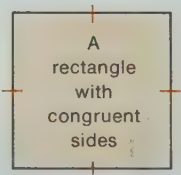


Trapezoid

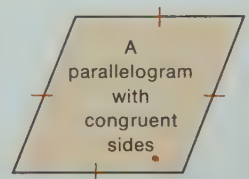


Parallelogram

Rectangle



Square



Rhombus

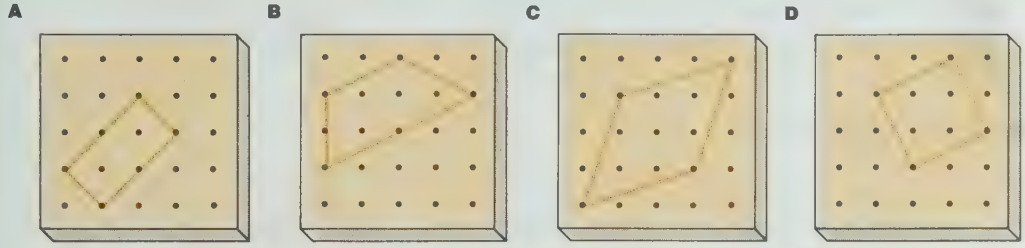
3. Can a quadrilateral have two right angles and not be a rectangle? Illustrate this with your geoboard or by drawing a picture.

4. Can a quadrilateral have two right angles and not be a trapezoid? Illustrate your answer by drawing a picture.

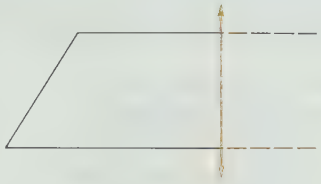
Using the Ideas

1. Of the five special kinds of quadrilaterals shown in Exercise 2 on the preceding page, tell which ones have
   - **A** at least one pair of parallel sides.
   - **B** two pairs of parallel sides.
   - **C** all right angles.
   - **D** all sides congruent.

2. Give the name of each special quadrilateral shown on the geoboards.



3. Draw a trapezoid that has a line of symmetry.



You have drawn an **isoceles trapezoid**; that is, a trapezoid with a pair of congruent sides.

4. How many lines of symmetry does a rectangle have? Draw a picture to show the lines of symmetry.

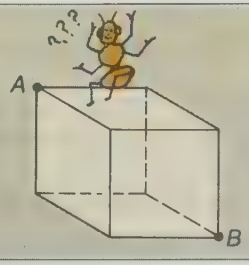
5. Draw a rhombus. Show any lines of symmetry.

6. Draw a square. Show all its lines of symmetry.

★ 7. Construct a rhombus by means of reflections.

think

Suppose a bug is at point *A*. What is the shortest path he can crawl in order to get to point *B*? Draw this path.



UTILIZATION

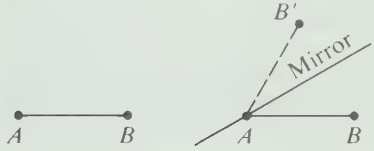
If geoboards are available, students might construct the quadrilaterals shown in Exercise 2. The figure in C is a rhombus since all sides are congruent. Some students may identify it simply as a parallelogram.

Students should have access to plastic mirrors for Exercises 4, 5, 6, and 7.

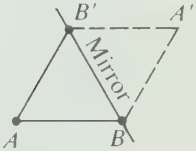
Construction, Exercise 7

Step 1:
Draw $\overline{AB}$.

Step 2:
Find B', the reflection image of B, and draw $\overline{AB'}$.

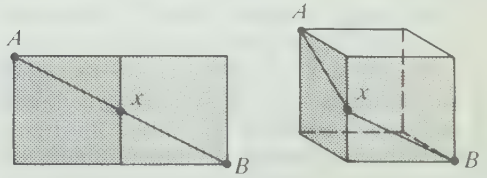


Step 3:
Locate A', the reflection image of A in $\overleftrightarrow{BB'}$, and draw $\overline{A'B}$ and $\overline{A'B'}$. $ABA'B'$ is a rhombus.



Think Solution

When the left side and front of the cube are spread flat (see figure 1), they form a rectangle. Thus, path AxB will be the shortest distance.



Assignments

Minimum 1–4. Average 1–6. Maximum 1–7.

Objective

The student can name the polygons of up to ten sides and determine the number of diagonals of each polygon.

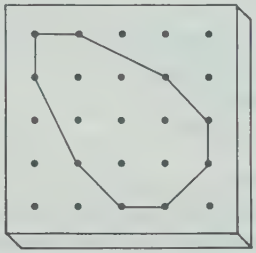
PREPARATION

Materials: Geoboards, dot paper or graph paper (*Duplicator Masters*, pages 82, 83).

Review the meaning of convex and concave figures and the special quadrilaterals before students begin the Investigation.

INVESTIGATION

The Investigation provides concrete experiences with polygons of many sides. Most students will be able to find a convex polygon of 8 sides on the geoboard. However, many students will have difficulty finding that the maximum number of sides of a convex polygon on a 5 by 5 geoboard is 9, as shown below.



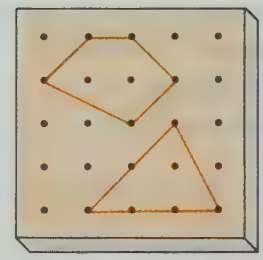
DISCUSSION

The Discussion Exercises focus on the definitions of polygon, regular polygon, and diagonals of polygons. Emphasize the fact that regular polygons have all sides congruent and all angles the same size. Exercise 3 illustrates that, if only one of these conditions is met, the polygon will not be a regular polygon. Although a rhombus has all sides congruent, it is not a regular polygon. A rectangle has all angles the same size, but since all sides are not necessarily congruent, it is not a regular polygon.

Other Polygons

Investigating the Ideas

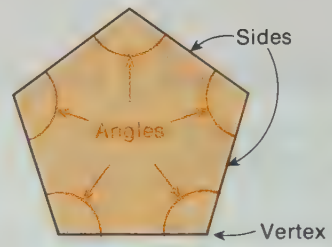
A triangle is a closed geometric figure with the fewest number of sides.



What is the greatest number of sides of any convex figure that you can show on a 5 by 5 nail geoboard?

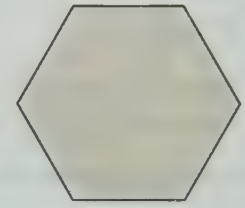
Discussing the Ideas

1. A **polygon** is a closed geometric figure whose boundary is composed of segments. The word polygon comes from
   poly meaning "many" and
   gonia meaning "angle."



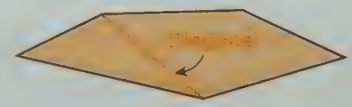
   Does every convex polygon have the same number of angles as it has sides?

2. A convex polygon is a **regular polygon** if all of its sides are congruent segments and all of its angles are congruent.
   - **A** What is the name of a regular polygon that has three sides?
   - **B** What is the name of a regular polygon that has four sides?



3. **A** Draw a polygon that has four congruent sides but which is not a regular polygon.
   B Draw a polygon that has four angles, all congruent to each other, but which is not a regular polygon.

4. A **diagonal** of a polygon is a segment connecting two nonconsecutive vertices of a polygon.



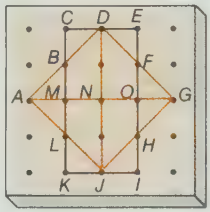
   - **A** Does a triangle have any diagonals?
   - **B** How many diagonals does a quadrilateral have?

Using the Ideas

1. Complete the table.

POLYGON	NAME	NUMBER OF SIDES	NUMBER OF DIAGONALS
	pentagon		
	hexagon		
	heptagon		
	octagon		
	nonagon		
	decagon		

2. Give the name of the polygon (**triangle**, **square**, and so on) shown on the geoboard whose vertices are named below.



- **A** *ADGJ*
- **B** *FGH*
- **C** *CEIK*
- **D** *MBDFO*
- **E** *DFHJ*
- **F** *BDFHJK*

3. Using the points on the geoboard in Exercise 2, name an octagon. Draw a picture of the octagon.

4. How many regular polygons are shown on the geoboard? Name them.

think

Use tracing paper to copy this figure. It is a **simple closed curve**. (Think of a loop of string that does not cross itself.) Within the colored circle, shade the interior of this simple closed curve.

UTILIZATION

You may want to emphasize the names of the polygons given in the table. Concentrate on the meaning of the prefixes: *tri-*, *quad-*, *penta-*, *hexa-*, etc.

Students may find the number of diagonals of some polygons difficult to count. Suggest that they look for patterns when filling out the table in Exercise 1. In general, for a polygon of n sides, the number of diagonals, d, is given by the formula

$$d = \frac{n(n-1)}{2} - n = \frac{n^2 - 3n}{2},$$

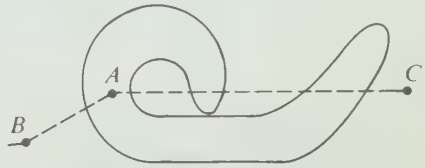
where n is the number of sides of the polygon.

EXTENSION

Enrichment: Ask students to try to find a method of constructing some of the regular polygons using straightedges and compasses.

Think Solution

The trick to the solution is based on an elementary concept. In crossing from the exterior to the interior of a simple closed curve, the curve itself must be crossed. This may be illustrated by a less complicated figure. Beginning at point B, which is in the exterior of the curve, the curve is crossed once to arrive at point A, which is in the interior of the curve



Beginning at point C in the exterior, crossing the curve *once*, places you in the interior; crossing a second time takes you to the exterior; the third crossing places you in the interior. Thus an odd number of crossings places you in the interior; an even number of crossings places you in the exterior.

Assignments

Minimum 1, 2. Average 1–3. Maximum 1–4.

Objective

The student can demonstrate the ability to work with the ideas presented in this module.

UTILIZATION

The review exercises can be assigned as independent work or discussed in large or small groups with students.

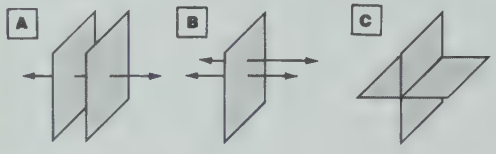
If you have omitted any of the topics in this module, you should not include the exercises dealing with those topics in the review assignments.

Additional exercises suitable for use in reviewing the concepts presented in this module are provided on page 84 of the *Workbook*.

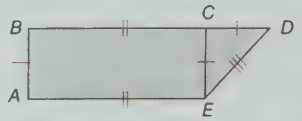
REVIEWING THE IDEAS

1. Give three names for this line.

2. Choose the correct figure suggested by the phrase "two planes intersect in a line."



3. Name the pairs of congruent segments in this figure.

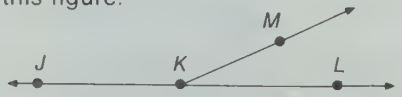


4. Complete each sentence.
   - A If $\overline{XY} \cong \overline{RS}$ and $\overline{RS} \cong \overline{MN}$, then $\overline{XY} \cong$? .
   - B If $\overline{CD} < \overline{JK}$ and $\overline{JK} \cong \overline{PQ}$, then ? .

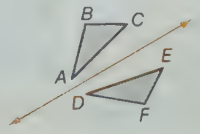
5. A Each of the two sides of a point on a line is called a ? .
   - B Each of the two sides of a line in a plane is called a ? .

6. Name three different rays.

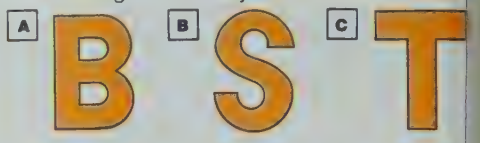
7. Name three angles shown in this figure.



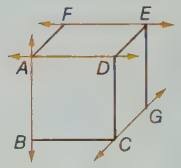
8. Are $\triangle ABC$ and $\triangle DEF$ reflection images of each other?



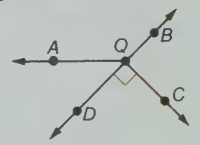
9. Which regions are symmetric?



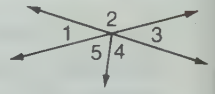
10. Name the kind of lines (**parallel**, **skew**, **perpendicular**) for the pairs shown on the cube.
    - A $\overleftrightarrow{AD}$ and $\overleftrightarrow{FE}$
    - B $\overleftrightarrow{AB}$ and $\overleftrightarrow{CG}$
    - C $\overleftrightarrow{AD}$ and $\overleftrightarrow{AB}$



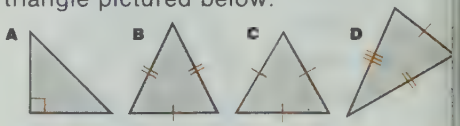
11. Tell which angles are
    - A acute
    - B obtuse
    - C right
    - D straight



12. Tell which pairs of angles are
    - A vertical
    - B supplementary

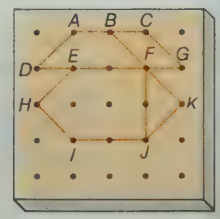


13. Give the name of each kind of triangle pictured below.



14. Which figure is convex?

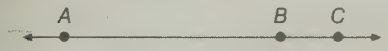
15. Find and name the following polygons on the geoboard.
    - A trapezoid
    - B parallelogram
    - C pentagon
    - D hexagon



TEST YOURSELF

1. Complete the sentence.
   If X and Y are points, there is exactly ? line that contains them.

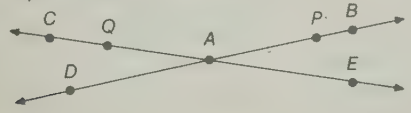
2. Name all the segments on the line.



3. Use the figure in Exercise 2. Which symbol ($<$, $\cong$, or $>$) goes in the ? $\overline{AB}$? $\overline{AC}$

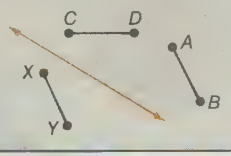
4. All points on one side of a line in a plane is called a ?

5. Refer to the figure below to complete each sentence.



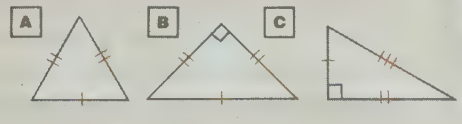
   A The two lines intersect at point ? .
   B The union of $\overrightarrow{AE}$ and $\overrightarrow{AD}$ is $\angle$? .

6. Which segment, $\overline{AB}$ or $\overline{CD}$, is the reflection image of $\overline{XY}$ in the line?

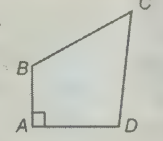


7. Name the kind of lines (**parallel**, **skew**, **perpendicular**) described.
   A Two lines that intersect to form right angles.
   B Two lines in a plane that do not intersect.
   C Two lines that are not in the same plane.

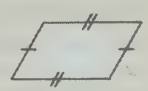
8. Which figure is an isoceles right triangle?



9. Name the type of angle for each part.
   A $\angle BAD$ is a(n) ? angle.
   B $\angle ABC$ is a(n) ? angle.
   C $\angle BCD$ is a(n) ? angle.



10. A Name this special quadrilateral.



    B Is it **convex** or **concave**?

RESEARCH PROJECTS

A Straight lines can be used to make some interesting patterns, designs, and figures. Make some designs based on straight lines. (See *Patterns and Puzzles* by Sylvia Horne; Chicago: Lyons and Carnahan, 1968. available from McGraw-Hill Ryerson, Ltd.)

B Make a large poster depicting the Snowflake Curve. This curve has the property that although it has a finite area, it can have infinite perimeter. (See *History of Mathematics* by Carl B. Boyer; New York: 1968, John Wiley and Sons Canada, Ltd., pp. 660–661.)

C Do you think that a straight line can be drawn without using a ruler or some other straight edge? Some devices, called **linkages**, have been invented with which to produce straight lines. One such linkage was invented by James Watt. In 1864, a linkage known as Peaucellier's Cell was invented which produced straight line motion. Find out and make some models of these linkages. (See *The Enjoyment of Mathematics* by Hans Rademacher and Otto Toeplitz; Princeton, New Jersey: Princeton University Press, 1957.)

TEST YOURSELF

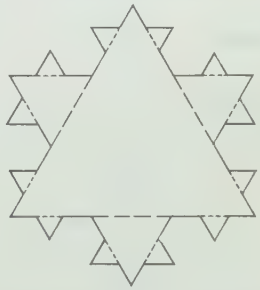
This self-evaluation test will give students a measure of their understanding of the major concepts of the module as set forth in the objectives listed at the beginning of the module.

RESEARCH PROJECTS

Students who undertake Project A will discover that attractive, durable straight-line designs can be made by using nails as points and stretching colored string or yarn between the nails. Such designs are often used in the works of professional artists and craftsmen.

Research Project B would be an appropriate undertaking for students who have shown a special interest in geometric constructions. A brief description of the principal construction steps follows.

Draw a large equilateral triangle on a piece of posterboard. Divide each side into 3 congruent segments. Construct an equilateral triangle outward on the middle third and erase the segment that was the middle segment. Now divide each side of the new curve into 3 congruent segments and again construct equilateral triangles outward, again deleting the middle third. Repeat this process as many times as possible. The resulting curve has an area less than the area of the circle circumscribing the original triangle, but the perimeter is unbounded.

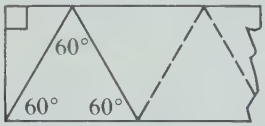


First Three Stages of the Snowflake Curve

Research Project C would be an excellent one for a mathematics fair. Attractive, workable models can be made from strips of wood or Meccano strips.

MATHEMATICAL RECREATION

To make the strip for the hexaflexagon, construct the first equilateral triangle on the strip by using a protractor to draw angles of 60° measure. Cut off the small right triangle at the left end of the strip. Now the remaining triangles can be formed by carefully folding the strip back and forth over the first triangle.

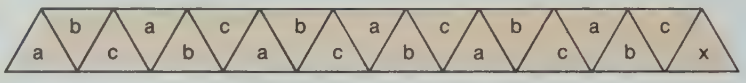


Since this model is somewhat complicated, you may want students to begin with a simpler model. A pattern having only three faces is presented in Donovan Johnson's *Paper Folding for the Mathematics Class*, Washington, D.C.: National Council of Teachers of Mathematics, 1957.

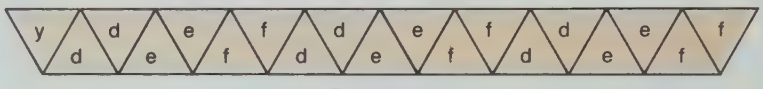
MATHEMATICAL RECREATIONS

A HEXAFLEXAGON

Use a strip of adding machine paper. Mark off 19 equilateral triangles on the strip. Label the front and back of the strip with the letters as shown in the diagrams below.

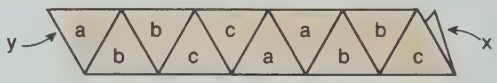


Front side



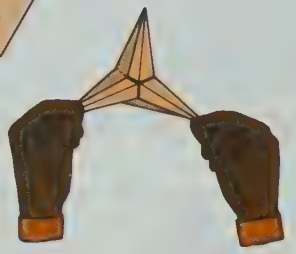
Back side

Fold the strip in a winding motion so that the same letters on the back side face each other; that is, d faces d, e faces e, and f faces f.



With the strip flat on your desk, fold the strip again with a winding motion so that all the pairs of faces labelled c come face to face. This should cause the strip to fold into a hexagon shaped with all the faces on one side of the hexagon labelled a and all the faces on the other side labelled b. The two faces labelled x and y are pasted face to face to complete the hexaflexagon.

Now flex your hexagon as shown in the figure by pinching it into a 3-cornered shape and letting it open at the centre. You will expose another face of the hexaflexagon with all of its faces having the same letter on them.



Can you flex your hexaflexagon so that you find all six faces?

MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. one **2.** $\overline{AB}$, $\overline{AC}$, $\overline{BC}$ **3.** $<$ **4.** half-plane **5.** **A** A; **B** DAE **6.** $\overline{CD}$ **7.** **A** perpendicular; **B** parallel; **C** skew **8.** **B** **9.** **A** right; **B** obtuse; **C** acute **10.** **A** parallelogram; **B** convex

UNIT C: GEOMETRY AND MEASUREMENT

Module 2: Measurement

General Objectives

To review and strengthen the students understanding of the process of measurement.
To place emphasis upon metric units of measure.
To provide experiences in measurement of length, area and volume.
To introduce metric measures of capacity.
To present the Pythagorean Theorem in terms of area of squares on the sides of a right triangle.
To develop skills in angle measurement with a protractor and to work with degrees, minutes, and seconds.
To show that the sum of the measures of the angles of a triangle is 180°

	Performance Objectives	Pupil Text	Reteach-Reinforce	Related Activities	
91	**RED** Given a segment, the student can give its length in metric units.	C-38, 39	**WB** 85		**SWM 1** 11
92	**RED** The student can find the perimeter of a polygon given the lengths of its sides.	C-40, 41	**WB** 86 **DM** 58	**SP** f-2 **DS** b-1	
93	**RED** Given a rectangular region, the student can find its area.	C-46, 47	**WB** 87 **DM** 59	**ASC** AP-7 **SP** f-3 **PD** e-2	**SWM 3** 118, 213, 256
94	**RED** Given a parallelogram, the student can find its area.	C-46, 47	**WB** 87 **DM** 59	**ASC** AP-8	
95	**RED** Given a triangular region, the student can find its area.	C-48, 49	**WB** 88 **DM** 60	**ASC** AP-9 **SP** f-3 **PD** e-2	**SWM 1** 28 **SWM 3** 119
96	**RED** Given a rectangular solid, the student can find its volume.	C-52, 53	**WB** 90 **DM** 61	**ASC** AP-12 **SP** f-4	
97	**RED** Given a rectangular solid, the student can find the surface area.	C-54, 55	**WB** 90 **DM** 61	**ASC** AP-11	
98	**RED** The student can find the capacity of a given container using metric units.	C-56, 57			**SWM 2** 264, 265
99	**RED** The student can measure angles with a protractor.	C-58, 59	**WB** 91		**SWM 2** 132, 133, 135
100	**YELLOW** Given a right triangle and the lengths of two of its sides, the student can apply the Pythagorean Theorem to find the length of the third side.	C-50, 51	**WB** 89	**DS** f-3, 4	
101	**YELLOW** The student can compute sums and differences of angle measures using degrees, minutes, and seconds.	C-60, 61	**WB** 92 **DM** 62	**SP** e-1	
102	***GREEN*** *Given a measurement, the student can give the greatest possible error in the measurement.*	C-42, 43			
103	***GREEN*** *Given a triangle or other polygon, the student can find the sum of the measures of its angles.*	C-62, 63	**WB** 93 **DM** 62	**SP** e-4 **PD** g-3, 4	
	Reviewing the Ideas	C-64	**WB** 94		

MATHEMATICS

The measurement process is basically a function in which a geometric object is assigned a number according to an agreed-upon unit. The steps of the measurement process are as follows:

If S stands for an object to be measured, then

1. Select a unit.
2. Divide S into units.
3. Count the number of units in S.

This number is the measure of S, $m(S)$.

To measure length, the unit we choose is a segment. To measure area, we choose a square as the unit. For volume, the unit is a cube.

A few basic formulas are developed in the module. $A = lw$, $A = \frac{1}{2}bh$, and $V = lwh$ are the most important of these.

TEACHING THE MODULE

Materials

Centimetre rulers, metre sticks, graph paper, small cubes, adding machine tape, geoboards, dot paper, scissors, posterboard, protractors.

Vocabulary

accurate
are
area
base
bisector
capacity
complementary
degree
gram
greatest possible error
hectare
height
length
litre
metre
minute
perimeter
polygon
polygonal path
precision
protractor
Pythagorean Theorem
relative error
second
supplementary
surface area
unit
- cube
- segment
- square region

volume

These are the three main goals for teaching the module:

1. To further the students' understanding of the measurement process.
2. To develop the students' ability to measure length, area, volume, and angles.
3. To improve the students' problem solving skills.

It is vital that students be provided with the necessary measurement tools for the work in this module. Centimetre rulers, metre sticks, cubes, and protractors are the basic measuring devices required for the module. Metric units of measure are used except for the cases in which the unit of measure is arbitrary.

While much of the material may constitute review for some of your students, there are many topics (such as greatest possible error; metric capacity; and angle measure using degrees, minutes, and seconds) that will be new to your students and deserve careful treatment.

Lesson Schedule

You should plan to allow about 14 days to cover this module, assuming coverage of one lesson per day. Additional time may be necessary if special projects are carried out in conjunction with the work of the module.

Evaluation

It is reasonable to expect that the student should be able to measure lengths using metric tools and to use the protractor in measuring angles. This aspect of students' progress is better evaluated by actual observation than by a paper and pencil test.

Students should be expected to compute areas of rectangles, parallelograms, and triangles and to find the volume of rectangular solids.

The chapter review and self-evaluation test should provide tools for assessment of student progress and can serve as models for a teacher-made test. Module Achievement Tests are also available from the publisher as part of the test package designed for this series.

RESOURCES FOR ACTIVE LEARNING

Activities

Developmental Math Cards, K15, 17; L1, Addison-Wesley

ESS:Senior Balancing, Webster, McGraw-Hill

Experiments in Mathematics, Stage 1, pp. 8–13; Stage 2, pp. 12–21, Houghton Mifflin

Experiments With Pentominoes, Midwest Publications

Let's Play Games in Metrics, National Textbook Co.

Measure and Find Out, Books 2 and 3, Scott Foresman

The Metric System, Addison-Wesley

Manipulative Devices

Centimetre cubes—1 gram mass (Addison-Wesley)

Games and Puzzles

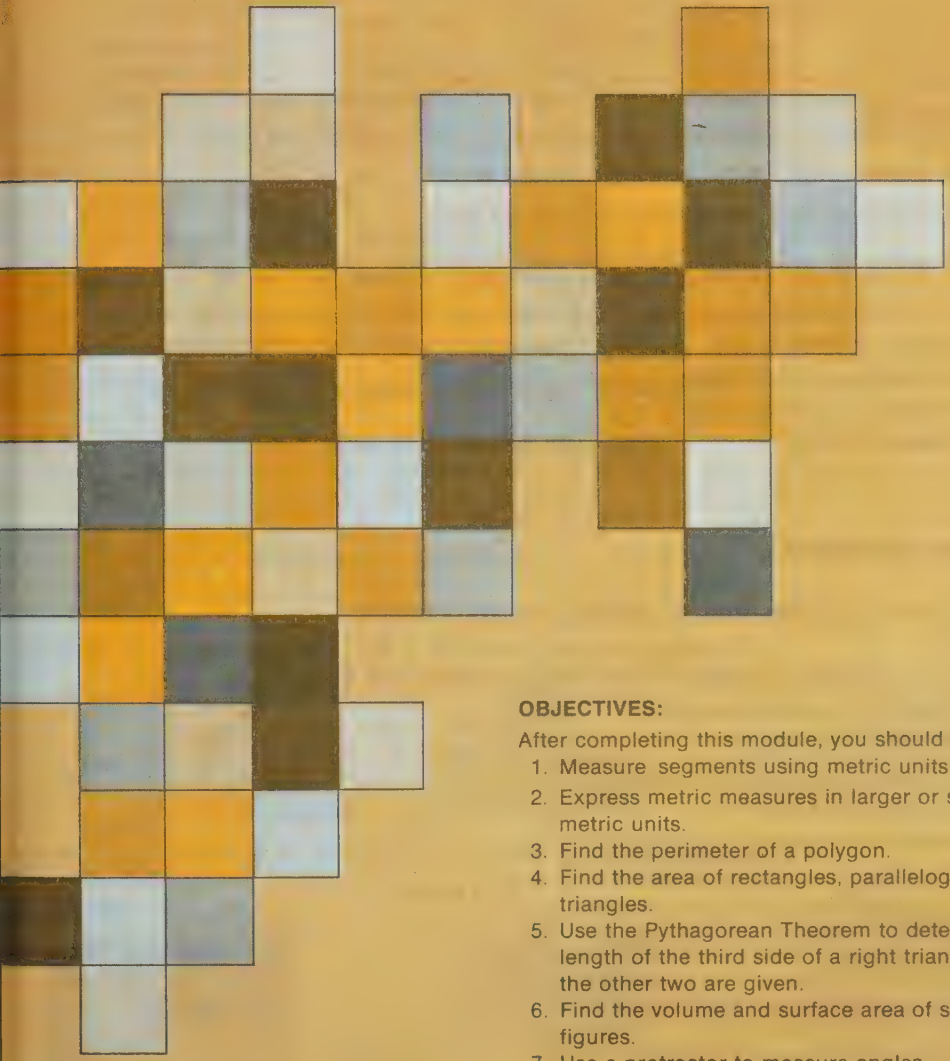
Hexed (Midwest Publications; Selective Educational Equipment)

Polyominoes (Lakeshore; Sigma)

Soma and other block puzzles (Creative Publications; Edmund Scientific)

UNIT C: *Geometry and Measurement*

MODULE 2: *Measurement*



OBJECTIVES:

After completing this module, you should be able to:

1. Measure segments using metric units of length.
2. Express metric measures in larger or smaller metric units.
3. Find the perimeter of a polygon.
4. Find the area of rectangles, parallelograms, and triangles.
5. Use the Pythagorean Theorem to determine the length of the third side of a right triangle when the other two are given.
6. Find the volume and surface area of space figures.
7. Use a protractor to measure angles.
8. Find sums and differences of angle measures given in degrees, minutes, and seconds.
9. Find the sum of the angle measures of triangles and other polygons.

Objective

The student can select an appropriate unit and use it to find the measure of an object.

PREPARATION

Materials: Small cubes, rulers, graph paper (*Duplicator Masters*, page 83).

Almost all students at this level will have had some measurement experiences involving length, area, and volume. You can discuss briefly the measurement process or proceed immediately to the Investigation.

INVESTIGATION

After students have had an opportunity to study the flow chart, allow them to select their own measurement activity if feasible, or have some students do the measurement of lengths, while others do the area or volume activity.

Do not specify the unit to be used, but simply guide the students into their choices. Some will use units on rulers, but some may use non-standard units. Provide graph paper for unit squares in the area activity and small cubes for units in the volume activity.

DISCUSSION

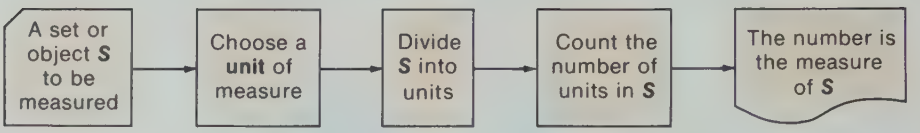
When discussing the measurement activities, refer to the flow chart in the Investigation and point out that regardless of which measurement activity was carried out, the measurement process was essentially the same.

To check their estimates of length and area in Exercises 2 and 3, the students can trace the units in the textbook. For Exercise 2, they should make a ruler. A grid of the squares in Exercise 3 could be used to check their area estimates, although some students will recall the area formula for rectangles and use it to check their estimate.

What is the Measurement Process?

Investigating the Ideas

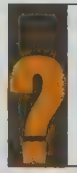
When you **measure** a set or object, you assign a number to it. The flow chart below shows the measurement process.


A set or object S to be measured
Choose a unit of measure
Divide S into units
Count the number of units in S
The number is the measure of S

A measure of a segment is called **length**.

A measure of a plane region is called **area**.

A measure of a space figure is called **volume**.



Can you choose a unit of measure and find the measure of at least one of these objects?

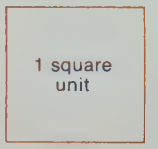
- **A** The length of one side of your classroom.
- **B** The area of a desk top or table top.
- **C** The volume of a box.

Discussing the Ideas

1. Explain how you measured the object that you chose?

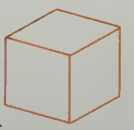
2. To measure **length**, a *segment* must be chosen as the unit.
   - **A** If $\overline{AB}$ is the unit, estimate the length of your desk top.
   - **B** Find a way to check your estimate.

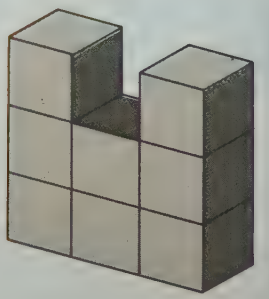
3. To measure the **area** of a plane region, a *square region* is chosen as the unit.
   - **A** Estimate the area of the front cover of your book using the square unit shown.
   - **B** Find a way to check your estimate.



4. To measure the **volume** of a space figure, a *cube* is chosen as the unit.

   Using the cubic unit shown, find the volume of the figure at the right.





Using the Ideas

1. When measuring each item described, would you be finding length, area, or volume?

- **A** The height of a flag pole.
- **B** The space inside of a drawer.
- **C** The size of a patio.
- **D** The distance of a bike trail.
- **E** The floor space in a cabin.
- **F** The amount of concrete needed for a sidewalk.

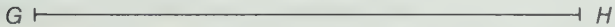
2. Tell whether **length**, **area**, or **volume** is being described in each part.

- **A** Joan claimed that she painted the entire inside of the closet.
- **B** Joan then said that she had completely filled the closet with boxes of clothes.
- **C** The closet was wide enough to accomodate three boxes.
- **D** She also noted that she had used two cans of paint for the job.
- **E** In putting the cartons into the closet, she figured that she had to walk back and forth across the room 115 times.
- **F** Joan's father noted that she had forgotten to paint the closet door.

3. If is the unit, how long is each segment.

A

B

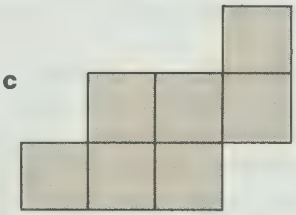
C 

D

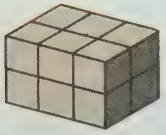
4. If is the unit, what is the area of each region?

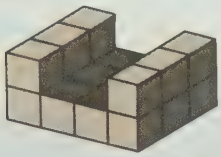
A

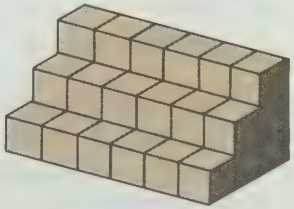
B

C 

5. If is the unit, what is the volume of each space figure?

A 

B 

C 

★**6.** Suppose a cube whose edges were only half as long as the cube in Exercise 5 was the unit of measure, what would be the volume of the figure in part **A**?

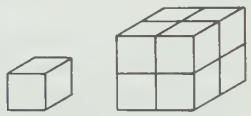
UTILIZATION

Exercises 1 and 2 may be discussed orally if you prefer.

The unit in Exercise 3 is the centimetre, and in Exercise 4 the unit is the square centimetre. In Exercise 3 the students can either use a centimetre ruler or mark off multiples of the unit along the edge of a sheet of paper and use that as their ruler.

Solution, Exercise 6

A cube with edges $\frac{1}{2}$ as long as those of another cube has only $\frac{1}{8}$ the volume of the other cube. A demonstration of this using 9 cubes of the same size shows that it will take 8 times as many of the smaller cube to equal the volume of the larger figure.



Therefore, $8 \cdot 12$, or 96, of the smaller cubic units are needed to make the figure shown in Exercise 5A.

EXTENSION

Enrichment: A pentomino is a plane region made of 5 squares each of which has at least one side in common with another square. Two pentominos are shown below.

Ask students to try to find all possible pentominos. (There are 12 in all.) Then ask them if they can arrange the 12 pentominos to form a rectangle.

Assignments

Minimum 1 and 2, oral; 3–5.
Average 1 and 2, oral; 3–5.
Maximum 1 and 2, oral; 3–6.

Objective

The student can measure and record lengths of segments using metric units.

PREPARATION

Materials: Centimetre rulers, metre sticks, adding machine tape.

The previous lesson introduced the general idea of measurement. This lesson focuses upon measurement of length using metric units. If your students have had considerable experience with metric units, you can treat this lesson as review. For students with little previous experience with the metric system, you will need to proceed slowly, carefully emphasizing the subunits of the metric system.

INVESTIGATION

The Investigation introduces the metric unit called a *decimetre*. Many students' hands will be approximately 1 decimetre wide. You might suggest that pairs of students work together and estimate each other's height in decimetres, using their hands as an approximate decimetre unit. After they have estimated the length or width of the classroom, suggest that they make a decimetre ruler from a strip of adding machine tape and use their ruler to check their estimate.

DISCUSSION

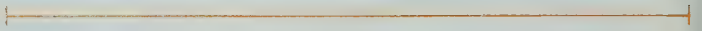
Students may already have made some decimetre rulers for the Investigation. Display a standard metre stick and have the students compare their homemade rulers with it.

The importance of understanding the prefixes of the metric system given in Exercise 3 cannot be stressed too strongly. Once the relation between the units is grasped, the metric system becomes quite simple to use.

Units of Length

Investigating the Ideas

This segment has a length of one decimetre.



A Is your hand more or less than one decimetre wide?

B Estimate your height in decimetres.

Can you estimate the length or width of your classroom in decimetres and then find a way to check your estimate?

Discussing the Ideas

1. The standard unit of length in the metric system is a **metre** (m). A **decimetre** is one-tenth as long as a metre.

   Cut a strip of paper or string to be one metre in length. Use the decimetre above to help you.

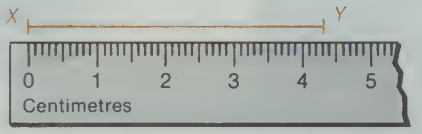
2. **A** About how many decimetres long is one of your arms?
   B What part of a metre is the length of one of your arms?

3. The table of prefixes will help you understand other units in the metric system. Fill in the screens.

PREFIX	MEANING	PREFIX	MEANING
deca-	ten	**deci-**	one tenth
hecto-	one hundred	**centi-**	one hundredth
kilo-	one thousand	**milli-**	one thousandth

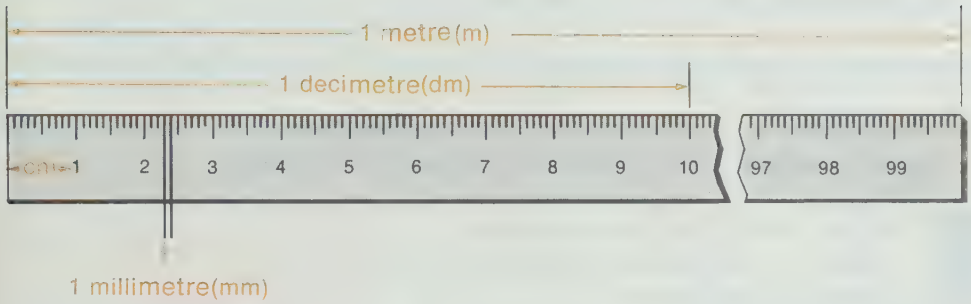
   A 1 kilometre (km) = ▒ m
   B 1 decametre (dam) = ▒ m
   C 1 centimetre (cm) = ▒ m
   D 1 millimetre (mm) = ▒ m
   E 1 decimetre (dm) = ▒ m
   F 1 hectometre (hm) = ▒ m

4. **A** What is the length of $\overline{XY}$ to the nearest whole centimetre?
   B What is the length of $\overline{XY}$ in millimetres?
   C If the length of $\overline{XY}$ in centimetres is 4.3 cm, what is its length in decimetres?
   D What is the length of $\overline{XY}$ in metres?



Using the Ideas

1. Give the missing number for each part.
The diagram may help you.



A 1 m = ▒ cm	E 1 m = ▒ mm	I 80 cm = ▒ dm
B 1 dm = ▒ cm	F 2 cm = ▒ mm	J 23 cm = ▒ mm
C 1 m = ▒ dm	G 30 mm = ▒ cm	K 7 m = ▒ cm
D 1 dm = ▒ mm	H 500 cm = ▒ m	L 1000 cm = ▒ m

2. Give the length of each segment in millimetres.
Then give the length in centimetres.

A ⊢——————⊣ C ⊢——————⊣

B ⊢————————⊣ D ⊢————⊣

3. Draw segments having these lengths.

A 9.4 cm	C 60 mm	E 39 mm	G 0.5 dm
B 7.5 cm	D 1.2 dm	F 0.02 m	H 0.083 m

4. Express each measure in metres.

EXAMPLE: 2 m 3 dm 8 cm = 2.38 m

A 5 m 6 dm 2 cm	D 67 cm	G 1 dam 8 m 7 dm
B 1 m 3 dm 4 cm	E 23 cm 8 mm	H 1 km 7 hm 5 dam
C 8 dm 5 cm	F 2 m 2 dm 2 cm 2 mm	I 1 km 5 cm 3 mm

5. A kilometre (km) is 1000 metres. Give the missing numbers for each part.

A 1 km = ▒ hm	C 1 km = ▒ m	E 1 km = ▒ cm
B 1 km = ▒ dam	D 1 km = ▒ dm	F 1 km = ▒ mm

UTILIZATION

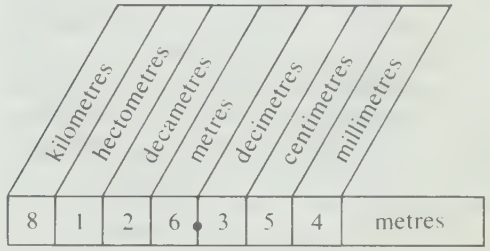
Students will need rulers scaled in centimetres for Exercise 2.

In practical applications of length, the most common units are kilometre, metre, centimetre, and millimetre. The other units, decimetre, decametre, and hectometre, are used less frequently. However, an understanding of these units will help the student understand the decimal nature of the metric system.

EXTENSION

See *Workbook* page 85 for further practice exercises.

Remedial: Write "8126.354 metres" on the chalkboard and then use a diagram like the one below to show how each digit in the numeral for the measure refers to the various metric units.



Enrichment: Measure a distance of 20 metres in a hallway (or a greater distance outside). Have students use a stopwatch to find the amount of time it takes them to walk the measured distance. Then have them compute the amount of time it would take them to walk a distance of 1 kilometre.

Assignments

Minimum 1–3. Average 1–4. Maximum 1–5.

Objective

The student can find lengths of polygonal paths and perimeters of polygons.

PREPARATION

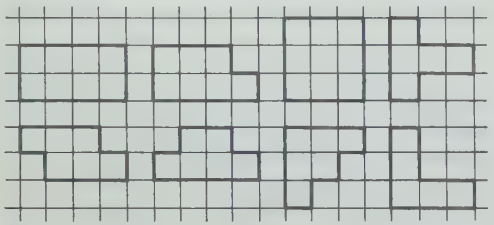
Materials: Centimetre rulers; geoboards, dot paper, or graph paper. (*Duplicator Masters*, pages 82, 83).

Most students will be familiar with the concept of perimeter of a figure as "the distance around" the figure. Therefore you might draw some polygons on the board and ask, "What is meant by the perimeter of this polygon?" Then have someone show how they could find the perimeter.

INVESTIGATION

Discuss the meaning of *polygonal path* and the length of the polygonal path. Have the students measure each segment of polygonal path *ABCDEF* with their centimetre rulers and compute the total length of the path.

You might limit students to about 15 minutes for carrying out the Investigation activity. Then, if time permits, collect and combine the drawings of the different shaped regions with perimeter 12 units that the students have found. A few sample regions are shown below.



DISCUSSION

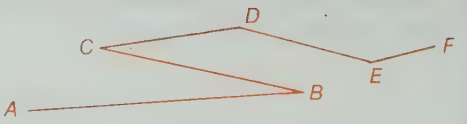
In Exercise 2, two equivalent ways of finding the perimeter might be suggested. First, double the length and double the width and then find the sum. Secondly, add the length and width together and then multiply the sum by 2.

Exercises 3 and 4 can be generalized:

If a regular polygon of n sides has each side of length s, the perimeter P is given by the formula $P = n \cdot s$.

Polygonal Paths and Perimeter

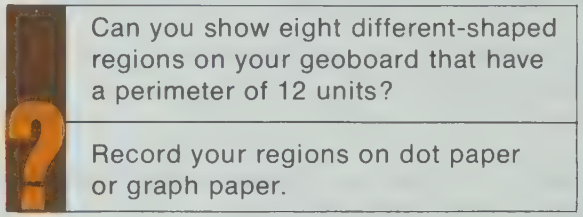
Investigating the Ideas



The **polygonal path** *ABCDEF* is the union of segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, and $\overline{EF}$.

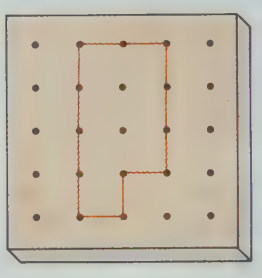
Its length is: length $\overline{AB}$ + length $\overline{BC}$ + length $\overline{CD}$ + length $\overline{DE}$ + length $\overline{EF}$.

If a region has a boundary which is a polygonal path, then the **perimeter** of the region is the length of the path.



Can you show eight different-shaped regions on your geoboard that have a perimeter of 12 units?

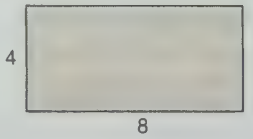
Record your regions on dot paper or graph paper.



Discussing the Ideas

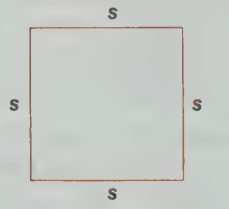
1. What is the length of the polygonal path *ABCDEF* in the Investigation? Use your centimetre ruler.

2. Explain how to find the perimeter of this rectangular region.



3. If the length of one side of a square is s units, then the perimeter P is given by the formula

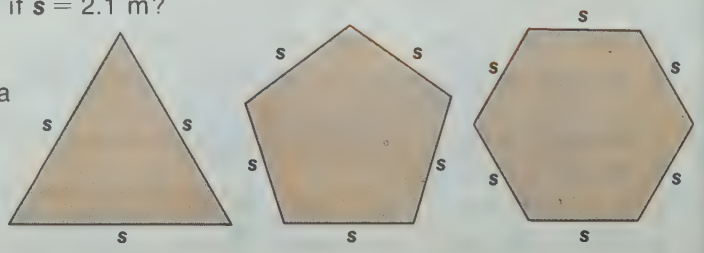
$$P = s + s + s + s = 4 \cdot s$$



   A What is the perimeter if $s = 7$ cm?

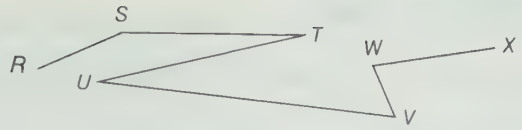
   B What is the perimeter if $s = 2.1$ m?

4. What would be a formula for the perimeter of each of the figures at the right? Each have sides with length s.

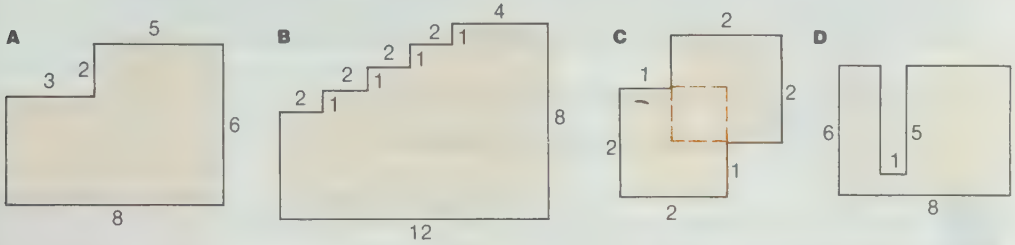


Using the Ideas

1. Find the length of the polygonal path *RSTUVWX*.

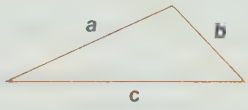


2. What is the perimeter of each region?



3. The formula for the perimeter of a triangle is

$$P = a + b + c$$



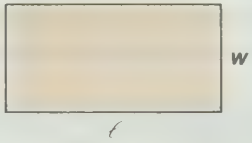
where a, b, and c are the lengths of the sides of the triangle. Use the formula to find the missing part in each row of the table.

	a	b	c	P
A	16	9	24	
B	$24\frac{1}{4}$	$19\frac{1}{2}$		67
C	23.7		31.3	78.4

4. A formula for the perimeter of the rectangular region would be

$P =$ ▮ $\cdot \ell +$ ▮ $\cdot w$ or $P =$ ▮ $\cdot (\ell + w)$

Complete the sentence by giving the number for the screens.

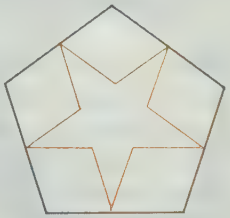


5. Use the formula for the perimeter of a rectangle to find the missing numbers in each part of the table.

	ℓ	w	P
A	12	7	
B	4.3	2.9	
★ C		31	200

think

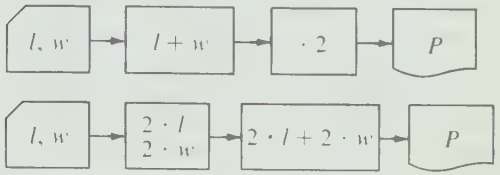
What is the perimeter of the large region? What is the perimeter of the star?



★ **6.** A rectangle has a perimeter of 24 centimetres. What could be the length and width of the sides if each side is a whole number of centimetres?

UTILIZATION

Formulas for perimeters of triangles and rectangles are presented in Exercises 3 and 4. Of course, it is the distributive principle that proves that the two formulas are the same, but flow charts might effectively be used to help students understand more clearly that the two formulas are equivalent.



Solution, Exercise 6

The sum of the length and width must be $\frac{1}{2}P$, or 12. Therefore any two whole numbers whose sum is 12 could be the length and width.

EXTENSION

Appropriate additional practice exercises are provided on page 86 of the *Workbook*.

Enrichment: Ask students whether they can show how to arrange 9 unit squares that do not overlap to form (1) a region with the smallest possible perimeter, and (2) a region with the greatest possible perimeter.

Solutions

9 unit squares
$P = 12$ units

9 unit squares
$P = 36$ units

Think Solution

Although the area of the star is considerably less than that of the pentagon, the two regions have the same perimeter. Note that the star can be formed by "pushing" each vertex of the pentagon inward.

Assignments

Minimum 1–4, 5A and B.
Average 1–5. Maximum 1–6.

Objective

The student can state the greatest possible error for a given measure of length.

PREPARATION

Materials: Centimetre rulers, metre rulers.

All actual measurement of length is subject to error and, in a certain sense, is always inexact. It is inexact because of the limitations of the devices we use for measuring. What is important is that when reporting the measure of some object we are able to state that the error in the measurement will not exceed a fixed amount—the greatest possible error. This greatest possible error is always $\frac{1}{2}$ of the smallest unit used in the measurement process.

INVESTIGATION

The measurement activities of the Investigation are designed to give students experience in rounding measurements to the nearest whole unit. Thus if some students find that the length of their desk top is between 7 and 8 dm, they must decide whether it is nearer to 7 dm or to 8 dm. In any case, the error will not exceed $\frac{1}{2}$ dm, or 5 cm.

DISCUSSION

The main point to be emphasized in the Discussion is that very few measurements are found to be exactly a whole number of units, even when the unit is very small. However, the measurement can be given as being nearer some number than another. Use Exercise 2 to help establish these ideas by asking:

"Between what two whole numbers is the thickness of the book?"

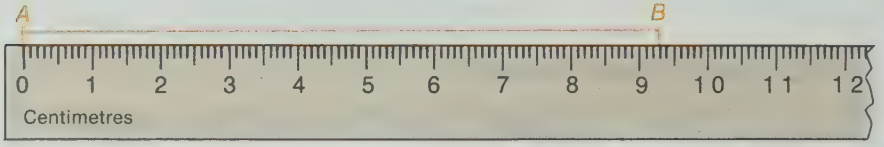
"Is the thickness nearer to 2 cm or to 3 cm?"

"If the thickness is reported as 3 cm, will the error be more than $\frac{1}{2}$ cm?"

These questions should aid the student in understanding why the greatest possible error for this exercise is 0.5 cm.

Investigating the Ideas

In measurement, the number that you get depends upon the unit you choose.



The length of $\overline{AB}$ **to the nearest decimetre** is 1.

The length of $\overline{AB}$ **to the nearest centimetre** is 9.

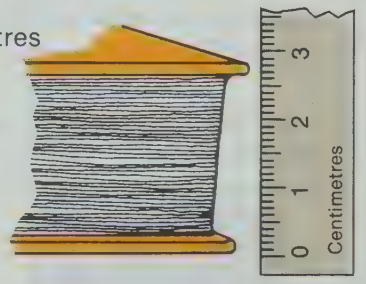
Can you measure one or more of the following objects and record their measures to the nearest unit?

- **A** The length of your desk top in decimetres.
- **B** The length of your room in metres.
- **C** The length of a book in centimetres.

Discussing the Ideas

1. Did any of the measurements that you made come out to exactly a whole number of units?
   How did you decide which number of units to record if the length was between two whole units?

2. Suppose a student said that a book was 3 centimetres thick, **to the nearest centimetre**.
   - **A** Could it be as thin as 2.1 centimetres?
   - **B** Could it be 2.8 centimetres?
   - **C** Could it be as much as 3.7 centimetres thick?



3. When a measurement is reported as 3 centimetres, to the nearest centimetre, the actual length must be between 2.5 centimetres and 3.5 centimetres.
   The **greatest possible error** is 0.5 centimetres.
   The greatest possible error of any measurement is half of the smallest unit of measurement used.

   What is the greatest possible error in each of the measurements you made in the Investigation?

Using the Ideas

1. For each reported measurement, give the largest and the smallest lengths possible for the actual length.

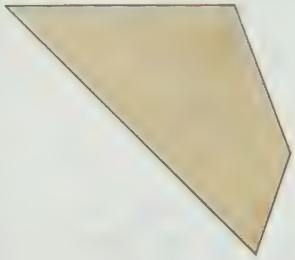
EXAMPLE: 8 cm
Answer: 8.5 cm, 7.5 cm

A 4 km **B** 19 m **C** 37 cm **D** 19 mm **E** 12 cm **F** 154 dm

2. **A** Measure each side of the figure to the nearest whole centimetre. Add the lengths to find the perimeter.

B Measure each side of the figure to the nearest millimetre and then add these lengths to find the perimeter.

C What is the difference in the perimeter for **A** and **B** ?



3. The **relative error** of a measurement is found by dividing the greatest possible error by the reported measurement. When you compare two measurements, the one with the smaller relative error is said to be **more precise**.

EXAMPLE: Reported measure: 10 centimetres
Greatest possible error: 0.5 centimetre
Relative error: $0.5 \div 10 = 0.05$ That is, the error may be as much as 5 parts in 100 or 1 part in 20.

Find the relative error of each measurement.

A 4 cm **B** 30 m **C** 25 mm **D** 18 km

Which is the most precise measurement?

4. Astronomers give the average distance from Jupiter to the sun as 773 million kilometres to the nearest million kilometres. Could the actual distance be as much as 773 700 000 kilometres? Why or why not?

5. An astronomer says that it is about 12 light years from Earth to a certain star. Taking a light year to be 9.4 trillion kilometres, how much could the actual distance (in kilometres) differ from the astronomer's figure?

UTILIZATION

Most students will probably gain more from an oral discussion of these exercises than from independent work. However, if you prefer to have them work independently, allow some time for discussion after they have completed the exercises.

The terms *relative error* and *more precise* are introduced in Exercise 3. These terms should not be overemphasized at this time. It is more important to be sure that your students understand the basic ideas involved than it is to have them attempt to memorize these specific terms and rules.

EXTENSION

Duplicator Masters page 58 provides appropriate exercises for further practice.

Enrichment: Have students work together in small groups, measuring as carefully as they can, to determine the perimeter of the classroom. Ask each group to record their measure of the perimeter on the chalkboard. When all groups have completed their measurements, compare the results and discuss ways in which the measure might be in error and the greatest possible error. Some students may suggest finding the average of all the measures in order to arrive at the "best" estimate of the actual perimeter.

Assignments

Minimum 1–3. Average 1–3. Maximum 1–6.

Objective

The student can find the area of a region by counting or estimating square units.

PREPARATION

Materials: Centimetre graph paper (*Duplicator Masters*, page 83).

The concept of area is probably familiar to most of your students. In this lesson, students determine the area of a region by counting unit squares or by estimating.

INVESTIGATION

Provide centimetre graph paper from which students can cut out a square decimetre. This region can be used by students to estimate the area of their desk tops. Some students will measure the length and width and compute the product to find the area.

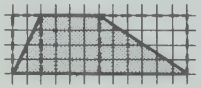
DISCUSSION

The word *area* is introduced in the first Discussion Exercise. Encourage students to estimate areas of a variety of plane regions.

For Exercise 2, have students use the square decimetre cut out for the Investigation. If the square was cut from centimetre graph paper, the relation between a square centimetre and a square decimetre will be obvious.

In Exercise 3, discuss ways of finding the area of the parallelogram and the trapezoid without having to count partial squares. We can find that the parallelogram has the same area as the rectangle because we can think of cutting off the triangular section indicated by the dashed segments and "gluing" it onto the other side of the parallelogram to form the rectangle.

Point out that the triangular portions of the trapezoid are just half of the rectangular regions indicated by the dashed segments.

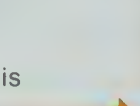


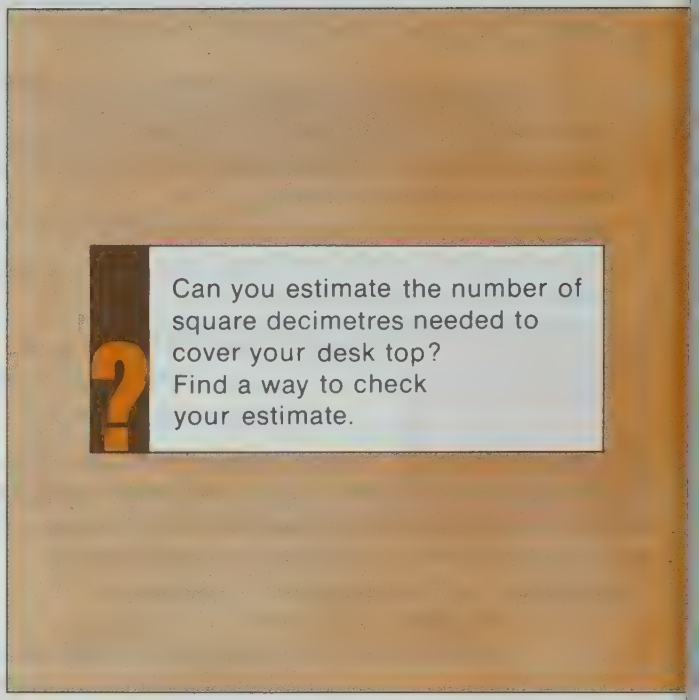
Units of Area

Investigating the Ideas

This region is one square centimetre.

This region is one square decimetre.



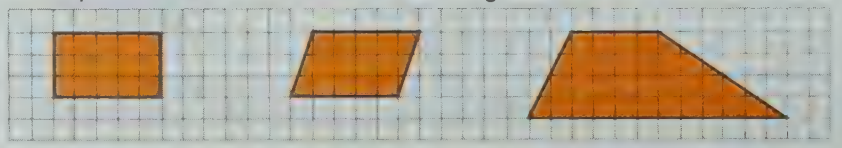


Discussing the Ideas

1. The number of square decimetres, or parts thereof, needed to cover your desk top is the **area** of the desk top. Estimate the area of some other regions using a square decimetre as the unit.

2. Think of a square one metre on each side.
   - A What is the area of the square metre if the square decimetre is the unit?
   - B What is the area if the square centimetre is the unit?

3. Unit squares are indicated on the grid. Count the squares to find the area of each figure.



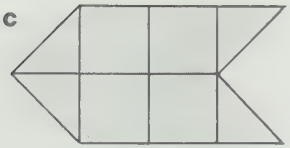
Using the Ideas

1. Each small square is a square centimetre.
What is the area of each region?

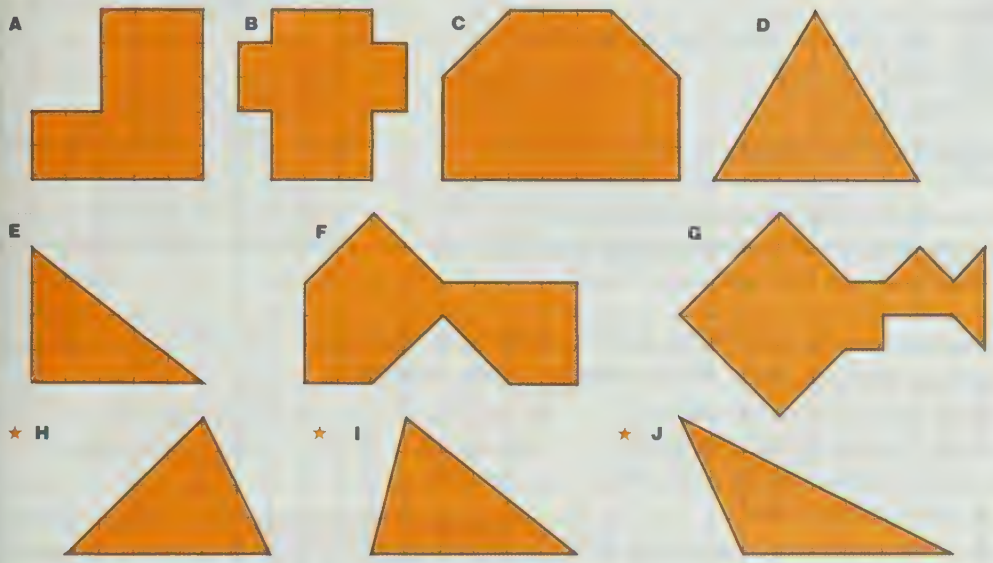
A

B

C



2. Find the area of each shaded region.
Unit squares are indicated by the grid.



3. An **are** (pronounced "air") is a unit of area. A square ten metres on each side has an area of one are.
How many square metres are there in one **are**?

4. A **hectare** is 10 000 square metres. How many ares are there in an hectare?

5. How many hectares are in one square kilometre?

★ 6. Suppose each small square represents an area of 125 000 square kilometres.
Estimate the total area of the country.

UTILIZATION

Exercises 1 and 2 involve finding area by counting. Some students may use formulas for area of rectangles or triangles. However, these will be covered in the following lesson, so there is no need to emphasize them here.

Exercises 3, 4, and 5 introduce metric units of area. The relation between the various metric units of area is given below.

1 are = 100 m^2
1 hectare = 100 ares = 10 000 M^2
1 km^2 = 100 hectares = 1 000 000 m^2

EXTENSION

Enrichment: To help students gain a clearer idea of the size of units of metric area, find a suitably large area in the gym or outside and mark out 1 m^2 and 1 *are*, and (if enough space is available) 1 hectare. Alternatively, you can have students find the number of *ares* in the floor of their math classrooms.

Assignment

Minimum 1, 2A–G. Average 1, 2A–G, 3, 4. Maximum 1–6.

Objective

The student can find area of rectangles and parallelograms by using area formulas.

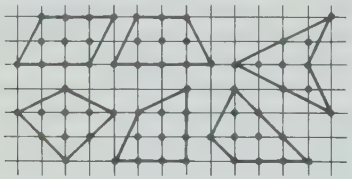
PREPARATION

Materials: Geoboards, dot paper or graph paper (*Duplicator Masters*, pages 82, 83).

Review the concept of area by showing a few simple figures on the geoboard or overhead projector and asking students to find the area by counting unit squares.

INVESTIGATION

The purpose of the Investigation is to allow students to discover that several different quadrilaterals all with area of 6 square units can be found on the geoboard. The regions below represent some possibilities.



DISCUSSION

During the Discussion, emphasize the formulas for area of a rectangle and area of a parallelogram. (Area of rectangle: $A = \ell \cdot w$; area of parallelogram: $A = b \cdot h$)

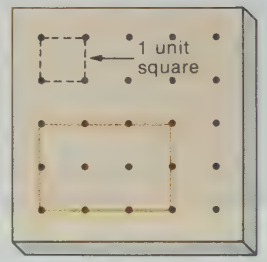
Another important point to be observed is that area is preserved under cutting and reassembling of the object to be measured.

Thus, if the area of region S is a given number and if region S is divided into two smaller regions, S_1 and S_2, and then put back together differently, the area remains the same. This is an important principle for developing the concept of area for various polygonal regions. In particular, the development of the formula for the area of any parallelogram in Exercise 4 depends upon this fact.

Finding Area of Rectangles and Parallelograms

Investigating the Ideas

Suppose each of the smallest squares on the geoboard is a unit square. Then the rectangle shown has an area of 6 square units.

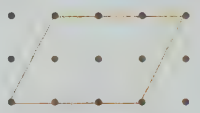


Can you find some different-shaped quadrilaterals that also have an area of 6 square units?

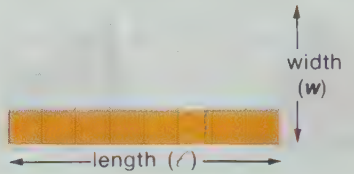
Record your regions on dot paper or graph paper.

Discussing the Ideas

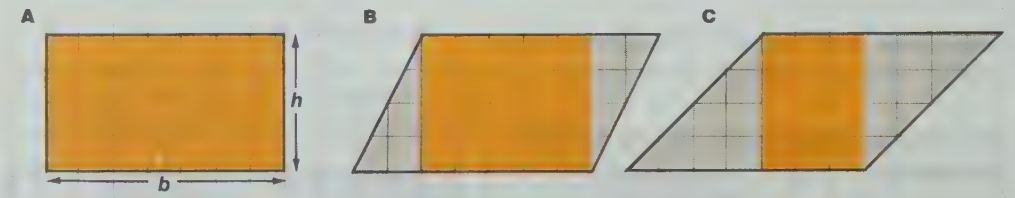
1. Would this parallelogram have the same area as the rectangle above? How can you decide?



2. Sometimes we say, "the area of a rectangle," when we really mean the **area of the interior of the rectangle**.
   - **A** In the rectangular region at the right, how many unit squares in the bottom row?
   - **B** How many rows?
   - **C** How many unit squares in all?
   - **D** Can you write a formula for the area of any rectangle with length ℓ and width w?



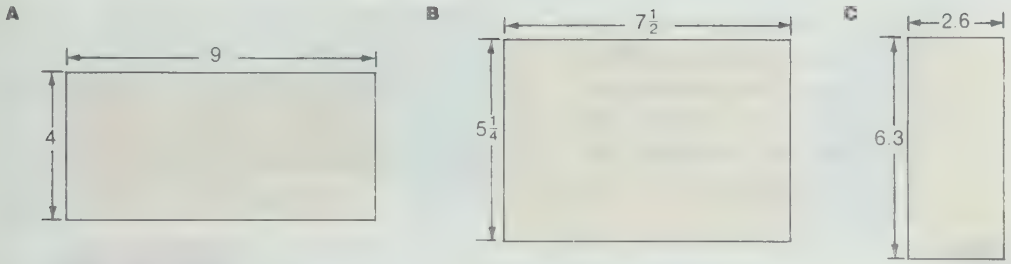
3. Can you find the area of each parallelogram?



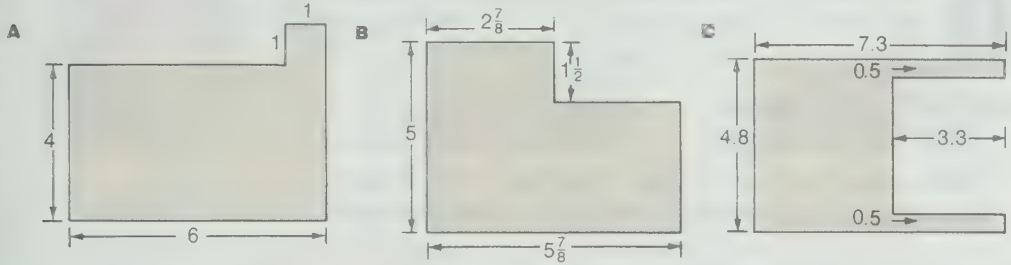
4. In Exercise 3, if b is the length of the **base** and h is the **height** of each of the parallelograms, can you write a formula for the area of a parallelogram?

Using the Ideas

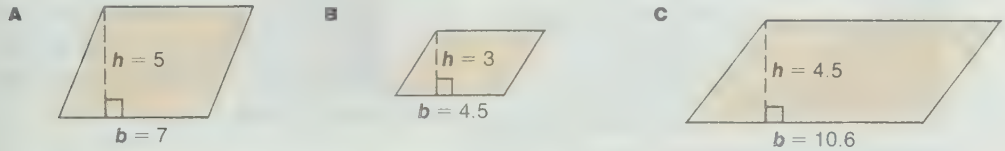
1. Find the area of each rectangle.



2. Find the area of each region.



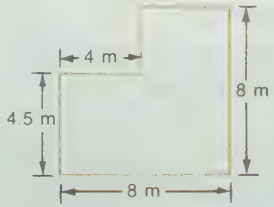
3. Use the formula $A = b \cdot h$ to find the area of each parallelogram.



4. Find the area of the floor of a rectangular room that is 4.5 metres long and 3.4 metres wide.

5. What would it cost to carpet the room in Exercise 4 if carpeting sells for $14.95 per square metre?

6. A certain rug cleaner charges $1.75 per square metre to clean rugs. How much would be charged to clean the wall-to-wall carpeting for the room shown at the right?



★7. A certain brand of tiling is available in squares that are 25 centimetres on each side and cost 29¢ each. About how much would it cost to tile a floor 2.8 metres by 3.4 metres?

More practice, page S-25, Set 42

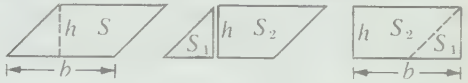
UTILIZATION

Exercises 1, 2, and 3 give practice in applying the area formulas. Exercises 4 through 7 deal with practical applications of area measured in metric units.

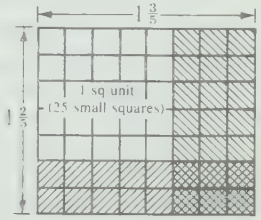
EXTENSION

To provide further practice, make selective assignments from Supplementary Exercise Set 42 on page S-25, *Workbook* page 87, *Duplicator Masters* page 59, and *Arithmetic Skill Cards* AP-7 and AP-8.

Remedial: Have students draw a parallelogram (region S), cut off a triangular region (S_1), and reassemble the pieces S_1 and S_2 to form a rectangle.



Enrichment: Some students may be interested in learning why the area formula for rectangles, $A = \ell w$, holds not only for whole numbers but for fractional numbers as well. A discussion of a diagram such as the one below could be used.



1 sq unit → 25 small squares

Each small square is $\frac{1}{25}$ of a square unit. $56 \cdot \frac{1}{25} = 2\frac{6}{25}$

→ 15 small squares

→ 10 small squares

Area = $2\frac{6}{25}$ square units

Notice that $1\frac{3}{5} \cdot 1\frac{2}{5} = \frac{8}{5} \cdot \frac{7}{5} = \frac{56}{25} = 2\frac{6}{25}$

→ 6 small squares

Total area → 56 small squares

Assignments

Minimum 1A–C, 3. Average 1–5.
Maximum 1–7.

Objective

The student can use the area formula, $A = \frac{1}{2} \cdot b \cdot h$, to find the area of any triangle.

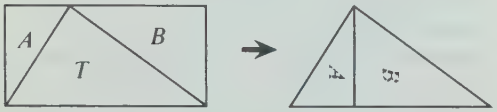
PREPARATION

Materials: Paper and scissors.

Review the method of finding area of rectangles and parallelograms. Understanding of the area of these figures is basic to the development in this lesson.

INVESTIGATION

The purpose of the Investigation is to show that any triangle whose base is the same as that of a rectangle and which has the same height as the rectangle will cover $\frac{1}{2}$ the area of the rectangle. When pieces *A* and *B* are rotated 180°, they exactly cover triangle *T*. Therefore the triangle covers $\frac{1}{2}$ the area of the rectangle.



DISCUSSION

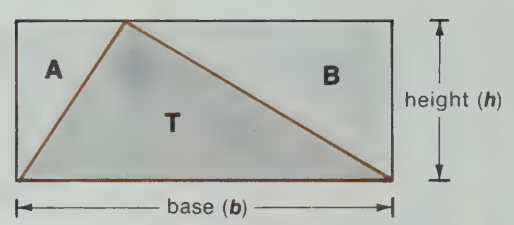
Exercise 2 reinforces and extends the ideas of the Investigation. Since the area of rectangle *ADBE* is 12 square units, the area of $\triangle ABD$ must be half as much, or 6 square units. Similarly, $\triangle BDC$ has half the area of rectangle *BDCF*, or 16 square units. Together the two triangles have an area of 22 square units, which is exactly $\frac{1}{2}$ the area of rectangle *ACFE*. Hence the area of the triangle can be found by taking $\frac{1}{2}$ the area of the rectangle. This justifies the area formula for triangles: $A = \frac{1}{2} \cdot b \cdot h$.

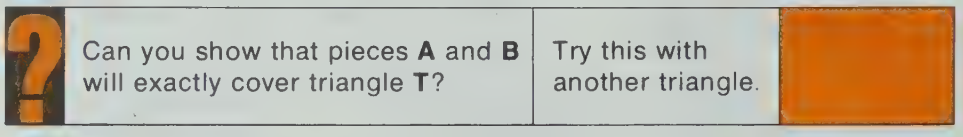
In the discussion of Exercise 3, make sure students understand that the height of a right triangle is the same as the length of one of the legs of the triangle.

Finding Area of Triangles

Investigating the Ideas

Draw a rectangle of any size. Then show a triangle with its base along one side of the rectangle and its height the same as the rectangle. Cut out the rectangle and pieces **A**, **B**, and **T**.

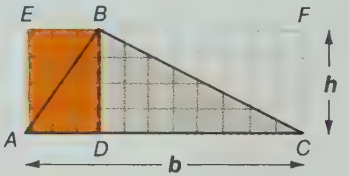




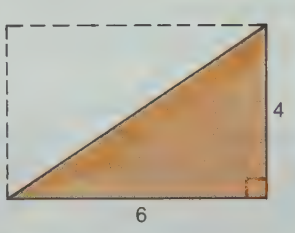
Discussing the Ideas

1. What does the Investigation show about the area of triangle **T** as compared to the area of the rectangle?

2. **A** What is the area of rectangle *ADBE*?
   B What is the area of $\triangle ABD$?
   C What is the area of rectangle *BDCF*?
   D What is the area of $\triangle BDC$?
   E What is the area of $\triangle ABC$?
   F Explain how the figure shows that the formula for the area of a triangle is $A = \frac{1}{2} \cdot b \cdot h$ where ***b*** is the base and ***h*** is the altitude.

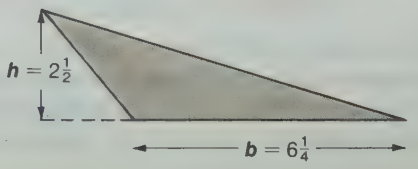


3. **A** What is the height of the right triangle?
   B How can you find its area?
   C What is its area?



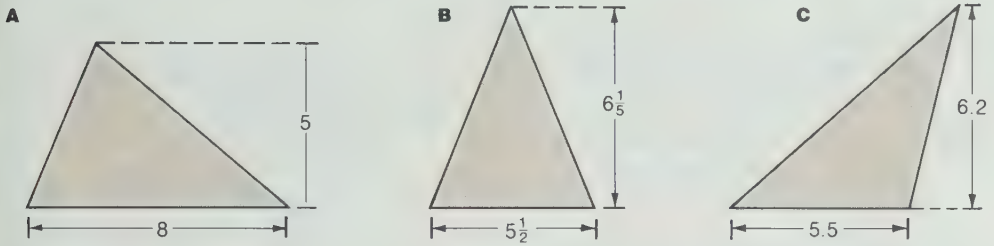
4. Complete the computation to find the area of this triangle.

$$A = \frac{1}{2} \cdot b \cdot h$$
$$= \frac{1}{2} \cdot 6\frac{1}{4} \cdot 2\frac{1}{2}$$
$$= \frac{1}{2} \cdot \frac{25}{4} \cdot \frac{5}{2} = \text{[redacted]} \text{ square units}$$

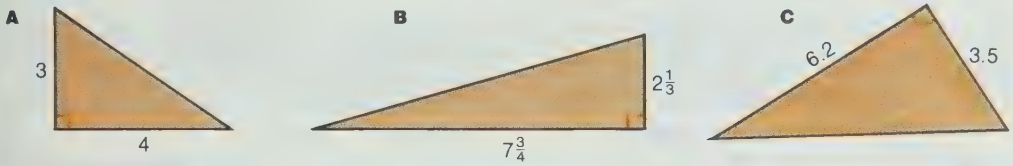


Using the Ideas

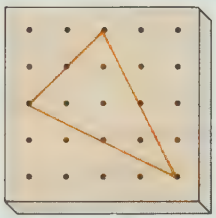
1. Find the area of each triangular region. Use the formula $A = \frac{1}{2} \cdot b \cdot h$.



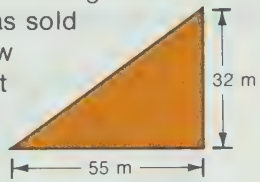
2. Find the area of each right triangular region.



3. What is the area of the largest triangle that can be formed on a 5 by 5 nail geoboard?



4. A triangular lot with a base of 55 metres and a height of 32 metres was sold for $15 960. How much did the lot cost per square metre?



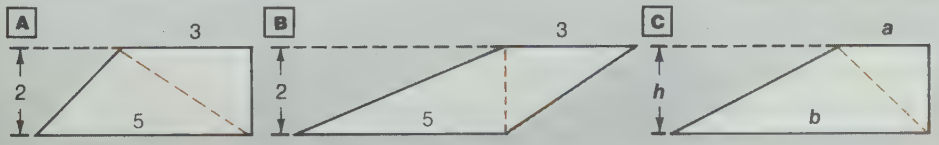
5. Find the areas of the triangles with the following dimensions:

A $b = 240$ mm, $h = 315$ mm
B $b = 3$ m, $h = 4\frac{1}{2}$ m
C $b = 4$ m, $h = 200$ cm
D $b = 6.3$ dm, $h = 28$ cm

6. Design a flow chart for finding the area of a triangle.

think

Find the area of each region.



More practice, page S-25, Set 43

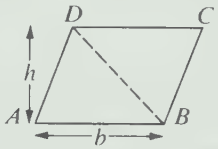
UTILIZATION

Exercises 1 and 2 are suitable for all students. In Exercise 2, however, you may need to remind some students that either leg of a right triangle can be the base, while the other leg is the height.

EXTENSION

Workbook page 88, *Duplicator Masters* page 60, and *Arithmetic Skill Card* AP-9 are recommended for use with students who would benefit from further practice with the concepts of this lesson.

Remedial: Have students try the following problem.



Cut parallelogram $ABCD$ along one diagonal and show that $\triangle ABD$ and $\triangle CDB$ are the same size. (Since the area of the parallelogram is $b \cdot h$, the area of each of the triangles is $\frac{1}{2} \cdot b \cdot h$.)

Enrichment: Show a triangle like this on the geoboard and challenge students to find its area. (*Solution:* $7\frac{1}{2}$ square units.)

Think Solution

These three examples involve finding areas of trapezoids by finding the sum of the areas of two triangles.

A. $\frac{1}{2}(2 \cdot 5) + (2 \cdot 3) = \frac{1}{2} \cdot 2(5 + 3) = 8$
B. $\frac{1}{2}(2 \cdot 5) + \frac{1}{2}(2 \cdot 3) = 5 + 3 = 8$
C. $\frac{1}{2}bh + \frac{1}{2}ah$ or $\frac{1}{2}h(b + a)$

Assignments

Minimum 1, 2. Average 1–5. Maximum 1–6.

Objective

The student can verify the Pythagorean Theorem for particular cases and can use the theorem to find the length of a side of a given right triangle.

PREPARATION

Materials: Scissors, graph paper (*Duplicator Masters*, page 83).

Lead a brief oral review of area of squares. Have students give the area when given the length of one side. Conversely, give the area of some square (making certain that it is a square number such as 100, 49, 64, and so on) and ask students to give the length of one side of the square.

INVESTIGATION

While the Investigation construction is quite general, students will probably find it easier to use graph paper as an aid in drawing the right triangles and the squares. If time permits, suggest that the students try the same activity with other right triangles.

DISCUSSION

At this stage, it is more important to provide students with the factual part of the Pythagorean Theorem (and some basic experiences which give them an opportunity to demonstrate visually the truth of the theorem) than it is to offer a rigorous proof of its validity.

In Exercise 3, little would be gained by interjecting the observation that one can find the length of the hypotenuse by finding the square root of 16. Rather, students should simply reason that the area of the square on the hypotenuse is 16 square units, so the length of the hypotenuse must be 4 units.

The Pythagorean Theorem

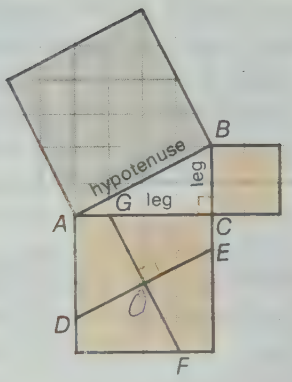
Investigating the Ideas

A Construct right $\triangle ABC$ with a square on each of its sides as shown in the figure.

B Locate the centre of the middle-sized square and call it point O.

C Draw $\overline{DE}$ through point O so that $\overline{DE} \parallel \overline{AB}$. Then draw $\overline{GF} \perp \overline{DE}$. (Four regions are formed in the middle-sized square.)

D Cut out the four regions of the middle-sized square and the small square region.

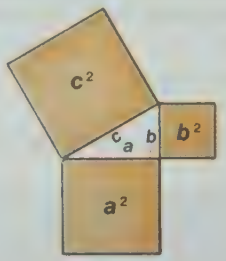


Can you place the five regions of the two smaller squares so that they will exactly cover the large square?

Discussing the Ideas

1. Try the Investigation again using a different-sized triangle for step A. Do the two squares on the legs of the triangle cover the square on the hypotenuse?

2. The Investigation illustrates the **Pythagorean Theorem**.
   In any right triangle, the sum of the areas of the squares on the legs is equal to the area of the square on the hypotenuse.
   If a = length of a leg
   b = length of another leg
   c = length of the hypotenuse
   then $a^2 + b^2 = c^2$



   Can you find three whole numbers a, b, and c, each less than 6, such that $a^2 + b^2 = c^2$?

3. If the area of the square on one leg of a right triangle is 10 square units and the area of the square on the other leg is 6 square units, what is the area of the square on the hypotenuse? How long is the hypotenuse?

Using the Ideas

1. Copy and complete the table below the figures.

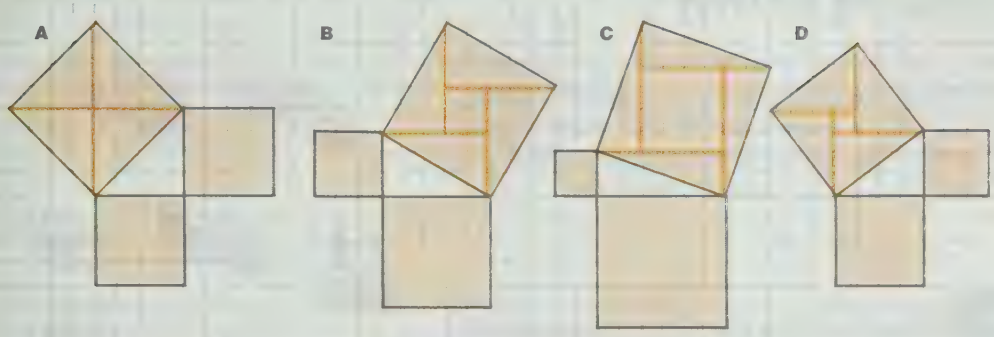


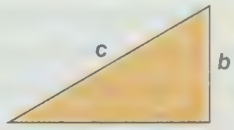
Figure	Area of each of the two small squares		Sum of areas of two small squares	Area of large square
A				
B				
C				
D				

2. Use the formula for the Pythagorean Theorem to complete each part.

   A If $a = 6$ and $b = 8$, then $a^2 + b^2 =$ ▒▒ and $c =$ ▒▒.

   B If $c^2 = 25$ and $b^2 = 16$, then $a^2 =$ ▒▒ and $a =$ ▒▒.

   C If $a = 5$ and $b = 12$, then $c^2 =$ ▒▒ and $c =$ ▒▒.



$a^2 + b^2 = c^2$

3. Complete the table using the Pythagorean Theorem.

	a	b	a^2	b^2	c^2	c
A	3	4				
B	8	6				
C	12					13
D			64		100	
E	12			81		

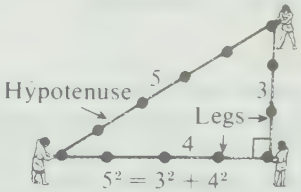
UTILIZATION

In discussing Exercise 1, call attention to the table of areas and the fact that the truth of the Pythagorean Theorem is demonstrated by finding the area of the three square regions and noting that the sum of the areas of the two small regions, in each case, is equal to the area of the large region. Also in Exercise 1, you may want to remind some students that they can find the area of the large square by adding the areas of its interior regions—the four small triangles and (in parts B, C, and D) the small square.

EXTENSION

The exercises on page 89 of the Workbook could be used to provide further practice with the concepts presented in this lesson.

Enrichment: Students might be interested in the legend that the ancient Egyptians, surveying their lands after the annual flooding of the Nile, constructed right angles by stretching ropes to form a 3-by-4-by-5 right triangle.



Suggest to the students that they use a piece of rope or string and make a loop with 12 equally spaced knots around it. Then three students should hold the loop at the appropriate knots to demonstrate how a right triangle is formed.

Assignments

Minimum 1. Average 1, 2. Maximum 1–3.

Objective

The student can find the volume of a rectangular solid by using the formula $V = \ell \cdot w \cdot h$.

PREPARATION

Materials: Small wooden cubes (or sugar cubes), at least 27 per group of students.

Sugar cubes can effectively be substituted for wooden cubes. They can be glued together and sprayed with hair spray or some other fixative for greater durability.

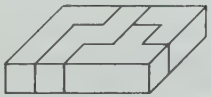
INVESTIGATION

Completion of the Investigation will require several minutes in addition to the time needed to construct the 7 pieces of the Soma puzzle. However, it is a highly motivating puzzle and the time devoted to it will be time well spent.

To answer the Investigation question, students will probably reason that the cube must be 3 units in length on each edge since there are 27 cubes altogether. The real challenge lies in constructing the cube.

DISCUSSION

Use the Soma Cube to motivate a discussion of the meaning of volume. For example, ask: "If one of the small cubes is the unit of volume, what is the volume of each piece?" "Can you arrange the 4 'flat pieces' into a rectangular block that is one unit high?" "What is the volume of such a block?" (15 cubic units)



Exercise 2 is important in that it develops the volume formula for rectangular solids. Be sure to bring out the basic idea that if all of the layers of a given figure are the same, then the area of each layer is the same as the area of the base of the figure, and this area can be multiplied by the height to give the volume.

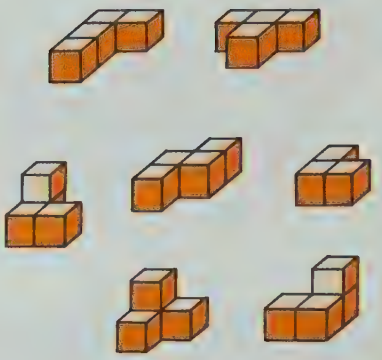
If the base of the figure is a rectangle, the area of the rectangle can be found by multiplying the length times the width, and the volume can be found by using the volume formula: $V = \ell \cdot w \cdot h$.

Finding Volume

Investigating the Ideas

These seven pieces of the **Soma Puzzle** are made from three or four cubes each. When placed together properly, they will make one large Soma Cube.

Can you do it?



Can you find the length of each edge of the Soma Cube?

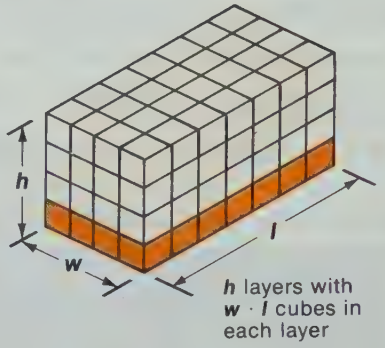
Discussing the Ideas

1. The number of unit cubes that it takes to make a space figure is called the **volume** of the figure.

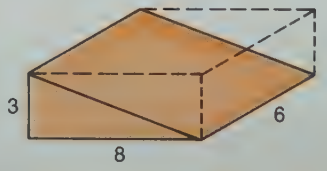
   What is the volume of the Soma Cube if each small cube is a unit of volume?

2. **A** How would you count the number of unit cubes in this rectangular box?
   B How many squares in the base?
   C How many unit cubes in the bottom layer?
   D How many layers?
   E How many unit cubes in all?
   F Explain each formula for the volume of a rectangular box:

   Volume = area of base times altitude
   and $V = l \cdot w \cdot h$



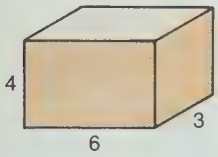
3. **A** What is the volume of the box shown by the dotted segments?
   B How does the volume of the wedge compare to the volume of the box?



Using the Ideas

Find the volume of each space figure using the dimensions given.

1.



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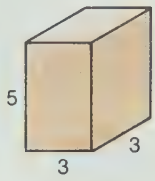

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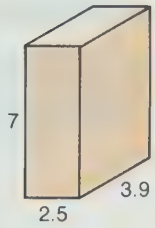

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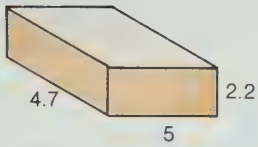

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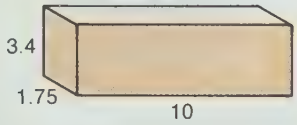

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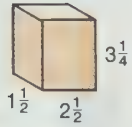

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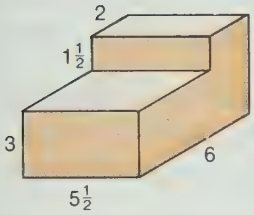

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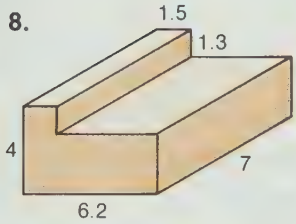

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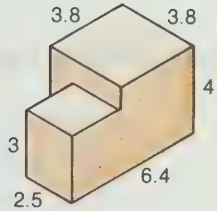

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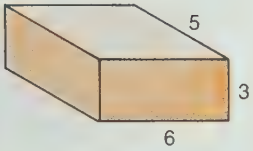

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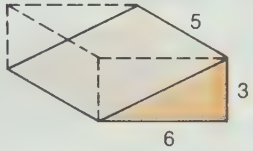

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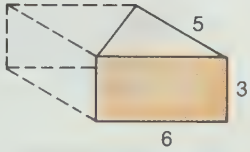
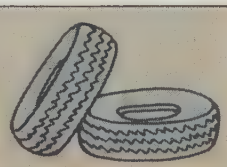


★ **13.** A A box is 5 centimetres long, 4 centimetres wide, and 3 centimetres high. What is its volume?

B If each side were twice as long, how many times greater would the volume of the box be?

think

A man bought five new tires for his car. By rotating the tires, he found that at the end of 40 000 kilometres of driving each tire had been used for the same number of kilometres. How many kilometres was this?



More practice, page S-26, Set 44

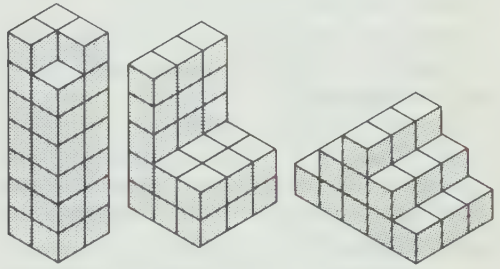
UTILIZATION

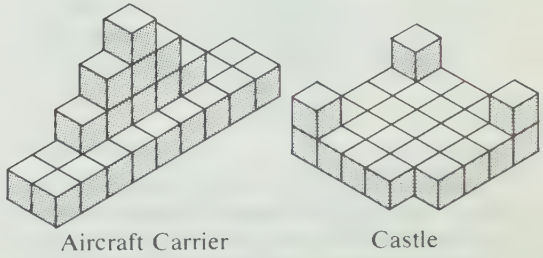
Since there is a considerable amount of arithmetic involved in finding the volumes in these exercises, an assignment of only part of the set may be sufficient for your students.

EXTENSION

Arithmetic Skill Card AP-12 is appropriate for supplemental use with this lesson.

Enrichment: Challenge students to use the Soma Cubes to make a variety of interesting space figures. Some sample figures are shown below.





Think Solution

Since four of the tires must have been on the car at all times during the 40 000 km, the total distance on all five tires was 160 000 km. This gives an average of 32 000 km for each tire.

Assignments

Minimum 1–6. Average 1–9. Maximum 1–13.

Objective

Given a rectangular solid, the student can find the volume and the surface area of the solid.

PREPARATION

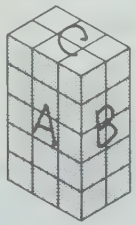
Materials: Models of cubes and rectangular solids.

Review the concept of volume of rectangular solids. If you choose, you might also discuss the meaning of surface area of a figure, even though this is introduced in the Investigation.

INVESTIGATION

Since no new mathematics is involved in the Investigation, students should be able to proceed independently.

If some students have difficulty with space visualization, provide a model of the figure in the Investigation with the unit squares marked on the faces, or provide cubes so that they can build a model.

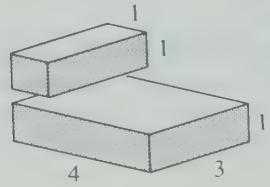


DISCUSSION

Exercise 1 discusses a way of finding the surface area of the Investigation figure. Some of your students may be able to understand a formula for surface area of a rectangular solid: If ℓ, w, and h are the dimensions, then

$$\text{Surface area} = 2(\ell \cdot w + \ell \cdot h + w \cdot h)$$

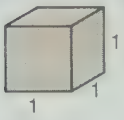
A model of Exercise 4 which can be separated into two rectangular solids may help students see how to compute the volume and surface area of such a figure.



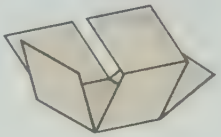
Volume and Surface Area

Investigating the Ideas

The diagrams below show that a cube with a volume of one cubic unit has a **surface area** of 6 square units.

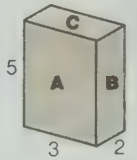


Volume: 1 cubic unit



Surface Area: 6 square units

Can you find the volume and the surface area of the space figure at the right?



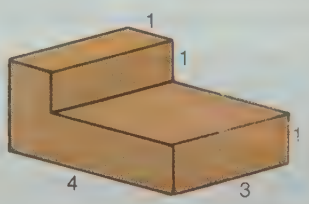
Discussing the Ideas

1. A What is the area of face A in the space figure above?
   B What is the area of face B?
   C What is the area of face C?
   D How can you find the area of all six faces (**total surface area**) of the space figure?

2. A What is the volume of this Soma Puzzle piece?
   B What is its surface area?

3. If you wanted to find the total surface area of the walls, the ceiling, and the floor of your mathematics classroom, what measurements would you make?

4. How would you find the volume and surface area of the space figure at the right?



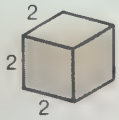
Using the Ideas

1. Find the volume and surface area of each space figure.

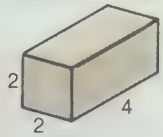
A B C D

2. Find the volume and surface area of each space figure.

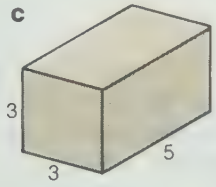
A



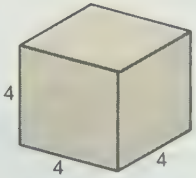
B



C



D

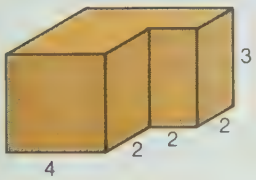


3. A cereal box is 28 centimetres high, 20 centimetres wide, and 9 centimetres thick.
   - A What is the volume of the box?
   - B What is the surface area of the box?

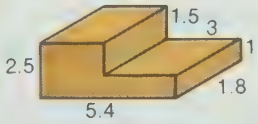
4. A shoe box is 1.7 decimetres by 0.9 decimetres by 3.1 decimetres.
   - A Find the volume of the box.
   - B Find the surface area of the box.

5. Find the volume and surface area of each space figure.

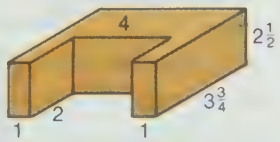
A



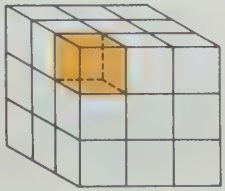
B



C



6. A When the colored cube is removed, the volume is decreased by 1 unit but the surface area is unchanged Explain this fact.
   B What other blocks can you remove from the figure and still keep the same surface area?



think

If a cube has the same number of units of volume as it has surface area, how large is the cube?

More practice, page S-26, Set 45

UTILIZATION

Exercise 1 can be used to determine whether students can discern a relation between the volume and surface area of n cubes placed in a row. Construct a table like the one below and have students complete it for several more cubes.

Volume (No. of cubes)	1	2	3	4				n
Surface area (square units)	6	10	14	18				

For n cubes in a row the surface area will be $4n + 2$ square units.

To aid students in Exercise 6, provide wooden cubes so that they can more readily visualize the problem.

EXTENSION

To provide further practice, make selective assignments from *Workbook* page 90, *Duplicator Masters* page 61, and *Arithmetic Skill Cards* AP-11 and AP-12.

Enrichment: Suggest to students that they find the surface area of some of the space figures made from the Soma Cubes in the last lesson.

Think Solution

Students can solve this problem by trial and error. However, an algebraic solution is possible:

Let x = length of one edge of the cube.
$6x^2$ = surface area
x^3 = Volume
If $x^3 = 6x^2$, then dividing both members by x^2, we have $x = 6$.
Hence, surface area $= 6 \cdot 6^2 = 216$ sq units, and Volume $= 6^3 = 216$ cubic units.

Assignments

Minimum 1–3. Average 1–4. Maximum 1–6.

Objective

The student can express the capacity of appropriate containers in liters or in subunits or multiples of litres.

PREPARATION

Materials: Lightweight posterboard, scissors, centimetre rulers.

Review the meaning of the metric prefixes.

deci- one tenth	*deca-* ten
*centi-*one hundredth	*hecto-*one hundred
milli- one thousandth	*kilo-* one thousand

INVESTIGATION

The purpose of the Investigation is to provide a concrete model of a cube whose capacity is 1 cubic decimetre = 1000 cm^3 or 1 litre. For students who have had little work with metric measures and know only the customary English units, this will be a valuable experience. Use posterboard or thin cardboard to make a durable model of the open-top cube.

DISCUSSION

A litre is a standard measure of capacity that is equal to the volume of 1 kilogram of water at 4°C and at standard atmospheric pressure of 760 millimetres of mercury. Since 1 gram of water occupies 1 cm^3, a litre contains 1000 cm^3 and, hence, is equivalent to 1 dm^3.

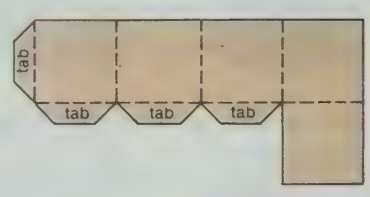
In most scientific work, a capacity of 1 cm^3 is referred to as 1 millilitre (ml), which is $\frac{1}{1000}$ litre.

In discussing Exercises 3 and 4, make certain that students understand the relation between the various units.

Metric Measures of Capacity

Investigating the Ideas

This is a pattern for an open-top cube.



Can you make a model of an open-top cube using a pattern in which each square is one decimetre on each side?

Discussing the Ideas

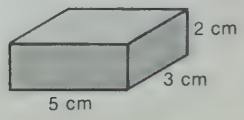
1. What is the volume of your cube in cubic centimetres?

2. If your cube could be filled with water, it would hold one **litre** (ℓ). A litre is a standard measure of **capacity**.
   - A What part of a litre is a millilitre?
   - B How large would a cube need to be in order to hold a millilitre?

3. What is the number for each screen?
   - A 1 kilolitre (kl) = ▒▒ ℓ
   - B 1 decilitre (dl) = ▒▒ ℓ
   - C 1 millilitre (ml) = ▒▒ ℓ
   - D 1 hectolitre (hl) = ▒▒ ℓ
   - E 1 centilitre (cl) = ▒▒ ℓ
   - F 1 decalitre (dal) = ▒▒ ℓ

4. Since 1 cubic centimetre (cm^3) has a capacity of 1 ml = 0.001 ℓ you can express units of volume in terms of capacity. Explain how to find the missing numbers.
   - A 500 cm^3 is a capacity of ▒▒ ℓ.
   - B 2749 cm^3 is a capacity of ▒▒ ℓ.
   - C 72 cm^3 is a capacity of ▒▒ ml.
   - D ▒▒ cm^3 is a capacity of 3.8 ml.

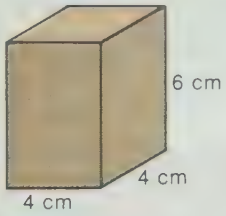
5. What is the capacity in millilitres of this container?



Using the Ideas

1. Find the capacity of each container in millilitres.
(Use the fact that 1 cm^3 is a capacity of 1 ml.)

A



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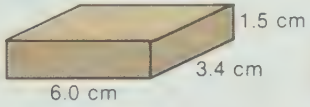
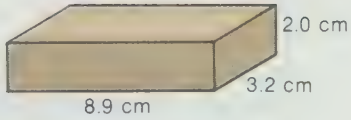
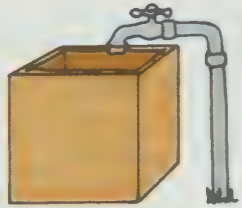


C



2. Express each capacity in Exercise 1 in litres.

3. A tank is in the form of a cube one metre on each edge. How many kilolitres of water will it take to fill the tank?



4. At standard pressure and temperature, **the mass of 1 millilitre of water is 1 gram.** What is the mass of

A 25 ml of water? B 1 ℓ of water? C 1 kl of water?

5. A soft drink bottle has a capacity of 236 millilitres. How many litres in a carton of six bottles?

6. A family uses about three litres of milk per day. If the cost is 29¢ per litre, how much do they spend for milk each week? each year?

7. Gasoline costs 11.9¢ per litre. How much (to the nearest cent) would it cost to fill a tank whose capacity is 60 litres?

think

Suppose that you have two containers, one that will hold 4 litres and one that will hold 9 litres. There are no markings on either container to indicate smaller quantities. How can you measure out 6 litres of water using only these two containers?

UTILIZATION

Exercises 1 through 4 give practice in finding capacity and mass of containers of water.

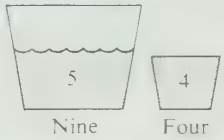
Exercises 5, 6, and 7 are applied word problems involving metric measures of capacity.

EXTENSION

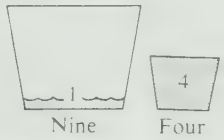
Think Solution

The necessary steps are these:

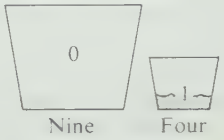
(1) Fill the 9-litre container and then pour 4 litres into the 4-litre container.



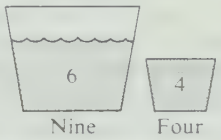
(2) Dump out the 4-litre container and pour 4 litres more from the 9-litre container into the 4-litre container.



(3) Again, dump out the 4-litre container and pour the 1 litre left in the 9-litre container into the 4-litre container.



(4) Refill the 9-litre container and then pour into the 4-litre container until the latter is full. Thus, the 9-litre container will have 6 litres remaining.



Assignments

Minimum 1–4. Average 1–6. Maximum 1–7.

Objective

Given a protractor and a picture of an angle, the student can find the measure of the angle.

PREPARATION

Materials: Protractors, straight edges.

Try to determine how many of your students are familiar with the use of a protractor. Most of them will probably benefit from a review of its use. If a clear plastic protractor is available, demonstrate its use by means of an overhead projector. Emphasize the importance of aligning the protractor properly and reading the measure from the appropriate scale, since many protractors have a double scale.

INVESTIGATION

The measures of $\angle CAB$, $\angle DAB$, $\angle EAB$, and $\angle FAB$ can be read directly from the protractor. There are six other angles, and their measures can be found by subtraction:

$m\angle CAD = 35°$ $m\angle DAE = 50°$
$m\angle CAE = 85°$ $m\angle DAF = 80°$
$m\angle CAF = 115°$ $m\angle EAF = 30°$

DISCUSSION

For Exercise 1, students should observe that the bottom edge of the protractor should be aligned with one of the rays of the angle so that the midpoint of the edge coincides with the vertex of the angle. Then the point at which the other ray intersects the outer edge of the protractor can be found and the angle measure can be read from the appropriate scale.

Exercise 3 brings out the idea that angle measure is additive for adjacent angles.

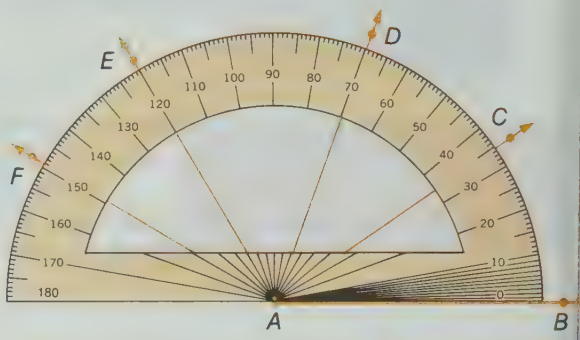
The *bisector* of an angle (Exercise 6) is a ray that divides the angle into two angles that have the same measure. Students should recognize that if they use a protractor to find the measure of a given angle its bisector will be the ray that intersects the outer edge of the protractor at the point halfway between the two rays of the angle.

How Are Angles Measured?

Investigating the Ideas

A familiar unit angle is called a **degree**. The size of a 1-degree angle is such that the measure of a right angle is 90 degrees.

You can measure an angle by using a **protractor** that has degree units marked around a semicircle.



The protractor shows that the measure of $\angle DAB$ is 70 degrees.
We write: $m\angle DAB = 70°$

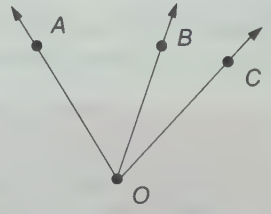
Using the angles shown on the protractor above, how many angles can you name and find the degree measure for?

Discussing the Ideas

1. Draw an angle on your paper or on the chalkboard. Explain how you would use a protractor to find the degree measure of the angle.

2. A degree is what fractional part of a straight angle?

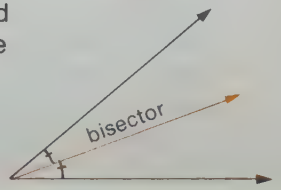
3. Measure each angle.
Does $m\angle AOC = m\angle AOB + m\angle BOC$?



4. Draw an angle and its supplement. Measure each angle, adding their measures. What do you observe?

5. Using your protractor, draw an angle of 45° and an angle of 135°. What can you say about these two angles?

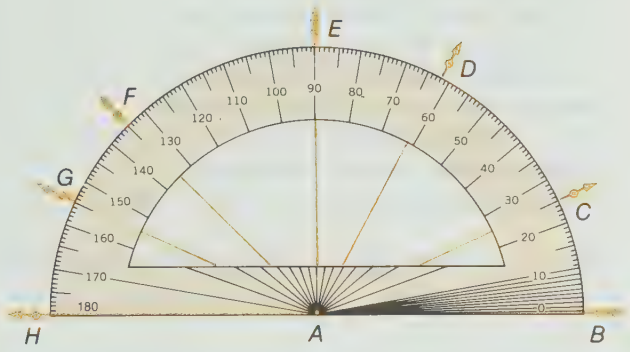
6. Explain how you can use a protractor to draw the **bisector** of an angle.



Using the Ideas

1. Using the picture of the protractor, give the measure of each angle in degrees.

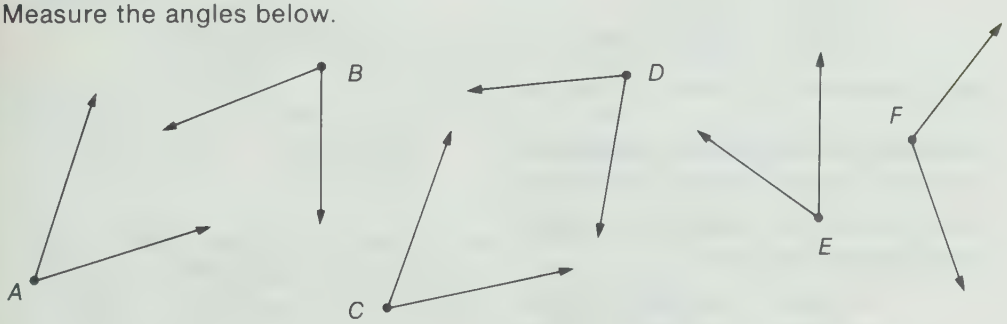
A	$\angle BAC$	L	$\angle DAE$
B	$\angle BAD$	M	$\angle DAF$
C	$\angle BAE$	N	$\angle DAG$
D	$\angle BAF$	O	$\angle DAH$
E	$\angle BAG$	P	$\angle EAF$
F	$\angle BAH$	Q	$\angle EAG$
G	$\angle CAD$	R	$\angle EAH$
H	$\angle CAE$	S	$\angle FAG$
I	$\angle CAF$	T	$\angle FAH$
J	$\angle CAG$	U	$\angle GAH$
K	$\angle CAH$	V	$\angle HAD$



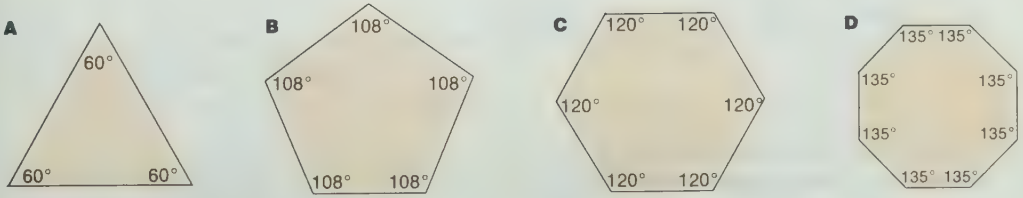
2. Use a protractor to draw angles with the following measures.

A 37° B 65° C 108° D 135° E 29° F 100°

3. Measure the angles below.



4. Use a protractor to help you draw each of these regular polygons. All the sides of each polygon should have the same length.



Equilateral triangle | Regular pentagon | Regular hexagon | Regular octagon

UTILIZATION

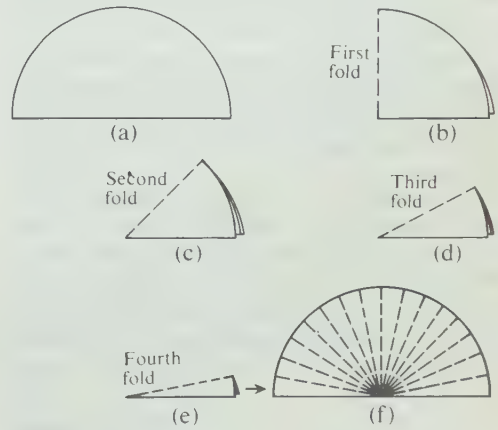
Protractors will be needed for all but the first exercise. In the figure for Exercise 1, the 7 rays form 21 distinct angles. Encourage students to observe that parts O and V refer to the same angle.

When students use their protractors to measure the angles in Exercise 3, you may need to remind some of them that they must think of the rays as extending indefinitely. Thus, they will realize that they can trace the angle from the text and then extend the pictured rays to help determine the angle measure. Note that Exercise 4 is intended primarily for more able students.

EXTENSION

See *Workbook* page 91 for exercises that may be assigned for further practice with the concepts of this lesson.

Enrichment: Have students design their own protractors and use them to measure some angles. A reasonably good protractor can be made by cutting out a semicircle of paper and folding it repeatedly as shown below. The fold lines become the divisions on the protractor. Four folds will result in 16 divisions. The smallest division can be used as the unit angle.



Assignments

Minimum 1, 3. Average 1–3. Maximum 1–4.

Objective

The student can find sums and differences of angle measures when the units are degrees, minutes, and seconds.

PREPARATION

Review angle measure. Point out that only whole degrees were used in the previous lesson. In this lesson smaller units, known as *minutes* and *seconds*, are used.

DISCUSSION

Discuss with the class the subdivision of the unit degree into the smaller units of minutes and seconds. To emphasize the smallness of a minute, ask the students to turn back to the previous lesson and look at the angle of one degree pictured there. Ask your students to imagine that this angle is divided into 60 congruent angles. Each small part is the unit of one minute.

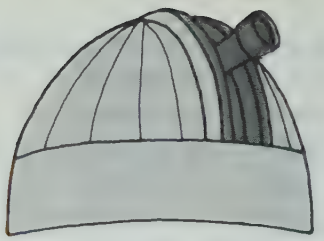
A second is $\frac{1}{60}$ of a minute. To stress the smallness of this unit, you might point out that if an angle has a measure of 1 second, each of its rays would have to be extended over a distance of 2 kilometres before the rays would have an "opening" of one centimetre.

Work through the examples of addition and subtraction, giving special attention to the work involving regrouping of units.

Point out that, in finding the measure of an angle supplementary to an angle whose measure is in degrees, minutes, and seconds, 180° can be rewritten as 179°59′60″ to facilitate subtraction.

Degrees, Minutes, and Seconds

Discussing the Ideas



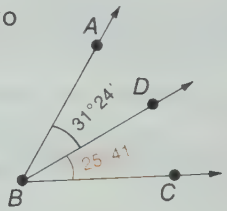
In astronomy, angles are often measured in degrees, minutes, and seconds.

Imagine that an angle of 1° is divided into 60 congruent angles. Each of the 60 small angles is called a **minute** (an angle of 1′). Now think about dividing an angle of 1′ into 60 congruent angles. Each of these smaller angles is called a **second** (an angle of 1″). 60 minutes (60′) = 1° 60 seconds (60″) = 1′

1. What part of a degree is each of the following?

A 30′ **B** 15′ **C** 10′ **D** 6′ **E** 45′ **F** 1′ **G** 30″ **H** 15″

2. The example below will help you understand how to add angle measures. The measure of $\angle ABC$ is the sum of the measures of $\angle ABD$ and $\angle DBC$.

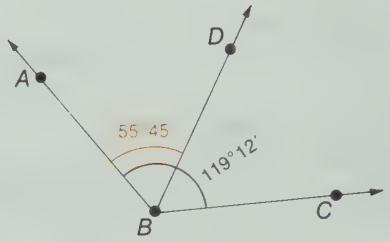


$$\begin{aligned} m\angle ABD &= 31°24' \\ m\angle DBC &= 25°41' \\ \hline m\angle ABC &= 56°65' = 57°5' \end{aligned}$$

What was done to rewrite 56°65′ as 57°5′?

3. When subtracting angle measures, you may have to regroup some units.

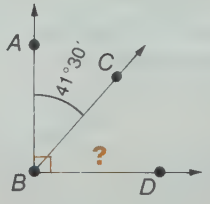
Find the missing number of minutes and then find $m\angle DBC$.



$$\begin{aligned} m\angle ABC &= 119°12' = 118°\square' \\ m\angle ABD &= \ \ 55°45' = \ \ 55°45' \\ \hline m\angle DBC &= \quad ? \quad \square° \square' \end{aligned}$$

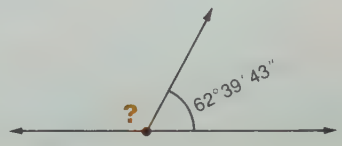
4. Two angles are **complementary** if the sum of their measures is 90°.

If $m\angle ABC = 41°30'$, what is $m\angle CBD$?



5. Two angles are **supplementary** if the sum of their measures is 180°.

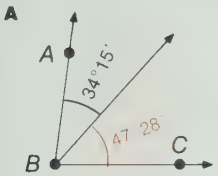
What is the measure of the supplement of an angle whose measure is 63°39′43″?



Using the Ideas

1. Give the measure of $\angle ABC$.

A



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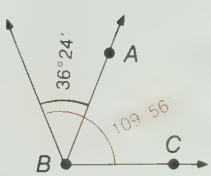

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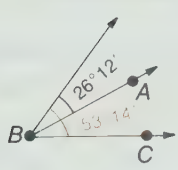
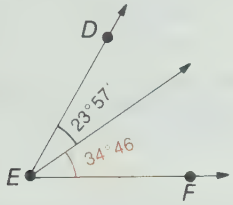


2. Give the measure of $\angle DEF$.

A



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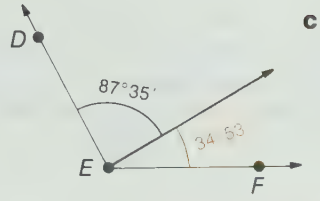
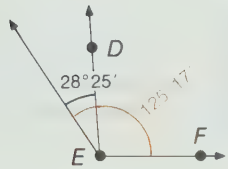


C



3. Find the sums of the angle measures.

EXAMPLE: $37°\ 48'35''$
$15°\ 57'46''$
$52°105'81'' = 52°106'21'' = 53°46'21''$

A $25°15'43''$ / $50°36'10''$

B $86°28'29''$ / $37°21'56''$

C $34°37'48''$ / $120°46'37''$

D $67°46'38''$ / $67°13'22''$

4. Find the **differences** of the angle measures.

EXAMPLE: $37°48'35'' - 15°57'46'' \rightarrow ?$
$37°47'95'' - 15°57'46'' \rightarrow ?\ 49''$
$36°107'95'' - 15°\ 57'46'' \rightarrow 21°\ 50'49''$

A $120°56'43''$ / $15°13'28''$

B $48°53'10''$ / $20°39'26''$

C $78°43'21''$ / $13°44'52''$

D $129°34'10''$ / $29°35'59''$

5. Find the measure of an angle that is complementary to an angle whose measure is

A $38°$ B $27°49'$ C $42°16'38''$ D $58°49''$ E $43'19''$

6. Find the measure of an angle that is supplementary to an angle whose measure is

A $56°$ B $105°20'$ C $84°26'39''$ D $125°58'43''$ E $4°58''$

UTILIZATION

The exercises furnish practice in using the newly introduced units of minutes and seconds in a variety of problems. Some skill in adding and subtracting with the new units should be an expected outcome of the lesson.

It may be helpful to point out that the division of the degree into minutes and seconds is analogous to the division of the hour into minutes and seconds.

Most of the difficulty in this lesson is likely to occur in addition and subtraction problems where "borrowing" or "carrying" seconds, minutes, and degrees is involved. While it is not incorrect to express the measure of an angle as $24°72'66''$, it is more desirable to express it by regrouping the units into the form $25°13'6''$.

EXTENSION

Workbook page 92 is recommended for extending practice with the ideas presented in this lesson.

Enrichment: We inherit the degree unit of measure from the ancient Sumerian civilization, which flourished about 3000 B.C. The Sumerians used a lunar calendar of 12 months, each of which had 30 days. They had to add extra days to the year at the end. If the year were exactly 360 days, the Earth in its annual passage around the sun would pass through 1° per day.

This might be a good time to call attention to Research Project B on page C-65, which concerns another unit of angle measure, the radian.

Other students might be encouraged to construct a metric protractor. Let a straight angle be the unit angle so that the measure of a straight angle is 1; that is, $m\ st\angle = 1$. Then, dividing the straight angle into ten congruent parts gives a deci-angle. Dividing this smaller angle into 10 parts would give a centi-angle.

Assignments

Minimum 1, 2. Average 1–4. Maximum 1–6.

Objective

The student can use the fact that the sum of the angle measures of a triangle is 180° to find measures of angles in triangles and other polygons.

PREPARATION

Materials: Protractor, scissors, straightedge.

This lesson extends the work of the previous lesson. You might review angle measure briefly before proceeding with the Investigation.

INVESTIGATION

The Investigation gives the student an opportunity to discover the geometric theorem that the sum of the measures of the angles of every triangle is 180°. Some students may already be familiar with this fact. Those who are not may obtain results that vary from 180° by 1 or 2 degrees because of inaccurate measurement. Record the angle sums on the chalkboard so that everyone can see that the results are "very close to 180°."

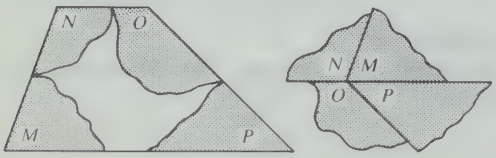
DISCUSSION

Exercise 1 illustrates an intuitive kind of proof of the angle sum theorem. The formal proof is based on the fact that alternate interior angles formed by a transversal of 2 parallel lines are congruent angles.

Another intuitive demonstration is illustrated below by means of paper folding.

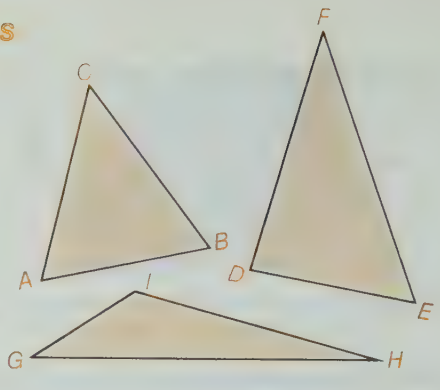
The angles at the three vertices fold together to form a straight angle.

In discussing Exercise 3, you might use a demonstration like that for Exercise 1. Tear off each corner of the quadrilateral and put the vertices together to show that the sum of all the angles is 360°.



Sums of Angle Measures of Polygons

Investigating the Ideas



A Draw three large differently-shaped triangles on your paper.

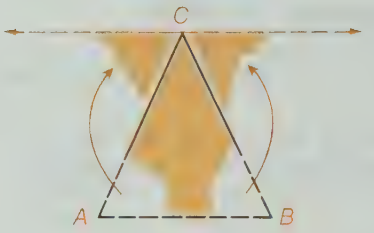
B Measure the three angles of each triangle to the nearest degree

C Find the sum of the angle measures of each triangle.

Do you think that the sum of the measures of a triangle depends upon the shape of the triangle?	Compare your sums of the angle measures with those of your classmates.

Discussing the Ideas

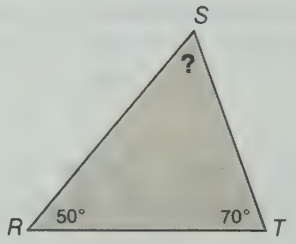
1. Cut out one of the triangles that you drew for the Investigation. Tear off two corners and place them as illustrated in the drawing.



What does this experiment suggest about the sum of the measures of the three angles?

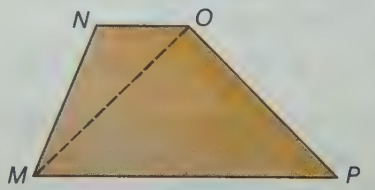
2. You should have convinced yourself of the truth of the following statement:

The sum of the measures of all the angles of any triangle is 180°.



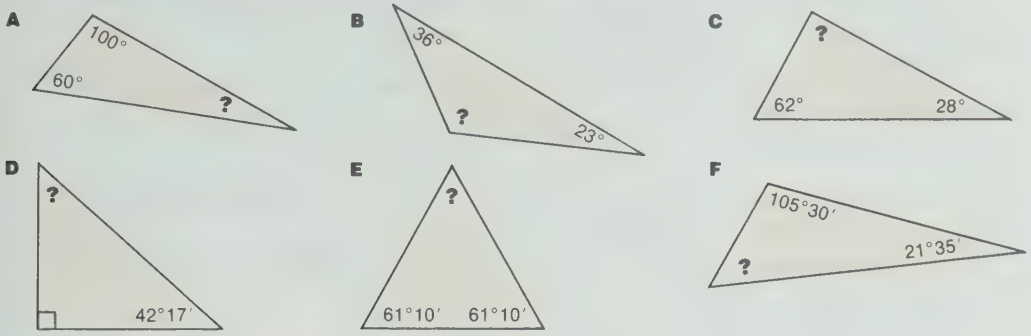
If you know that one angle of a triangle has a measure of 50° and a second angle has a measure of 70°, how could you find the measure of the third angle without using a protractor?

3. Every quadrilateral can be divided into two triangles. Since the sum of the angle measure in each triangle is 180°, what is the sum of the measures of the four angles of quadrilateral *MNOP*?



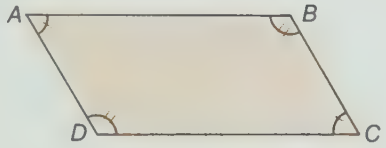
Using the Ideas

1. Find the degree measure of the angle whose measure is not given.



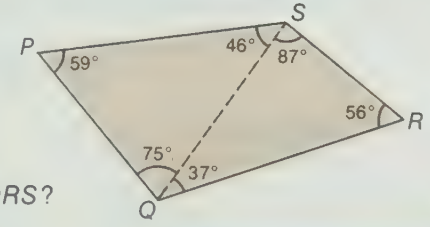
2. What is the measure of each angle of equilateral $\triangle RST$?

3. In parallelogram $ABCD$ congruent angles are marked by the same number of slashes.

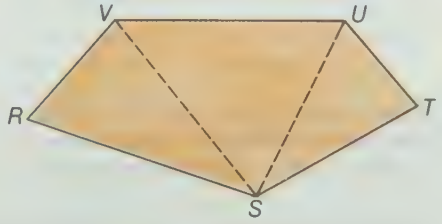


If $m\angle A = 59°10'$ and $\angle D$ is supplementary to $\angle A$, what is the measure of angles B, C, and D?

4. What is the sum of the measures
   - A of the angles of $\triangle PQS$?
   - B of the angles of $\triangle QRS$?
   - C of the angles of quadrilateral $PQRS$?



5. What is the sum of the measures
   - A of the angles of $\triangle RSV$?
   - B of the angles of $\triangle VSU$?
   - C of the angles of $\triangle UST$?
   - D of the angles of pentagon $RSTUV$?



★6. What is the sum of the measures of the angles of a hexagon?

★7. Can you write a formula for the sums of the angle measures of a polygon having any number n of sides?

UTILIZATION

All of the Exercises are based on the fact that the sum of the measures of the angles of a triangle is 180°.

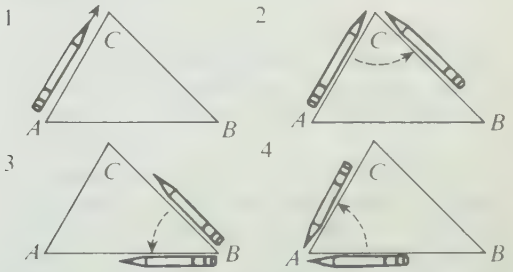
By using the fact that any convex polygon of n sides can be divided into $(n - 2)$ triangles, the sum of the measures of the angles of any convex polygon can be found. The latter exercises in the set are designed to lead to student discovery of this fact.

EXTENSION

To provide further practice, make assignments from page 93 of the *Workbook* and page 62 of the *Duplicator Masters*.

Enrichment: The sum of the angles of a polygon can be demonstrated by means of rotations as described below.

Draw a large triangle, $\triangle ABC$. Place a pencil with the eraser end at vertex A. Slide the pencil along $\overline{AC}$ until the point reaches C (Step 1). Rotate the pencil about C until it coincides with $\overline{BC}$ (Step 2). Slide the pencil along $\overline{BC}$ until the end reaches B. Rotate about B until pencil reaches $\overline{AB}$. (Step 3). Slide pencil to point A. Rotate about A until pencil coincides with $\overline{AC}$ (Step 4).



Now observe that in going around $\triangle ABC$ the pencil has made exactly $\frac{1}{2}$ turn or it has rotated exactly 180°.

A similar demonstration with a quadrilateral will show that the pencil makes a complete revolution; hence the sum of the angle measures must be 360°.

Assignments

Minimum 1–3. Average 1–5. Maximum 1–7.

Objective

The student can demonstrate the ability to work with the ideas presented in this module.

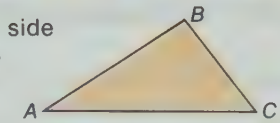
UTILIZATION

Centimetre rulers and protractors should be available for students to use in these exercises. Allow time for discussion of the Exercises, especially those which students have most difficulty in working.

Additional review exercises are provided on page 94 of the *Workbook*.

REVIEWING THE IDEAS

1. Measure each side of the triangle in millimetres.

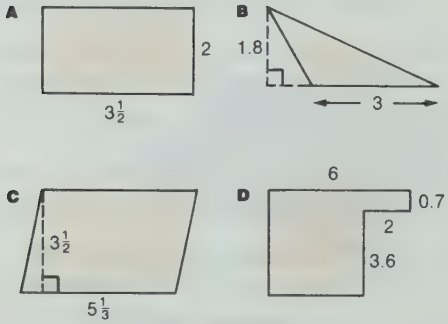


2. What is the perimeter of the triangle in Exercise 1?

3. **A** 1 metre = ▨ centimetres
B 1 centimetre = ▨ millimetres
C 1 decimetre = ▨ metre
D 23 cm = ▨ dm
E 47 dm = ▨ m
F ▨ m = 5 km

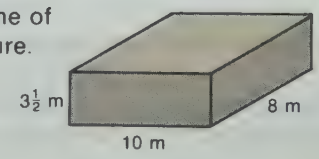
4. When the measure of an object is reported as 8 centimetres (to the nearest centimetre), what is the largest and the smallest measure for its actual length?

5. Find the area of each region.



6. If the lengths of the legs of a right triangle are 6 and 8 metres, what is the length of the hypotenuse?

7. Find the volume of this space figure.



8. Find the surface area of the space figure in Exercise 7.

9. One litre of water has a volume of ▨ cubic centimetres.

10. What is the capacity in ml of a cube 4 cm on each edge?

11. Give the measure of each angle.

12. Find the **sums** of the angle measures.

A
$25°15'43''$
$50°35'36''$

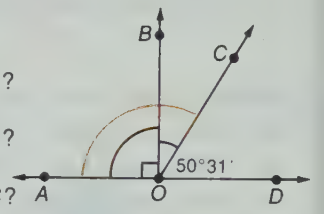
B
$86°28'29''$
$37°49'32''$

13. Find the **differences** of the angle measures.

A
$120°56'23''$
$75°13'48''$

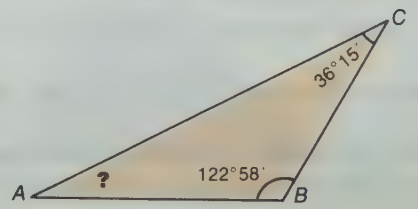
B
$98°23'10''$
$29°50'26''$

14. **A** What is $m\angle AOB$?
B What is $m\angle AOC$?
C What is $m\angle BOC$?



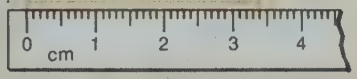
15. What is the measure of an angle which is supplementary to an angle which measures $103°42'$?

16. What is the measure of $\angle A$?



TEST YOURSELF

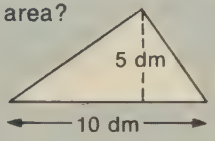
1. How many millimetres long is the segment?



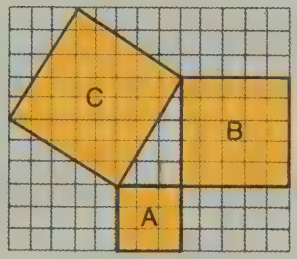
2. 73 centimetres = ▥ millimetres

3. A rectangle is $7\frac{1}{2}$ units long and $2\frac{1}{4}$ units wide.
   - A What is the perimeter?
   - B What is the area?

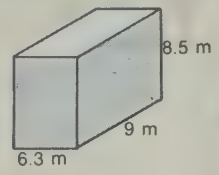
4. What is the area of the triangle?



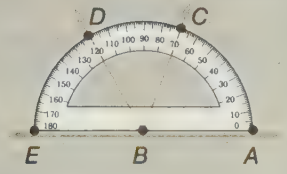
5. A Find the areas of squares A and B.
   B What is the area of square C?



6. A Find the volume of the figure.
   B Find the surface area.



7. What is $m\angle CBD$?



8. Find the sums of the angle measures.
   A

   B $46°34'45''$ + $19°28'25''$

9. Find the differences of the angle measures.
   A $158°35'$ − $87°48'$
   B $106°48'20''$ − $73°35'41''$

10. If two angles of a triangle each have a measure of 54 degrees, what is the measure of the third angle?

RESEARCH PROJECTS

A Can you find a formula for the area of any region on the geoboard when you know the number of nails on the boundary of the figure and the number of nails inside the figure? The formula for the area is called **Pick's Theorem**. (See *Geoboards and Motion Geometry* by John J. Del Grande; Glenview, Illinois: Scott, Foresman, 1972 available from Gage Educational Publishing, Ltd.)

B Another unit of angular measure that is often used in advanced mathematics is called a **radian**. Find out how the radian compareswith the degree, how you can construct an angle whose measure is one radian, and how this unit is used in mathematics.

C There are many different proofs of the **Pythagorean Theorem**. Find an explanation of one or more of them. Make some diagrams or charts that will help explain the proof. (See *The Pythogorean Proposition* by Elisha Scott Loomis; Washington, D.C.: NCTM, 1968.)

D The numbers 3, 4, and 5 are called **Pythagorean numbers** because

$$3^2 + 4^2 = 5^2.$$

Can you find some other triples which are Pythagorean numbers? Look up this topic in a reference book to see how you can find many other triples.

TEST YOURSELF

Have students work through the self-evaluation test independently and then use the answers on the following page to check their own work. If students miss a problem, they should note the corresponding objective and then attempt to analyze their mistake by reviewing the appropriate lesson of the module.

RESEARCH PROJECTS

Project A may be especially attractive to students who like to figure things out for themselves. By starting with very simple figures and using trial-and-error methods, they can discover Pick's formula:

$$A = \tfrac{1}{2} \cdot b + i - 1$$

where A is the area, b is the number of nails on the boundary, and i is the number of nails in the interior. They might also develop a formula for a region that has a "hole" in it:

$$A = \tfrac{1}{2}b + i.$$

For students who are interested in Research Project B, you might point out that any secondary-level geometry or trigonometry text will contain information on *radians*. A radian is defined as an angle which, if its vertex is at the center of a circle, will intercept an arc of the circle whose length is the same as the radius of the circle.

Project C is recommended for your more able students. You should not expect them to achieve complete understanding of the proofs of the theorem, especially those that are algebraic.

Students who explore Project D might be encouraged to try to construct a table of Pythagorean triples (a, b, and c where $a^2 + b^2 = c^2$). Such a table can be constructed by letting $a = p^2 - q^2$, $b = 2 \cdot pq$, and $c = p^2 + q^2$, where p and q are positive integers and $p > q$.

MATHEMATICAL RECREATIONS

If playing cards are available, use the 13 cards of any one suit for this activity.

The solution to this stacking problem lies in thinking about the inverse operations to the steps given in the rules. The inverse operations are as follows:

Pick up card 13 first and place it face down in your hand. Pick up card 12 and place it face down on 13. Slide the bottom card to the top. Pick up card 11, place it face down on the stack, and slide the bottom card to the top. Continue this procedure until all cards have been picked up.

The final order of the cards in the stack will be 1, 12, 2, 8, 3, 11, 4, 9, 5, 13, 6, 10, 7.

When dealt out in the manner stated in the rules, the cards will come out in consecutive order.

MATHEMATICAL RECREATIONS

A STACKING PROBLEM

Print the numerals 1 through 13 on index cards.

1	2	3	4	5	6

7	8	9	10	11	12	13

The problem is to devise a way of stacking the cards in such a way that when you place the stack in your hand face down, you can turn up the cards 1 through 13 in order following the rules below.

RULES:

1. With the stack of cards face down in your hand, turn the first card face up and place it on your desk. It must be card number 1.

2. Slide the next card (that is on top of the stack) to the bottom of the stack without turning it over.

3. Turn up the next (third) card and place it face up on your desk. It must be card number 2.

4. Slide the next card that is on top to the bottom of the stack.

5. Repeat the steps, alternating by turning up a card and then moving the following top card to the bottom of the stack.

How should the cards be "stacked" so that they can turn up in 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13 order?

MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. 34 mm **2.** 730 **3.** A $19\frac{1}{2}$ units; B $16\frac{7}{8}$ sq units **4.** 25 dm^2
5. A A: 9 sq units; B: 25 sq units; B C: 34 sq units **6.** A 481.95 m^3; B 373.5 m^2 **7.** 50° **8.** A 119°8′; B 66°3′10″
9. A 70°47′; B 33°12′39″ **10.** 72°

C–66

UNIT C: GEOMETRY AND MEASUREMENT

Module 3: Rigid Motions and Constructions

General Objectives

To introduce motion geometry by a study of some rigid motions.
To provide experience with translations, rotations, and reflections.
To examine some aspects of rotational, translational, and reflectional symmetry.
To provide motivation through work with translations.
To relate the concepts of rigid motions to congruence.
To provide experience with geometric constructions by means of reflection constructions and compass ans straightedge constructions.

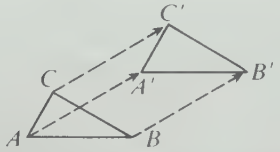
	Performance Objectives	Pupil Text	Reteach-Reinforce	Related Activities	
104	**RED** The student can give the coordinates of a point and can locate a point when co-ordinates are given.	C-68, 69		**SP** b-3	**SWM 1** 92
105	**RED** The student can find a translation image of a given figure in the co-ordinate plane.	C-70, 71 C-72, 73	**WB** 95 **DM** 63		**SWM 3** 164, 165
106	**RED** The student can find a rotation image of a given figure after a $\frac{1}{4}$, $\frac{1}{2}$, or $\frac{3}{4}$-turn about a point.	C-74, 75	**WB** 96 **DM** 64		**SWM 2** 64, 65
107	**RED** The student can determine if a given figure has rotational symmetry.	C-76, 77	**DM** 64		**SWM 1** 115 **SWM 3** 160
108	**RED** Given a geometric figure and a line in the coordinate plane, the student can find the reflection image of the figure in the line.	C-78, 79	**WB** 97 **DM** 65		**SWM 2** 62, 63, 269
109	**RED** Given a pair of congruent triangles, the student can identify corresponding sides and angles.	C-86, 87	**WB** 98 **DM** 66	**PD** b-2	
110	**YELLOW** The student can apply reflections or ruler-and-compass methods to construct various geometric figures.	C-90 through 97	**WB** 99-101 **DM** 67, 68		**SWM 1** 116 **SWM 2** 69
111	***GREEN*** *The student can decide whether a figure can tessellate a plane.*	C-82, 83 C-84, 85			**SWM 3** 168, 169
	Reviewing the Ideas	C-98	**WB** 102		

MATHEMATICS

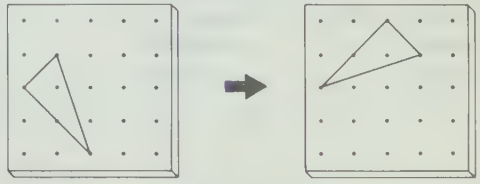
This module introduces the student to three kinds of motions in geometry: translations, rotations, and reflections. These motions are called *rigid motions* or *isometries* because the distance between any two points of a figure before and following any of these motions is unchanged. Rigid motions are thus said to be *distance-preserving motions*.

Intuitively, a *translation*, or a slide, is a motion that "moves" each point of the plane the same distance along parallel lines. Thus, $\triangle A'B'C'$ is said to be the slide image of $\triangle ABC$.

The arrow from C to C' can be thought of as indicating the distance and direction of the slide.



A *rotation* is rigid motion in which each point of the plane is rotated about some fixed point in the plane. The fixed point is called the centre of rotation. Most of the work on rotations in this module deals with rotations that are $\frac{1}{4}$, $\frac{1}{2}$, or $\frac{3}{4}$-turns since these are easily demonstrated on the geoboard or co-ordinate plane.



The figure above illustrates a $\frac{1}{4}$-turn clockwise. Work with rotations of figures in the co-ordinate plane gives experience in naming new co-ordinates of points after a given rotation.

Reflections were introduced in Module 1 of this learning unit. The material on reflections in this module involves reflections in the co-ordinate plane.

As an extension of the work with rigid motion, we define congruent figures in terms of the rigid motions. Two geometric figures are congruent if one is the image of the other by means of one of the rigid motions or by a combination of rigid motions.

Simple geometric constructions involving perpendiculars are introduced by means of reflections. Ruler and compass methods are used in the study of other traditional constructions.

TEACHING THE MODULE

Materials

Graph paper, co-ordinate grid paper, compass, ruler, scissors, geoboards, dot paper, plastic mirrors, tracing paper, crayons, colored pencils, index cards, posterboard.

Vocabulary

altitude
bisector
centre of rotation
centroid
circumcentre
compass
congruent
construction
co-ordinates
corresponding
incentre
line of symmetry
median
midpoint
order of symmetry
perpendicular
perpendicular bisector
reflection
reflection image
rigid motion
rotation
rotational symmetry
slide
slide image
straight edge
tangram
tessellation
translation

If some of your students have studied from earlier books in the *Investigating School Mathematics* series, they may have had a brief introduction to the ideas of this module. Keep the work on an intuitive level, stressing the ideas that are presented. We have purposely avoided introducing unnecessary terminology and symbolism associated with rigid motions.

Most of the work with the rigid motions is done in the first quadrant of the co-ordinate plane. This presentation is intended both to facilitate the work and to provide additional practice with co-ordinate geometry.

The material dealing with combinations of motions, symmetry, and tessellations is designed to extend the ideas of motion geometry.

The work with congruence centres on congruent triangles, and relates congruence to rigid motions. The material is informal, and the only proofs involved are intuitive in nature.

The latter part of the module introduces geometric constructions, first by using a plastic mirror and reflections and then by using traditional tools—compass and straightedge.

Lesson Schedule

The time you spend on the module will depend on the background of your students and the depth of coverage you give to the module. Average coverage will require about 12 days, but a longer period might be desirable for slower students or a more in-depth study of the material.

Evaluation

Use the students' daily performance as a guide in evaluating their progress in this module. There are not a great many facts or skills that the students should master. Rather, they should become acquainted with the ideas of rigid motions in geometry and become more motivated to pursue these ideas in depth.

The module review and the self-evaluation test can be used as models for your own evaluative instrument for the module, or you might wish to use the Module Achievement Tests which are available from the publisher.

RESOURCES FOR ACTIVE LEARNING

Activities

ESS:Mirror Cards; Pattern Blocks; Tangrams, Webster; McGraw-Hill

Experiments in Mathematics, Stage 1, pp. 14–15, 36–39; Stage 3, pp. 12–13, Houghton Mifflin

Geoboards and Motion Geometry, Scott Foresman

Mira Activities, Activities 81–100, 127–138, Creative Publications

Nuffield Project; *Problems*-Red Set, Nos. 19–21; *Symmetry,* Wiley

Relationshapes Activity Cards, Nos. 76–79, Addison-Wesley

Rolling; Tessellations, Cuisenaire Co.

Manipulative Devices

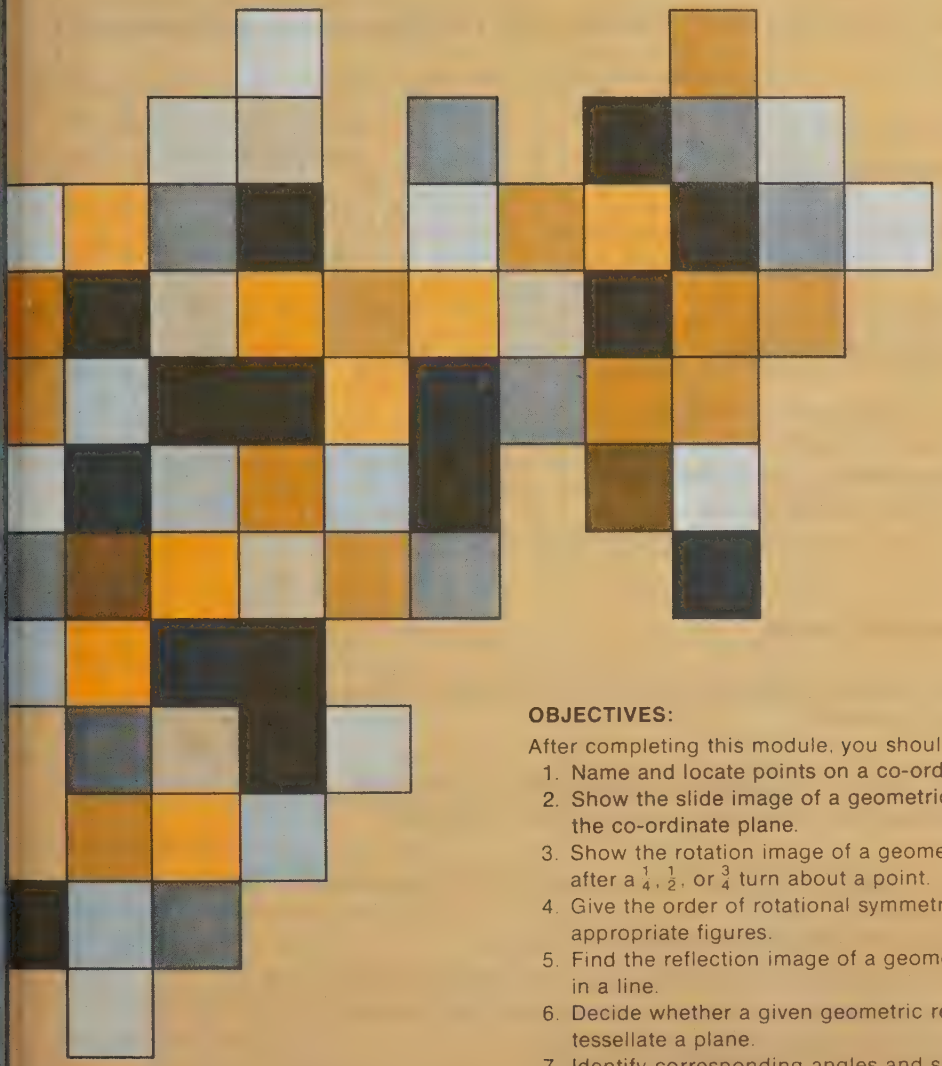
Circle Master Compass (Creative Publications)
Geoboards (Addison-Wesley)
Mira (Creative Publications; Cuisenaire Co.)
Pattern Blocks and stickers (Webster, McGraw-Hill)
Protractors (Geyer, Sigma)
RelationShapes (Addison-Wesley)
Templates (Pickett)

Games and Puzzles

Configurations (Wff'N Proof)
Madagascar Madness (Mind/Matter)
Pythagoras (Midwest Publications)
Tangrams (Addison-Wesley; Creative Publications)

UNIT C: *Geometry and Measurement*

MODULE 3: *Rigid Motions and Constructions*



OBJECTIVES:

After completing this module, you should be able to:

1. Name and locate points on a co-ordinate grid.
2. Show the slide image of a geometric figure in the co-ordinate plane.
3. Show the rotation image of a geometric figure after a $\frac{1}{4}$, $\frac{1}{2}$, or $\frac{3}{4}$ turn about a point.
4. Give the order of rotational symmetry of appropriate figures.
5. Find the reflection image of a geometric figure in a line.
6. Decide whether a given geometric region will tessellate a plane.
7. Identify corresponding angles and sides of congruent triangles.
8. Make the following constructions using reflections or ruler-and-compass method:
   - copy an angle
   - copy a triangle
   - bisect an angle
   - construct perpendicular lines
   - construct perpendicular bisector of a segment
   - construct a triangle given at least two segments

Objectives

Given the co-ordinates of points in the plane, the student can locate the points in the plane.

Given points in the plane, the student can give the co-ordinates of the points.

PREPARATION

Materials: Compass and ruler, graph paper or co-ordinate grid paper (*Duplicator Masters*, pages 83, 85).

You should plan to review graphing of points in the co-ordinate plane, especially if this topic was not covered in the previous learning units. Use a transparency of the first quadrant on the overhead projector (or draw a grid on the chalkboard). Identify the co-ordinates of various points and mark points after stating the ordered pairs for the points.

INVESTIGATION

The Investigation provides 4 problems in locating points in the plane. Part D will require a little more thought than the other parts. Students should reason that, since the area of the square in part A is 5 units, the rectangle in D which is one unit in height must have a length of 5 units if it is to have the same area as the square. Although the rectangle could be constructed along the horizontal axis or the vertical axis, most students will construct it along the horizontal axis.

DISCUSSION

The Discussion questions focus on the results of the Investigation and provide an opportunity for students to compare the co-ordinates they have found with those found by other students.

In Exercise 3, although the length of the radius of the circle constructed in part C is an irrational number ($\sqrt{8}$), there are three other points on the circle which have whole number co-ordinates: (7, 7), (3, 7), and (3, 3).

Note that the co-ordinates of the vertices of the rectangle discussed in Exercise 4 will vary if some students constructed the rectangle with its long side on the vertical axis.

Locating Points

Investigating the Ideas

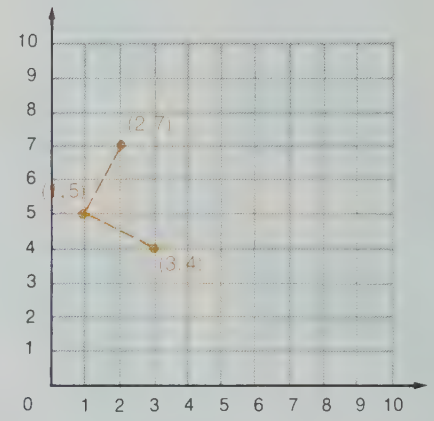
How many of the situations below can you show on a co-ordinate grid?

A A square with three of its vertices at (1,5), (2,7), and (3,4).

B The segment with an endpoint at (7,5) and its midpoint at (8,7).

C A circle with its centre at (5,5) and passing through (7,3).

D A rectangle with one vertex at (0,0), a height of one unit, and an area equal to the area of the square in part **A**.



Discussing the Ideas

1. What are the co-ordinates of the fourth vertex of the square for part A above?

2. Explain how you found the other endpoint of the segment for part B. What are the co-ordinates of this endpoint?

3. Can you give other co-ordinates of points on the circle for part C?

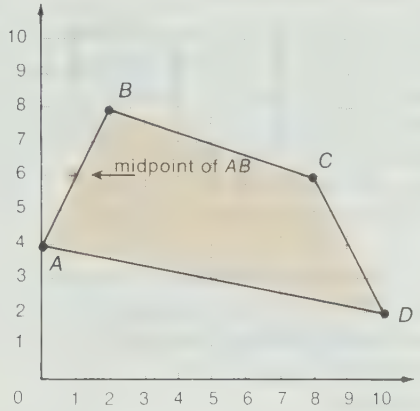
4. **A** What is the area of the square in part A?
   B What is the length of the rectangle for part D?
   C What are the co-ordinates of all the vertices of the rectangles that you drew for part D?

5. **A** Give the co-ordinates of a square whose sides are each twice as long as the square for part A and which also has one of its vertices at (1,5).
   B How does the area of this square compare with the area of the square for part A?

Using the Ideas

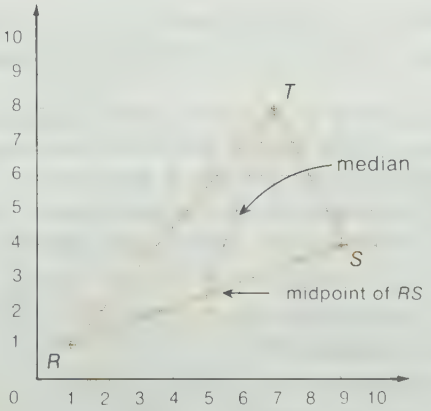
Refer to quadrilateral *ABCD* on the grid below for Exercises 1 through 4.

1. Give the co-ordinates of each vertex of quadrilateral *ABCD*.
2. Give the co-ordinates of the **midpoint** (middle point) of each side of the quadrilateral.
3. Find the co-ordinates of the midpoint of each diagonal.
4. Estimate the co-ordinates of the point of intersection of the diagonals.



Refer to triangle *RST* on the grid below for Exercises 5 through 8.

5. Give the co-ordinates of the vertices *R*, *S*, and *T*.
6. Give the co-ordinates of the midpoint of each side of the triangle
7. A **median** of a triangle is the segment from a vertex to the midpoint of the opposite side of the triangle.
   Draw $\triangle RST$ on graph paper. Then draw the three medians of the triangle.
8. The three medians of a triangle intersect at one point called the **centroid** of the triangle. Estimate the co-ordinates of the centroid of $\triangle RST$.



UTILIZATION

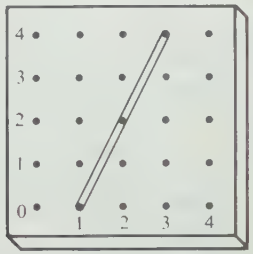
The exercises provide practice in giving co-ordinates of points in the plane and in estimating co-ordinates which are not whole numbers. The students will need grid paper for these exercises.

Allow the students to use their own methods in finding the co-ordinates of midpoints. Some may measure the segments and then mark the midpoints. To find the midpoint of diagonal $\overline{AC}$, some students may be able to reason that, since the diagonal "goes over 8 and up 2," the midpoint is found by starting at point A and going "over 4 and up 1."

In Exercise 4, the exact point of intersection is $(5\frac{1}{2}, 5\frac{3}{8})$, but of course any reasonably close estimate should be accepted. Similarly, answers will vary for the centroid of $\triangle RST$ in Exercise 8. The actual co-ordinates of the centroid are $(5\frac{2}{3}, 4\frac{1}{3})$.

EXTENSION

Remedial: Guide students in setting up a co-ordinate system on the geoboard. Then give co-ordinates of pairs of points such as (1, 0) and (3, 4) and have the students connect the points with a rubber band.



Enrichment: Ask students to try to create a list of co-ordinates of points such that when they are connected in order by segments they will form a picture.

Assignments

Minimum 1–4. Average 1–6. Maximum 1–8.

Objective

Given a geometric figure in the co-ordinate plane, the student can show the slide image of the figure after a given translation.

PREPARATION

Materials: Co-ordinate grid paper or graph paper (*Duplicator Masters*, pages 83, 85).

Review the work of the previous lesson on co-ordinates of points.

INVESTIGATION

The purpose of the Investigation is to introduce the student to the idea of a *translation* or *slide*.

You might suggest that the students make a table such as the one below as an aid in listing the new co-ordinates of each point.

Original point	*A*	*B*	*C*	*D*	*E*	*F*
Co-ordinates	(3, 6)	(5, 7)	(5, 9)	(3, 10)	(1, 9)	(1, 7)
New co-ordinates	(7, 1)	(9, 2)	(9, 4)	(7, 5)	(5, 4)	(5, 2)

DISCUSSION

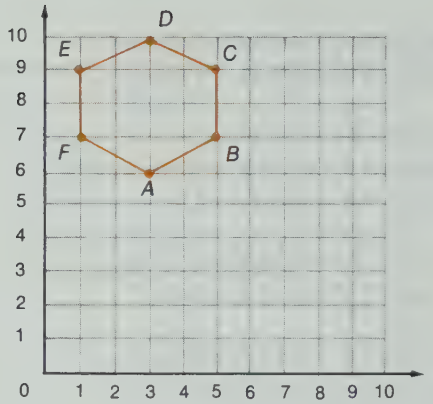
In discussing the results of the Investigation, you might demonstrate the slide by cutting out a hexagon of appropriate size and placing it on top of a grid shown on the overhead projector. First show it in the original position and then slide it to the position given by the co-ordinates obtained by adding 4 and subtracting 5 from the numbers in each pair.

Emphasize that a translation is a *rigid motion* because the slide image is congruent to the original figure. This means that all distances between points in the slide image are the same as those between the corresponding points in the original figure and that all angle measures remain the same. A rigid motion is called a "distance preserving motion" or an *isometry*.

Translations

Investigating the Ideas

Draw a hexagon, like the one shown, on a 10 by 10 co-ordinate grid. List the co-ordinates for the vertices of the hexagon. Make another list of co-ordinates in which you add 4 to the first number of each pair and subtract 5 from the second number of each pair.



Can you draw the polygon formed by joining the new number pairs with line segments?

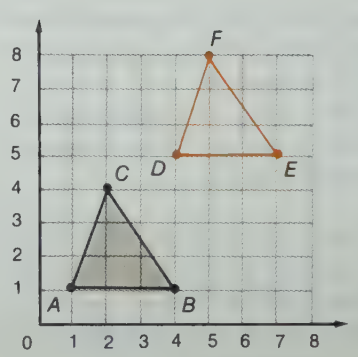
Discussing the Ideas

1. Describe the effect of "adding 4 and subtracting 5" to the coordinates of the original hexagon.

2. The motion produced by "adding 4 and subtracting 5" to the co-ordinates of every point in the plane is called a **slide** or **translation**. Describe some other translations.

3. A motion is a **rigid motion** if it does not change the size and shape of a figure. Do you think that a translation is a rigid motion?

4. The figure that you drew in the Investigation is called a **slide image** of the original hexagon. It could be called a "right 4, down 5" translation. Draw some figures on your graph paper. Then show a "left 2, up 3" translation.

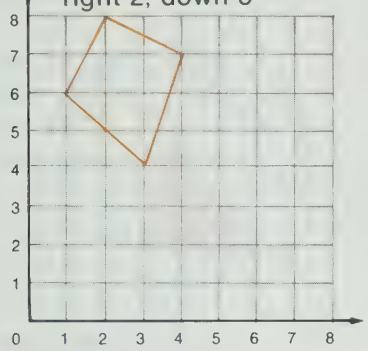


5. **A** Is $\triangle DEF$ a slide image of $\triangle ABC$? How can you tell?
   B The translation that slides $\triangle ABC$ to $\triangle DEF$ is "right ▮▮▮, up ▮▮▮."
   C How would you describe the translation that slides $\triangle DEF$ to $\triangle ABC$?

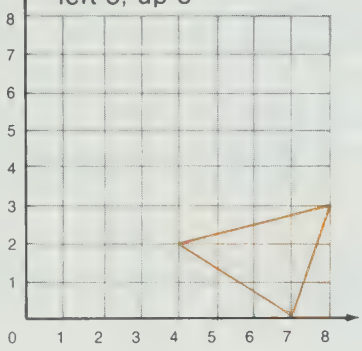
Using the Ideas

1. Copy each figure on a co-ordinate grid. Then show the slide image for the given translation.

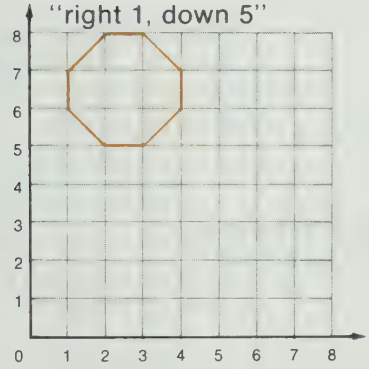
A "right 2, down 3"



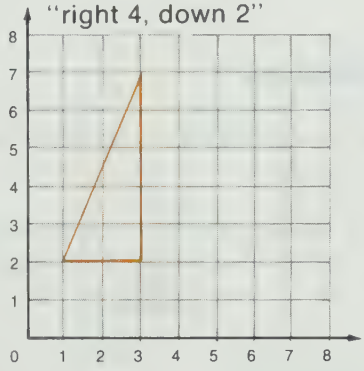
B "left 3, up 3"



C "right 1, down 5"

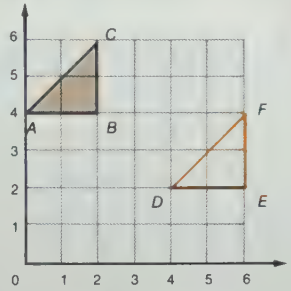


D "right 4, down 2"

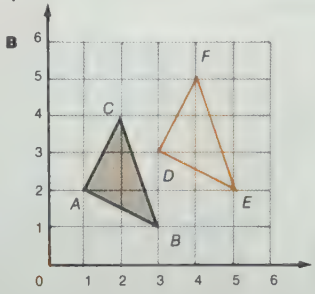


2. In each part, $\triangle DEF$ is the slide image of $\triangle ABC$. Describe the translation for each part using the words "right", "left", "up", or "down".

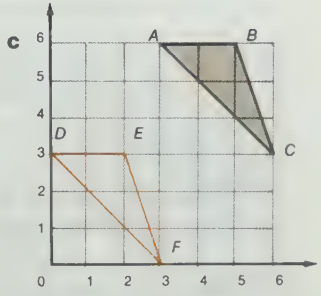
A



B



C

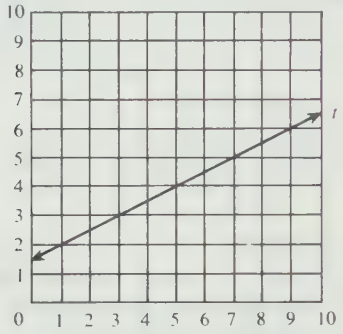


3. If a line is translated "right 2, up 1," what is the relation between the original line and its slide image?

UTILIZATION

Provide co-ordinate grid paper or graph paper for the exercises.

In Exercise 3, most students will respond that if a line is translated the slide image is parallel to the original line. However, in the case of a line such as t below and a translation of "right 2, up 1," the slide image falls again on line t. Hence, the complete answer is that either the line and its slide image are parallel or they are identical.



EXTENSION

Additional practice is provided in the *Workbook* on page 95.

Enrichment: Play a game of "Translations" on a 10 by 10 grid. Use two cubes. Label the faces of an "over cube" with numerals 0 through 5. Label a different colored cube, the "up cube," with numerals 0 through 5. Each player starts at (0, 0) and takes turns tossing the cubes. Using a counter, each player moves from (0, 0) the amount given by the "over and up" cubes. The object of the game is to reach (10, 10) before the other player. The rules might stipulate that a player who reaches either the top or the right edge of the grid, tosses only one cube until reaching (10, 10).

Assignments

Minimum 1. Average 1, 2. Maximum 1–3.

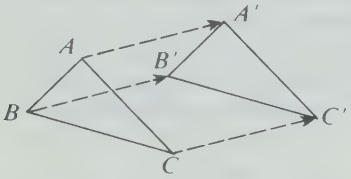
Objective

The student can show the slide image of a figure after at least two translations.

PREPARATION

Materials: Co-ordinate grids or graph paper.

Show two triangles in the co-ordinate plane which are slide images of each other. Draw arrows from each vertex of one triangle to the corresponding vertex of the other triangle. Point out that any one of the arrows is sufficient to show the "over and up" motion of the translation.



INVESTIGATION

Provide co-ordinate paper or graph paper for the Investigation. If possible, prepare a transparency or chalkboard drawing of the grid, including $\triangle ABC$ in its original position. When the students have completed the Investigation, have selected students show the slide images for the translation given by arrows *b*, *c*, and *d*.

DISCUSSION

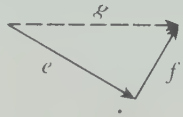
Be sure to emphasize that each arrow indicates a slide of a given amount in a given direction. Conversely, given a translation, it can be indicated by an appropriate arrow.

Discussion Exercise 2 should bring out two ideas about translations:

(1) The combination of 2 translations can be replaced by a single translation.

(2) If two translations are performed in succession, changing the order of the translations does not change the position of the final slide image.

You may want to show that the combination of arrows *e* and *f* has the same effect as the single translation given by arrow *g* below:

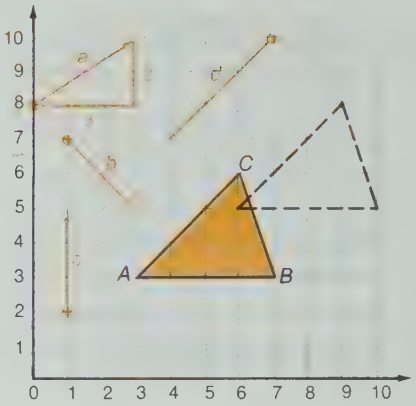


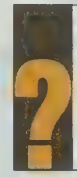
Combinations of Translations

Investigating the Ideas

The dotted triangle shows the slide image of $\triangle ABC$ when translated "right 3, up 2."

The arrow ***a*** is another way of indicating the translation "right 3, up 2."





Can you show how the triangle will be translated for arrows ***b***, ***c***, and ***d***?

Discussing the Ideas

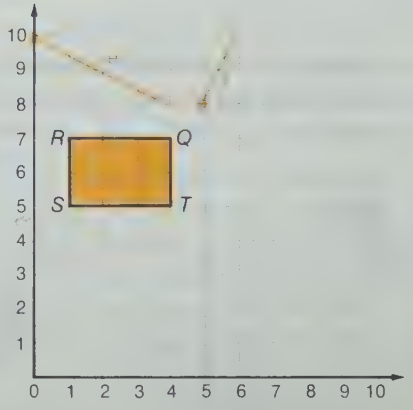
1. How would you describe the translation for arrow ***b***? arrow **c**? arrow ***d***?

2. Suppose that rectangle *QRST* was translated first according to arrow ***e*** and the image then translated by arrow ***f***.

   A On your graph paper, show the location of the final image of rectangle *QRST*.

   B What single arrow would have translated *QRST* to its final position in only one move?

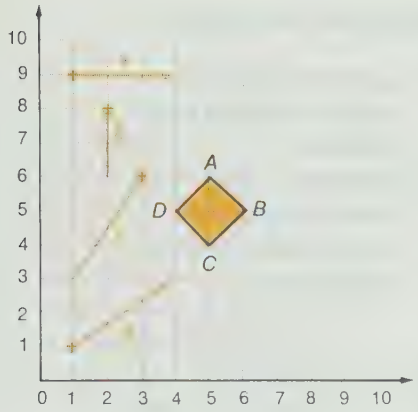
   C Suppose that *QRST* was first translated by arrow ***f*** and then by arrow ***e***. Is the final translation image in a different position than the one you drew for part **A**?



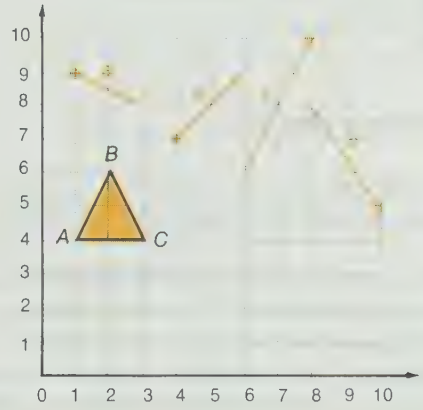
Using the Ideas

1. Show the slide image of square *ABCD* for each arrow.

2. Show the slide image of square *ABCD* for each combination of translations.
   - A ***a*** followed by ***b***
   - B ***a*** followed by ***c***
   - C ***b*** followed by ***d***
   - D ***c*** followed by ***d***
   - E ***b*** followed by ***a***
   - F ***c*** followed by ***a***



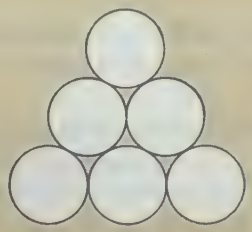
3. Show the slide image of triangle *ABC* for each combination of translations.
   - A ***a*** followed by ***d***
   - B ***b*** followed by ***c***
   - C ***b*** followed by ***d***
   - D ***a*** followed by ***b***
   - ★ E ***a*** followed by ***b***, followed by ***d***
   - ● F ***a*** followed by ***b***, followed by ***c***, followed by ***d***



think

Fill the circles with the numerals 1, 2, 3, 4, 5 and 6 so that the sum along each three circles in a row is 9.

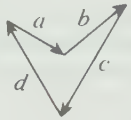
Can you arrange the numerals to get a sum of 10? A sum of 11? A sum of 12?



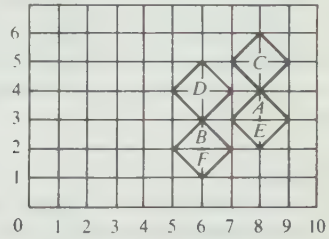
UTILIZATION

Sheets of co-ordinate grid paper will facilitate students' work on these exercises. Only one translation or combination of translations should be shown on each grid; otherwise, the figures may overlap and cause confusion.

Parts E and F of Exercise 3 call for more than two successive translations. In Part F students may be surprised to find that the final image coincides with the original figure. If the four translation arrows are connected as shown below, it can be seen that the combination of the four translations is the "identity translation"; that is, it translates, or *maps*, the figure onto itself.



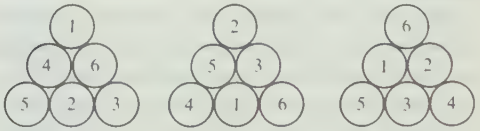
Solutions, Exercise 2



EXTENSION

Page 63 of the *Duplicator Masters* may be assigned to provide additional practice with the concepts of this lesson.

Think Solution



Assignments

Minimum 1, 2A–C. Average 1, 2, 3A–D. Maximum 1–3.

Objective

Given a geometric figure in a co-ordinate plane, the student can show the rotation image of the figure when the centre of rotation is given.

PREPARATION

Materials: Scissors, graph paper, co-ordinate grids (*Duplicator Masters*, pages 83 and 85).

On the overhead projector or the chalkboard, show a drawing of a special clock face like this:

Clarify the meaning of the terms that will be used in the Investigation by asking: "Where will the hand point after a $\frac{1}{4}$-turn clockwise? after a $\frac{3}{4}$-turn clockwise? after a full turn clockwise?" and so on.

INVESTIGATION

Have the students prepare the materials as suggested in the Investigation. Suggest that the students record the new co-ordinates of points *P*, *Q*, and *R* to facilitate the subsequent discussion.

DISCUSSION

The turn of the square grid is made about the centre of the square, (5, 5). While any point in the plane could be the centre of rotation, we have purposely chosen a central point in order to keep the work at a basic level.

With a little experience, students can read off co-ordinates by turning their book $\frac{1}{4}$, $\frac{1}{2}$, or $\frac{3}{4}$ turn and mentally renumbering the points on the axes.

Some of your more able students might be interested in the mathematical way of finding the new co-ordinate for a rotation about (5, 5): If a point has co-ordinates (a, b), then after a $\frac{1}{4}$-turn clockwise about (5, 5), the new co-ordinates will be $(b, 10 - a)$; for a $\frac{1}{2}$-turn, they will be $(10 - a, 10 - b)$; and for a $\frac{3}{4}$-turn, they will be $(10 - b, a)$.

Emphasize that rotations, like translations, are rigid motions.

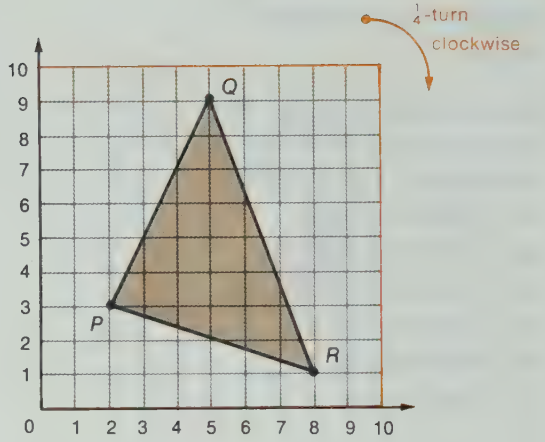
Rotations

Investigating the Ideas

A Mark the sides of a 10 by 10 square grid in color.

B Mark the points *P*, *Q*, and *R* as shown.

C Cut out the 10 by 10 square grid.



Can you find the new co-ordinates of points *P*, *Q*, and *R* if the cut-out square is given a $\frac{1}{4}$ turn in a clockwise direction and placed back inside the colored square?

Discussing the Ideas

1. When the square was given a $\frac{1}{4}$-turn, all of the points on the grid, except one, moved to a new position. What are the co-ordinates of that point?

2. The turn that you gave the square in the Investigation is called a **rotation**. The point whose co-ordinates did not change (the centre of the square) is called the **centre of rotation**. The **rotation** image of the point *P*, whose co-ordinates are (2, 3), is the point whose co-ordinates are (3, 8).

   A Give the rotation images of some other points for a $\frac{1}{4}$-turn rotation.

   B Can you find the co-ordinates of some rotation images without actually rotating the square?

3. Do you think that a rotation is an example of a **rigid motion**?

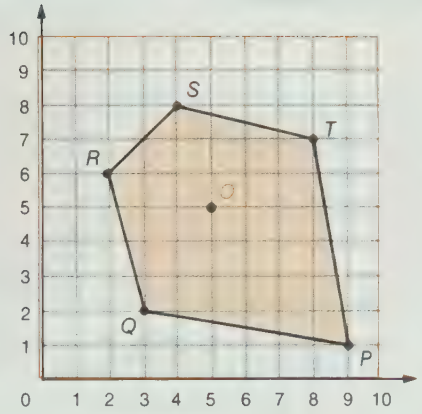
4. What are the new co-ordinates of points *P*, *Q*, and *R* for each of these rotations?

   A $\frac{1}{2}$-turn clockwise **B** $\frac{3}{4}$-turn clockwise **C** full turn

Using the Ideas

1. Suppose that the 10 by 10 square is given a $\frac{1}{4}$-turn clockwise with centre at point O. What are the co-ordinates of the rotation images of each of these points?

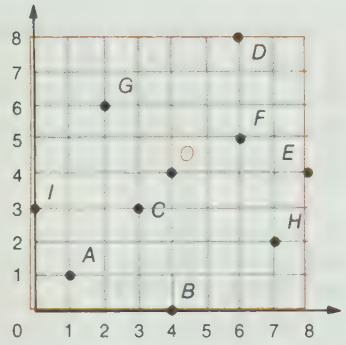
   A P **C** R **E** T
   B Q **D** S **F** O



2. Give the rotation images of each of the points P, Q, R, S, and T if the square is given a $\frac{1}{2}$-turn clockwise.

3. Repeat Exercise 2 for a $\frac{3}{4}$-turn clockwise of the square.

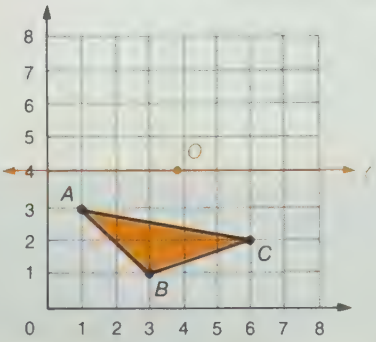
4. An 8 by 8 square with centre O at (4,4) is given a $\frac{1}{4}$-turn clockwise around O. Give the co-ordinates of the rotation images for the points with the co-ordinates shown.



5. Repeat Exercise 4 for a $\frac{1}{2}$-turn clockwise.

6. Repeat Exercise 4 for a $\frac{3}{4}$-turn clockwise.

7. **A** What are the co-ordinates of the vertices of the reflection image of $\triangle ABC$ in the line ℓ?
   B What are the co-ordinates of the vertices of $\triangle ABC$ after a $\frac{1}{2}$-turn clockwise about point O?
   C Is the half turn about O the same motion as a reflection in line ℓ?



UTILIZATION

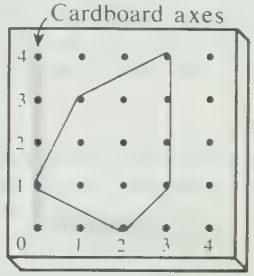
Students will need graph paper for the exercises.

Suggest to the students that they cut out the 10 by 10 square for Exercises 1, 2, and 3 and actually perform the rotations. An alternate approach might be to use a piece of tracing paper, trace the points, and then, holding the tracing paper fixed on (5, 5), turn the paper to the prescribed positions and read off the co-ordinates as required. A similar method can be used in Exercises 4, 5, and 6 with the centre of rotation at (4, 4).

Exercise 7 is designed to show that a $\frac{1}{2}$-turn is not the same rigid motion as a reflection in a line.

EXTENSION

Remedial: Establish a 5 by 5 co-ordinate system on a geoboard. Have the students read off the co-ordinates of some figure on the geoboard before and after various rotations around the centre nail (2, 2).



Assignments

Minimum 1–3. Average 1–6. Maximum 1–7.

Objective

Given a geometric figure and a centre of rotation, the student can determine the order of rotational symmetry of the figure.

PREPARATION

Materials: Geoboards, dot paper or graph paper (*Duplicator Masters*, pages 82, 83).

This lesson is an extension of the previous lesson. Therefore, you may want to review the concept of rotations before beginning the Investigation.

INVESTIGATION

The Investigation will require only a few minutes to complete. Its purpose is to enable the student to discover that some geometric figures have the property that, after a given rotation, the rotation images of all the points coincide with the points of the original figures. For example, the parallelogram coincides with itself every $\frac{1}{2}$-turn; therefore we say the parallelogram has rotational symmetry of order 2. That is, in one complete rotation, the points of the parallelogram coincide 2 times.

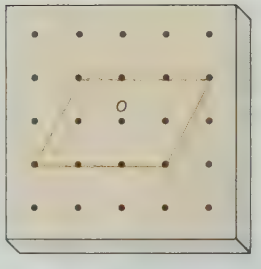
DISCUSSION

Exercise 1 introduces the term *rotational symmetry*, while Exercise 2 introduces the *order* of rotational symmetry. Note that an equilateral triangle has rotational symmetry about its centroid of order 3. A turn of $\frac{1}{3}$ about the centroid maps the triangle onto itself. If possible, demonstrate this by means of a transparency on the overhead projector.

Rotational Symmetry

Investigating the Ideas

Make a parallelogram like the one shown and in the same position on your geoboard or dot paper.

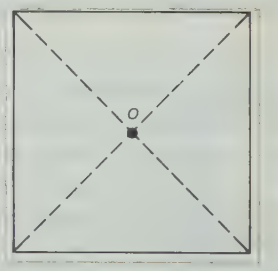


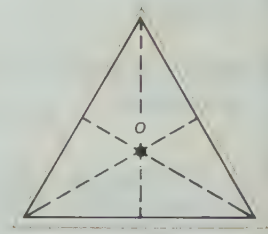
What is the smallest rotation with centre *O* that you can make so that the parallelogram will appear in the same position on the geoboard?

Discussing the Ideas

1. The parallelogram is said to have **rotational symmetry** with respect to a rotation about point *O* because all the points of the parallelogram fall on the parallelogram again after a $\frac{1}{2}$-turn. What are some other geometric figures that have rotational symmetry?

2. A square is said to have **rotational symmetry of order 4** because the square will fit into its outline 4 times in one complete turn about point *O*. What is the rotational symmetry order for the equilateral triangle about its centre *O*?



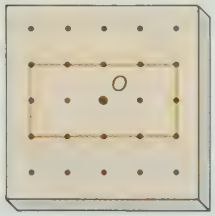


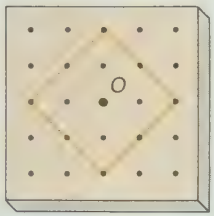
3. What is the order of rotational symmetry for the parallelogram shown in the Investigation?

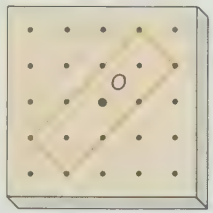
4. Form some polygons on your geoboard or on dot paper. See if they have rotational symmetry of any order.

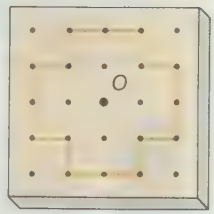
Using the Ideas

1. Each figure on the geoboards has rotational symmetry about centre *O*. Give the **order** of the symmetry.

A 

B 

C 

D 

2. Give the order of rotational symmetry for each regular polygon with centre at point *O*.

A

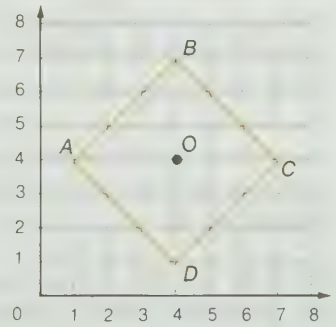
B

C

D

3. Suppose square *ABCD* is rotated about centre *O*. Complete the table for each clockwise rotation.

ROTATION	CO-ORDINATES OF POINTS			
	A	*B*	*C*	*D*
$\frac{1}{4}$	(4,7)			
$\frac{1}{2}$				
$\frac{3}{4}$				
1				



Think

The quotient of **two consecutive even whole numbers** is $1\frac{1}{48}$.
Find this pair of whole numbers. $a \div b = 1\frac{1}{48}$

UTILIZATION

If feasible, suggest that students form the figures in Exercise 1 on their geoboards in order to check for rotational symmetry. It is also possible for the student to make tracings or rotate the pictures in the textbook to determine the type of symmetry.

Students should be able to generalize from their experience with Exercise 2 that a regular polygon of n sides will have rotational symmetry of order n.

EXTENSION

To provide more practice, suggest page 96 of the *Workbook* and *Duplicator Masters*, page 63.

Enrichment: Have the students make designs on the geoboard or on dot paper that exhibit various orders of rotational symmetry.

Think Solution

When $1\frac{1}{48}$ is written as a quotient of two whole numbers, it becomes $\frac{49}{48}$; we see that only one of these numbers is even, so we could multiply each by 2. This will give the quotient $\frac{98}{96}$, which gives the answer.

Assignments

Minimum 1, 2. Average 1–3. Maximum 1–3.

Objective

The student can find reflection images of points in the co-ordinate planes.

PREPARATION

Materials: Plastic mirrors, co-ordinate grid paper or graph paper (*Duplicator Masters*, pages 83, 85).

Reflections in a line were introduced in Module 1 of this unit. A brief review of reflections before beginning the Investigation will benefit most students.

INVESTIGATION

Students should have plastic mirrors available for use in the Investigation even though some may be able to find the reflection images without using the mirror. Since reflections were discussed in an earlier module, you may point out that here we are considering reflections in the co-ordinate plane.

DISCUSSION

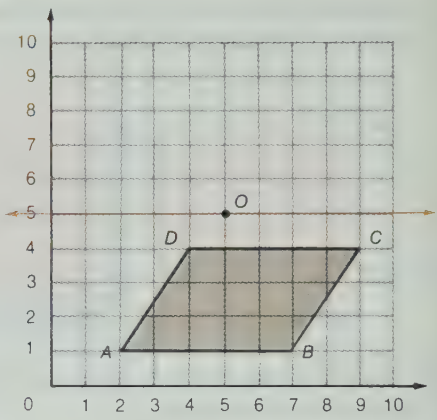
Exercises 1, 2, and 3 bring out some important ideas about reflection. A reflection is another example of a rigid motion. It is a motion that is different from a translation or a rotation.

If an overhead projector is available, the discussion of Exercises 1 and 2 will be enhanced by a demonstration that neither a slide nor a rotation will give the same image as the reflection of parallelogram *ABCD*.

Reflections in the Coordinate Plane

Investigating the Ideas

Suppose the colored line represents a mirror placed upright to the grid.



Can you find the reflection image of each point *A*, *B*, *C*, and *D* and then connect these points with line segments?

Discussing the Ideas

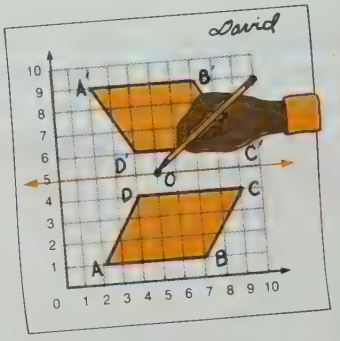
1. A How does the reflection image of figure *ABCD* compare with the original figure *ABCD*?

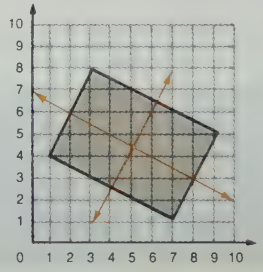
   B Do you think that reflections are examples of rigid motions?

2. Suppose that the reflection image of *ABCD* is labeled *A′ B′ C′ D′*. Do you think there is a translation that would slide *ABCD* to *A′ B′ C′ D′*? Explain your answer.

3. David thought that a reflection in the colored line had the same effect as a $\frac{1}{2}$-turn about a point. In order to check this, he put a piece of tracing paper over his drawing, traced figure *ABCD*, then placed his pencil at point *O* and rotated his tracing a $\frac{1}{2}$ turn.

   What do you think that David found out about the rotation compared to the reflection?



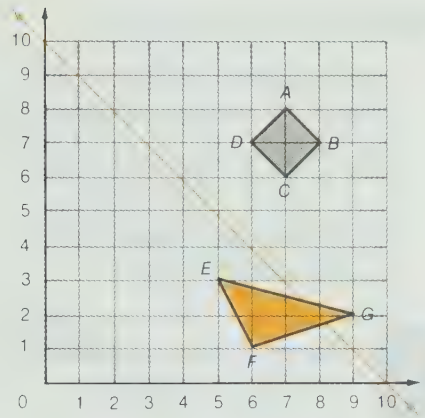


4. Draw a figure on a co-ordinate grid paper that you think is a symmetric figure. Show one or more lines of symmetry for your figure.

Using the Ideas

1. **A** Draw square *ABCD* as shown. Then find its reflection image in the diagonal line.

B Give the co-ordinates of the reflection images for points *A*, *B*, *C*, and *D*.



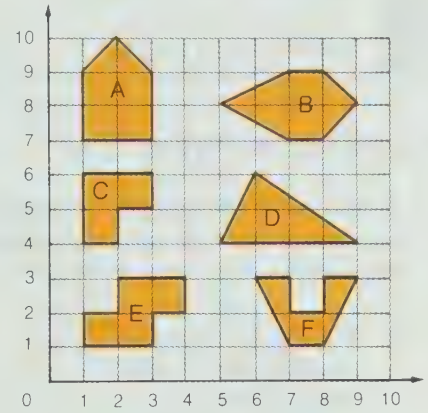
2. **A** Draw triangle *EFG* as shown. Find its reflection image in the diagonal line.

B Give the co-ordinates of the reflection images for points *E*, *F*, and *G*.

3. **A** Draw each shape on a 10 by 10 co-ordinate grid.

B Which shapes have a line of symmetry?

C Draw the line(s) of symmetry for the symmetric figures.



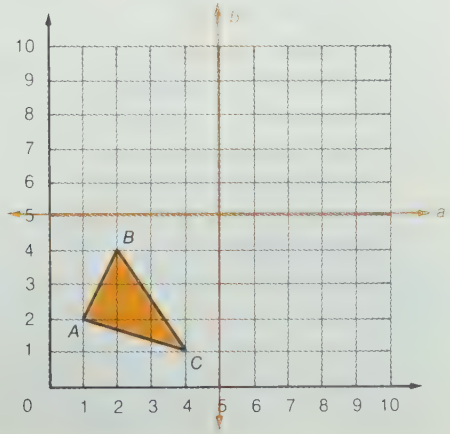
4. **A** Draw $\triangle ABC$ in the position shown on a 10 by 10 co-ordinate grid at the right.

B Find the reflection image of $\triangle ABC$ in line *a*. Call it $\triangle A'B'C'$.

C Find the reflection image of $\triangle A'B'C'$ in line *b*. Call it $\triangle A''B''C''$.

D Are $\triangle ABC$ and $\triangle A''B''C''$ reflection images of each other?

E Can you find a rigid motion so that $\triangle ABC$ and $\triangle A''B''C''$ will be images of each other?



UTILIZATION

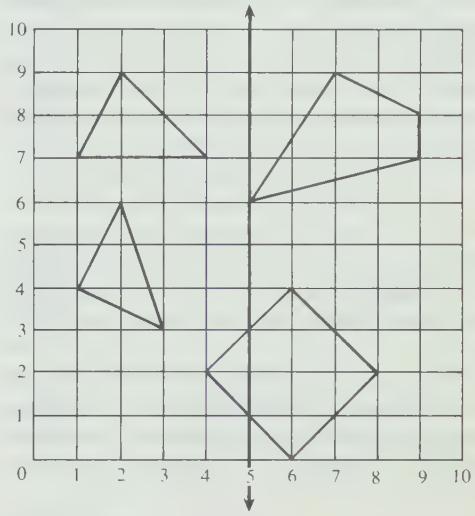
Students should have their plastic mirrors available as aids in locating the reflection images in Exercises 1 and 2.

Exercise 4 illustrates an interesting property of a combination of reflections. The combination of a reflection in line *a* followed by a reflection in line *b* is equivalent to a half turn about the point of intersection of the two lines.

EXTENSION

Page 97 of the *Workbook* may be assigned to provide additional practice.

Remedial: Ask students to show the reflection image of each figure in the vertical line.



Enrichment: Have the students use their mirrors and show pairs of figures that are reflection images of each other. Students can exchange papers and then, using the mirror, locate and draw the reflecting lines for the pairs of figures.

Assignments

Minimum 1–3. Average 1–3, 4A–D. Maximum 1–4.

Objective

Given a geometric figure in the co-ordinate plane, the student can find the image of the figure after a combination of rigid motions.

PREPARATION

Materials: Plastic mirrors, tracing paper, co-ordinate grid paper or graph paper (*Duplicator Masters*, pages 83, 85).

Review briefly the three kinds of rigid motions studied in this module: translations, rotations, and reflections. In this lesson, combinations of two rigid motions are considered.

INVESTIGATION

Students will need co-ordinate grid paper, graph paper, and their mirrors to carry out the Investigation. In carrying out the instructions for the Investigation, students should take care to use the image obtained by reflection in line ℓ as the figure to be reflected in line m.

The effect of successive reflections in two intersecting lines is a rotation about the point of intersection of the two lines and through a turn twice as large as the directed angle (measured clockwise) from the first line to the second. Since the measure of the directed angle from m to ℓ is 135°, the result of the two reflections is a turn of 270° or a $\frac{3}{4}$-turn. By placing a piece of tracing paper over the original figure and marking the figure and point O, the student can check to see that the reflections do yield a $\frac{3}{4}$-turn.

DISCUSSION

Discussion Exercise 1 shows that the order in which the reflections are considered makes a difference in the final image. Reversing the order, in this case, gives only a $\frac{1}{4}$-turn (twice the measure of directed angle from ℓ to m).

In general the Discussion Exercises should convince the student that the *order* in which the various rigid motions are performed is important because the position of the final image may be different for the same combination of motions if they are performed in different orders.

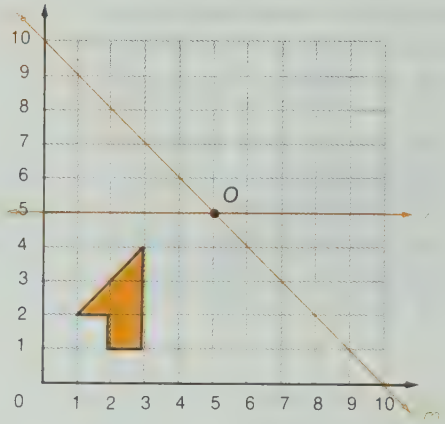
Combinations of Motions

Investigating the Ideas

Draw a region like the one shown on the co-ordinate grid.

A Find the reflection image of the figure in line m.

B Reflect the image found for part **A** in line ℓ.



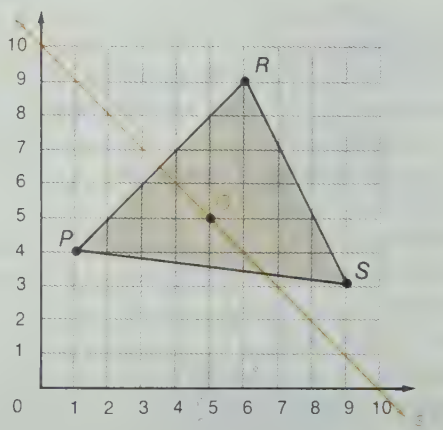
Can you describe a single translation, reflection, or rotation that will give the same final image as the combination of reflections **A** and **B**?

Discussing the Ideas

1. Would the final reflection image be in the same final position if the first reflection were in line ℓ and the second reflection in line m?

2. **A** What would be the co-ordinates of the image of point P after a clockwise rotation of 90° $\left(\frac{1}{4}\text{-turn}\right)$ about O and then a reflection in line s?

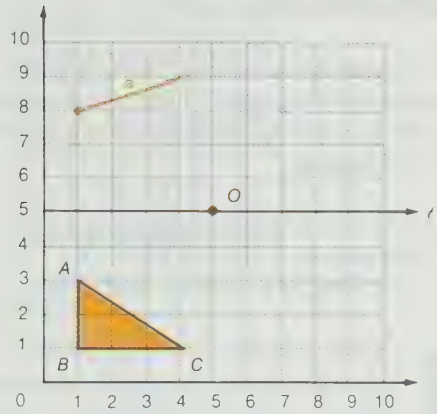
   B Find the co-ordinates of the image of points R and S after the motions given in part **A**.



3. Repeat the motions of Exercise 2, but perform them in reverse order. Do you get the same final image points?

Using the Ideas

Refer to the co-ordinate grid containing $\triangle ABC$ for Exercises 1, 2, and 3.

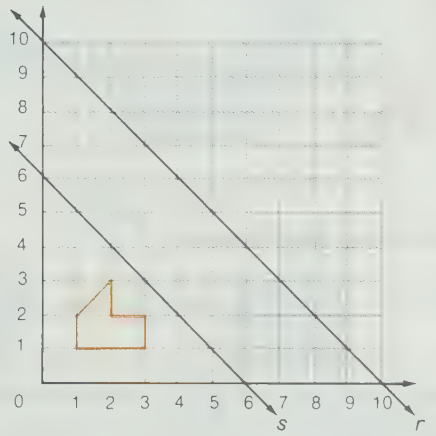


1. Show the final image of $\triangle ABC$ after a translation by arrow ***a*** and a reflection in line ℓ.

2. Show the final image of $\triangle ABC$ after a clockwise rotation of 180° $\left(\frac{1}{2}\text{-turn}\right)$ about O and a reflection in line ℓ.

3. Show the final image of $\triangle ABC$ after a translation by arrow ***a*** and a clockwise rotation of 90°.

4. Draw parallel lines r and s and the region as shown.
   - **A** Find the reflection image of the region in line s followed by a reflection in line r.
   - **B** Can you find and describe one rigit motion that has the same effect as the two reflections for part **A**.

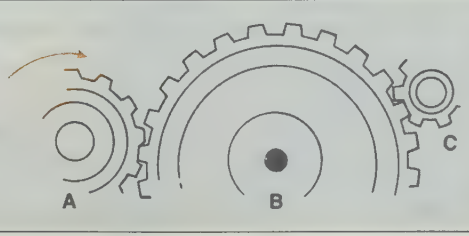


5. Draw a geometric figure on a 10 by 10 co-ordinate grid. Choose any two motions (translation, rotation, reflection). Then, show the final image of your geometric figure after the two motions.

think

Gear **A** has 24 cogs (teeth) and turns clockwise at 20 revolutions per second. Gear **B** has 48 cogs and gear **C** has 12 cogs.

How fast and in which direction do gears **B** and **C** turn?



UTILIZATION

Exercises 1, 2, and 3 provide reinforcement of the ideas of combinations of motions.

In discussing the exercises, it may prove interesting to compare Exercise 4 with the Investigation. Whereas a combination of reflections in a pair of intersecting lines is equivalent to a rotation, a combination of reflections in 2 parallel lines is equivalent to a translation. In this case, the translation is a "right 4, up 4" translation.

EXTENSION

Assign page 65 of the *Duplicator Masters* to provide more practice.

Enrichment: Suggest that students try Exercise 4 again, varying the distance between the parallel lines. See if they can guess the amount of translation for a reflection in two parallel lines. The translation is twice the directed distance between the parallel lines and along a line perpendicular to the parallel lines.

Think Solution

Since gear A rotates clockwise, gear B must turn counterclockwise. This makes gear C turn clockwise. Since gear A has 24 teeth and gear B has 48, gear B will make only 1 revolution for every 2 revolutions of A. Hence, gear B will turn at 10 revolutions per second. Gear C will make 4 revolutions for every one of gear B. Hence, B will turn at 40 revolutions per second.

Assignments

Minimum 1–3. Average 1–4. Maximum 1–5.

Objective

Given a geometric region, the student can determine if the region will tessellate the plane.

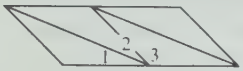
PREPARATION

Materials: Scissors, crayons or colored pencils, index cards, graph paper (*Duplicator Masters*, pages 83, 88).

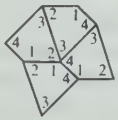
If possible display some pictures of linoleum patterns or wallpaper designs which illustrate tessellations.

INVESTIGATION

Encourage students to use as many different shapes as possible to form tessellations. Every triangle can tessellate the plane because the sum of its angle measures is 180°. By placing 3 triangles so that the three different angles meet at the same vertex a strip region can be formed. Repetition of the strip will tessellate the plane.



Similarly, any quadrilateral will tessellate the plane because the sum of the angle measures of a quadrilateral is 360°. Each of the 4 angles must be placed so as to have a common vertex:



In general, polygons of more than 4 sides will not tessellate the plane, although certain special polygons of more than 4 sides will.

DISCUSSION

The main point of the discussion should be to help students see that every triangle and quadrilateral can be used to tessellate the plane. You may want to discuss this fact, using the arguments given in the Investigation section.

If the student constructs a template of a region, the work of drawing tessellations is simplified. Templates cut from sheets of acetate are easier to align on graph paper in order to draw the tessellation.

Tessellations

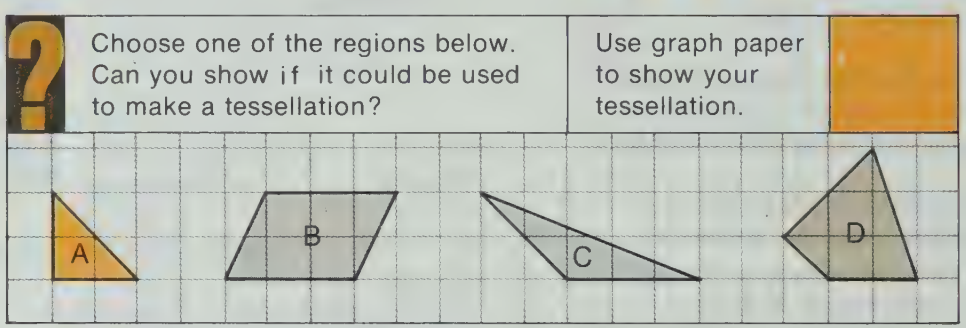
Investigating the Ideas

A **tessellation** is a repeated pattern of geometric figures which will completely "cover a plane" without any gaps or overlapping.

A checkerboard is an example of a tessellation. You can think of continuing the pattern endlessly.

Choose one of the regions below. Can you show if it could be used to make a tessellation?

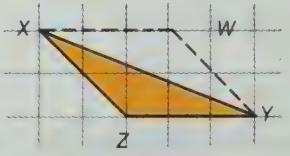
Use graph paper to show your tessellation.



Discussing the Ideas

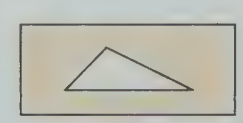
1. Do you think that every triangle can tessellate the plane?

2. In order to see that triangle C above will tessellate the plane, think about turning $\triangle XYZ$ until it is in the position of $\triangle XYW$.
How would you complete the tessellation?

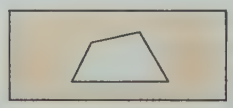


3. Some tessellations may be easier to construct if you would use a **template** or pattern to help you draw the shapes.
Cut out geometric shapes like those shown below from a small index card. Try each of these shapes to form a tessellation.

A



B

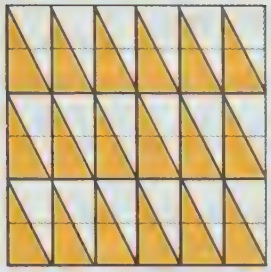


C

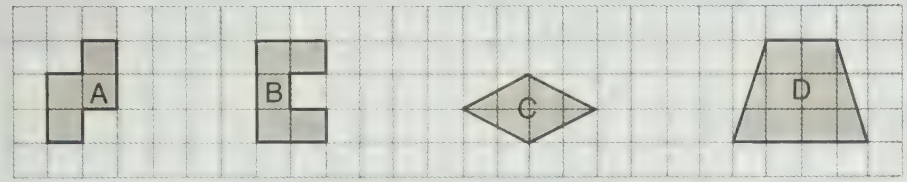
4. Which shape in Exercise 3 do you think will not tessellate the plane?

Using the Ideas

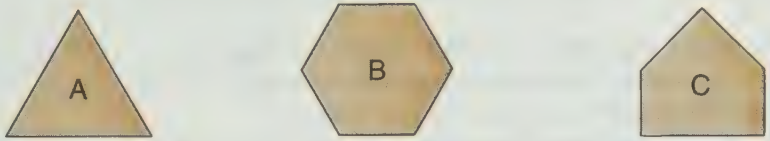
1. The tessellation shown at the right was made from a certain triangle. Graph paper was used to help make the tessellation.
   Show a different tessellation using the same shape and size of triangle



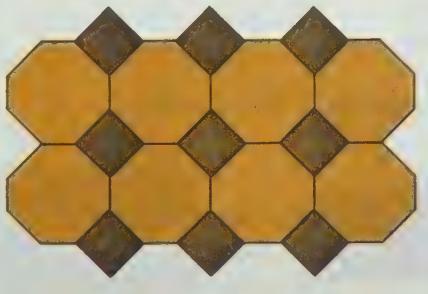
2. Show that each of the shapes will tessellate the plane. Draw enough of the pattern to convince yourself that the pattern will form a tessellation.



3. Make a template for each figure. Use it to form a tessellation.

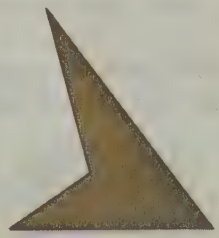


4. The figure below shows a tessellation made up of squares and octagons. Design a tessellation using squares and triangles



think

Can you make a tessellation with this shape?



UTILIZATION

The main purpose of the exercises is to provide practice in making some tessellations. Arranging the regions in different ways and coloring the regions so that the pattern stands out will yield attractive tessellations.

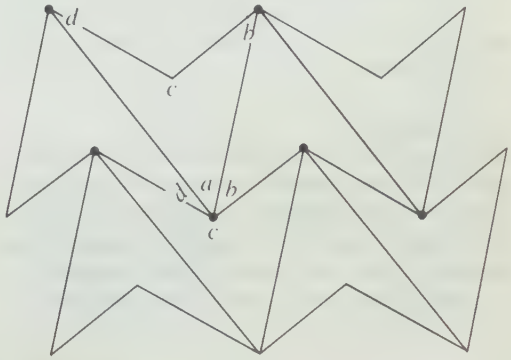
Except in Exercise 4, only tessellations using a single geometric shape are considered. You should encourage students to be creative in their design for Exercise 4.

EXTENSION

Enrichment: This would be an appropriate time to encourage students to undertake Research Project C on page C-99. Making tessellation designs is an activity in which all students should be able to participate. To heighten interest in the project, you might present it as a contest for which each student creates and colors a design to be displayed in the classroom. Then the entire class could vote on which designs are the most original and the most attractive.

Think Solution

Tessellations can be made of any quadrilateral, whether convex or concave. Note that the corners marked *a*, *b*, *c*, and *d* can be arranged to completely cover the region around a point.



Assignments

Minimum 1, 2. Average 1–3. Maximum 1–4.

Objective

Given tessellation patterns, the student can identify certain types of symmetry in the patterns.

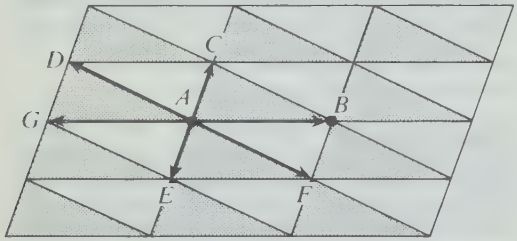
PREPARATION

Materials: Plastic mirrors, tracing paper.

This lesson is an extension of the previous lesson. You can distribute the materials for the Investigation and have students begin work on it at once.

INVESTIGATION

The students should quickly discover at least 6 translations that will make the tessellation pattern fit on itself again. If we imagine sliding point *A* to any one of the six points *B*, *C*, *D*, *E*, *F*, or *G* shown below, the pattern will fit on itself.



There are actually infinitely many such tessellations since the translation could be made by any multiple of the lengths of these six arrows of translation.

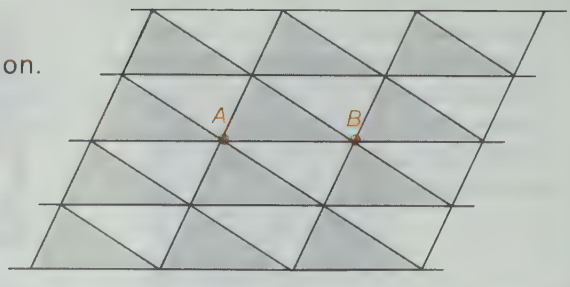
DISCUSSION

To check for reflectional symmetry, the student will need his plastic mirror. Students can use their tracing of the tessellation to check for rotational symmetry by placing a pencil point on the tracing on the centre of rotation and turning the tracing until it fits the pattern again.

Tessellations and Motions

Investigating the Ideas

Make a copy of this tessellation. on tracing paper. If you slide your copy from *A* to *B*, the tracing will again fit on the tessellation. (Assume that the tessellation covers the complete plane.)



How many slides in different directions along the lines can you find that will make the tracing fit the tessellation again?

Do not turn or flip the tracing!

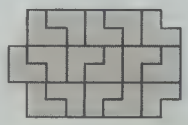
Discussing the Ideas

1. The tessellation above has **translational symmetry** because certain slides (translations) of the points of the tessellation makes the tessellation pattern fit again on itself. Do you think that the tessellation has **reflectional symmetry**?

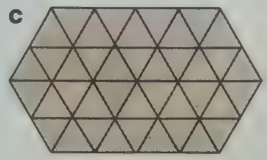
2. A Can you find a translation of the tessellation that is not along one of the lines so that the pattern falls again on itself? Describe the translation.
   B Can you find the shortest translation that is not along a line that makes the tessellation fall again on itself?

3. A Is point *A* a centre of rotation for a turn that will fit the tessellation back on itself? If it is, then the pattern has **rotational symmetry**.
   B What other centres of rotational symmetry can you find?

4. What kind of symmetry (**rotational**, **translational**, **reflectional**) can you find in these tessellations?

A 

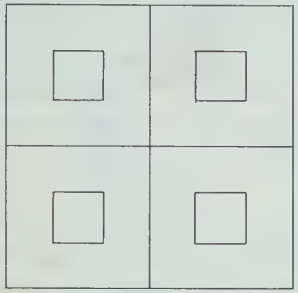
B

C 

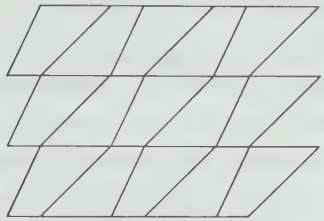
Using the Ideas

A small portion of a tessellation is shown for Exercises 1–6. Tell which kinds of symmetry (**rotational**, **translational**, or **reflectional**) each figure has.

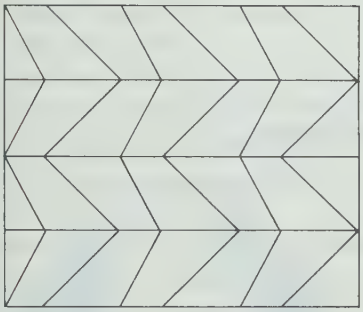
1.



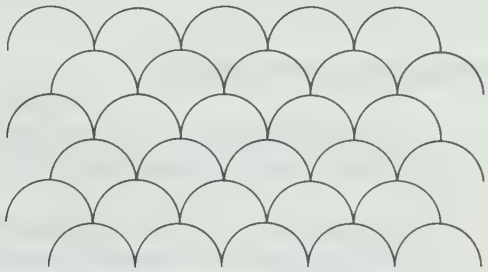
2.



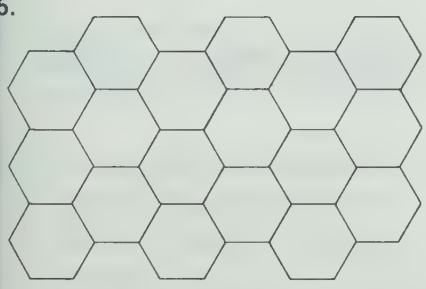
3.



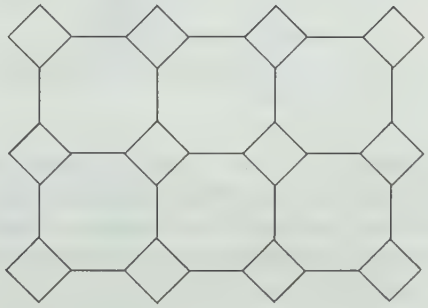
4.



5.



6.



7. Design and color some tessellations of your own. Find what kind(s) of symmetry your tessellations have.

UTILIZATION

The exercises provide practice in recognizing various types of symmetry. Use tracing paper and the plastic mirrors to check for symmetry. In general, give this material a light touch.

Encourage students to design and color their own tessellations, as suggested in Exercise 7, if they have not already done so in connection with the Enrichment suggestion for the preceding lesson. If students have already created a variety of designs, encourage them to identify the kind(s) of symmetry in the tessellations.

EXTENSION

Enrichment: Make a collection of wallpaper patterns, cloth designs, or linoleum patterns that illustrate designs with reflectional, rotational, or translational symmetry.

Assignments

Minimum 1–4. Average 1–4, 7.
Maximum 1–7.

Objective

The student can name corresponding parts of congruent figures that are images of each other by one of the rigid motions.

PREPARATION

Materials: Tracing paper, plastic mirror, graph paper or co-ordinate plane paper (*Duplicator Masters,* pages 83, 85).

Since no new material is introduced in the Investigation, students can begin it without any introductory comments, after the necessary materials have been distributed.

INVESTIGATION

The purpose of the Investigation is to review the three rigid motions and observe that each motion produces an image that is the same size and shape as the original figure.

DISCUSSION

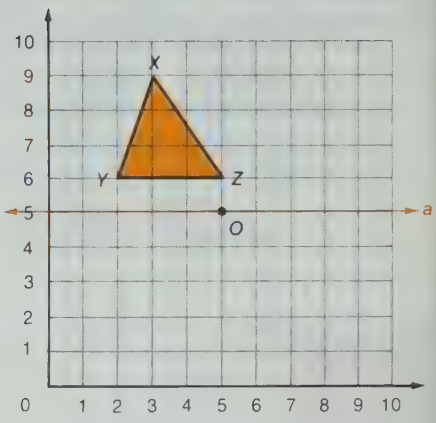
It is quite likely that many of your students will already be familiar with congruence of triangles from previous work in geometry. If so, you may proceed quickly through the discussion.

Exercise 3 brings out the correspondence of specific parts that is indicated by the order of the symbols we use to denote congruent triangles. In discussing Exercise 4, it should be observed that congruent segments, which were studied earlier, are simply segments having the same length. Similarly, congruent angles are angles having the same degree measure. Also, in Exercise 4, emphasize again the implications of the notation $\triangle ABC \cong \triangle DEF$: $\angle A \cong \angle D$, $\angle B \cong \angle E$, $\angle C \cong \angle F$, $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\overline{AC} \cong \overline{DF}$.

Rigid Motions and Congruence

Investigating the Ideas

Draw $\triangle XYZ$ and line ***a*** through point O on a 10 by 10 grid.



Can you find the images of $\triangle XYZ$ for each of the rigid motions below?

1. The reflection image in line ***a***.
2. The slide image after a "right 3, down 6" translation.
3. The rotational image after a $\frac{1}{4}$-clockwise turn about centre O.

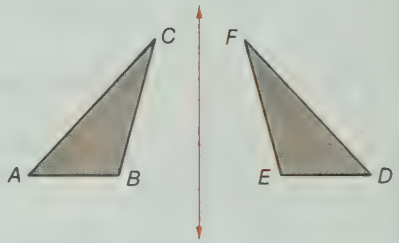
Discussing the Ideas

1. Are the three images that you found in the Investigation the same size and shape as $\triangle XYZ$?

2. Two geometric figures are **congruent** if one of the figures is the image of the other by one of the rigid motions, a reflection, a translation, or a rotation, or by any combination of rigid motions.

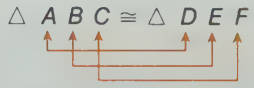
   A Which rigid motion makes $\triangle DEF$ an image of $\triangle ABC$?

   B What is the image of each vertex of $\triangle ABC$?

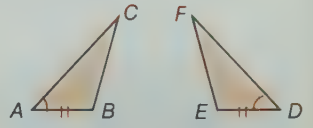


$\triangle ABC$ is congruent to $\triangle DEF$.
We write: $\triangle ABC \cong \triangle DEF$

3. When you write $\triangle ABC \cong \triangle DEF$, you pair the **corresponding points** that are images of each other. (See diagram at right.) Which points are images of each other if $\triangle RTS \cong \triangle QEK$?

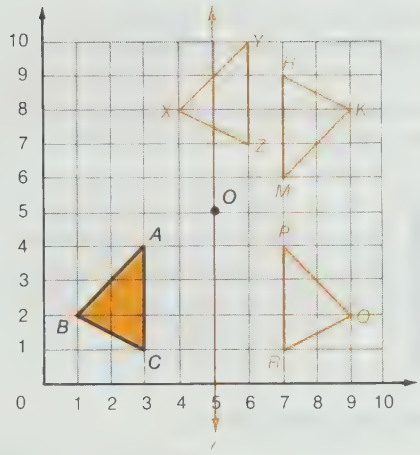


4. If $\triangle ABC \cong \triangle DEF$, then **corresponding segments** and **corresponding angles** are congruent. Thus, $\overline{AB} \cong \overline{DE}$ and $\angle A \cong \angle D$. What other pairs of segments and angles are congruent?



Using the Ideas

Use the co-ordinate grid at the right for Exercises 1, 2, and 3.



1. **A** $\triangle ABC \cong \triangle YXZ$ because $\triangle YXZ$ is a translation image of $\triangle ABC$ by a translation of "right ▮▮▮, up ▮▮▮."

B $\angle A \cong \angle$▮▮▮ $\quad \angle B \cong \angle$▮▮▮ $\quad \angle C \cong \angle$▮▮▮

C $\overline{AB} \cong$ ▮▮▮ $\quad \overline{BC} \cong$ ▮▮▮ $\quad \overline{AC} \cong$ ▮▮▮

2. $\triangle HKM$ is the rotation image of $\triangle ABC$ by a $\frac{1}{2}$-turn about point O.

A Which pairs of segments are congruent?

B Which pairs of angles are congruent?

3. $\triangle PQR$ is the reflection image in line ℓ of $\triangle ABC$.

A Which pairs of segments are congruent?

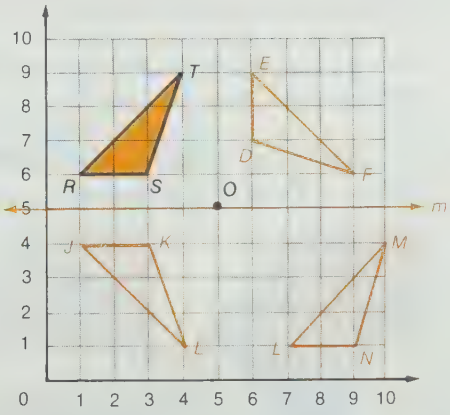
B Which pairs of angles are congruent?

4. Describe the rigid motion that makes

A $\triangle JKL$ an image of $\triangle RST$.

B $\triangle EDF$ an image of $\triangle RST$.

C $\triangle LMN$ an image of $\triangle RST$.



5. Write a statement of congruence for $\triangle RST$ and each of the other triangles.

6. Draw a quadrilateral on a grid.

A Show a translation image.

B Show a rotation image about a point.

C Show a reflection image in a line.

D Write a statement of congruence between the quadrilateral and its images.

UTILIZATION

The exercises stress the idea that the image of a figure under one of the rigid motions is congruent to the original figure. Furthermore, the exercises provide practice in recognizing congruent parts, that is, the corresponding sides and corresponding angles of congruent figures.

Exercise 6 introduces congruence for a quadrilateral. If quadrilateral $ABCD$ has an image $EFGH$ by some rigid motion, we can write quad $ABCD \cong$ quad $EFGH$ and this means that pairs of vertices are images of each other by the correspondences

$A \leftrightarrow E$, $B \leftrightarrow F$, $C \leftrightarrow G$, and $D \leftrightarrow H$.

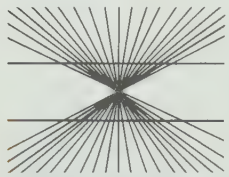
With this agreement, we can read the pairs of congruent angles and pairs of congruent sides.

EXTENSION

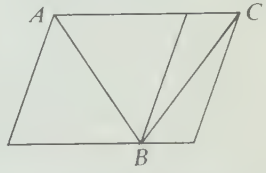
Additional practice is provided in the *Workbook* on page 87 and on page 66 of the *Duplicator Masters*.

Remedial: Write some statements of congruence between pairs of triangles, but do not represent them pictorially. Have the students tell which pairs of angles and which pairs of sides are congruent.

Enrichment: It is often difficult for students to understand why visual inspection is not always sufficient for determining whether two geometric figures are congruent. Optical illusions, such as those below, might be displayed on the chalkboard to show students how inaccurate their visual perceptions can be. Ask your students to answer the questions accompanying the figures.



Are the heavy lines straight, or are they curved?



Which is longer, $\overline{AB}$ or $\overline{BC}$?

Assignments

Minimum 1–3. Average 1–5. Maximum 1–6.

Objective

The student can use the pieces of the tangram puzzle to construct congruent regions.

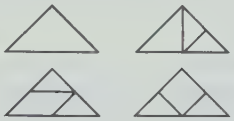
PREPARATION

Materials: Scissors, tracing paper, poster board.

If possible, prepare duplicated copies of the tangram puzzle of the same size pictured in the text. Distribute these copies and have students paste the shapes to poster board and cut out the pieces. Otherwise, have the students trace the pieces from the book and cut them out.

INVESTIGATION

The students should make a drawing of each arrangement they find that is congruent to one of the large triangles. There are essentially four possibilities, including the other large triangles, as shown below.



DISCUSSION

The Discussion Exercises suggest the construction of several other pairs of congruent figures. Students should superimpose one figure on another to check for congruence.

The trapezoids that the students may construct for Exercise 7 may vary in size. Three possibilities are shown below. No two are congruent.

The Tangram Puzzle

Investigating the Ideas

These seven regions are the pieces of an ancient puzzle called the Tangram Puzzle.

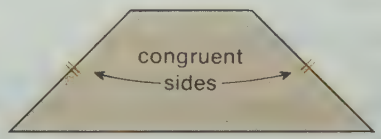
Trace the pieces and cut them out.

How many different ways can you find to make a triangle that is congruent to one of the large triangles?

Discussing the Ideas

1. Are the two large triangles congruent? How can you decide?
2. What other pair of pieces of the tangram puzzle are congruent?
3. Which pieces will form a square congruent to the small square piece?
4. Can you make a figure congruent to the parallelogram? Which pieces must you use?
5. Form a triangle congruent to the middle-sized triangle. Which pieces must you use?
6. A trapezoid has a pair of parallel sides. Use the tangram pieces to show two congruent trapezoids like the ones below.

7. Can you form an **isoceles trapezoid** from some of the tangram pieces?

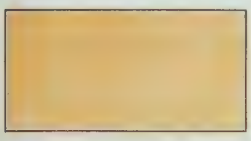


Isoceles trapezoid

Using the Ideas

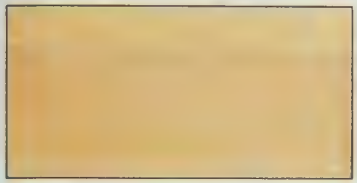
Make shapes congruent to each of these below using the tangram pieces that you cut out for the Investigation. Make a sketch to show your solution.

A



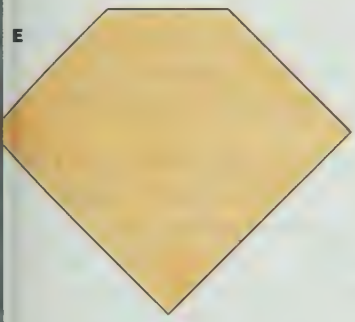
B

C

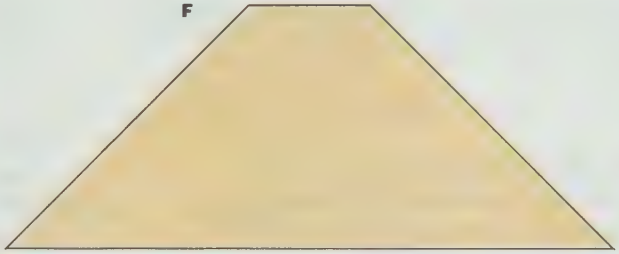


D

E



F

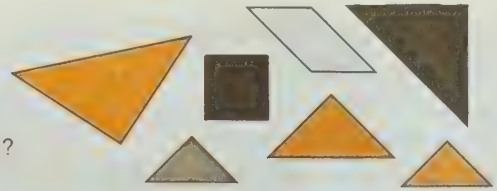


Use one of the small triangles and the square to form a shape. Use other pieces to form a shape congruent to the first shape.

Make a square using the two large triangles. Then make a square congruent to it by using the remaining pieces.

Think

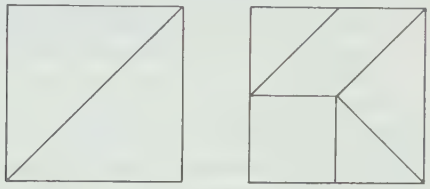
How many different-sized squares can you form using two or more of the tangram pieces for each square?



UTILIZATION

Each of the regions in Exercise 1 can be constructed in more than one way. In part A, for example, the rectangle can be formed by using the square and the two small triangles or by using the middle-sized triangle and the two small triangles.

Solution, Exercise 3



EXTENSION

Enrichment: Ask students to use all 7 tangram pieces to make the figures described below.

A. Rectangle
B. Parallelogram
C. Right triangle
D. A design of your own choice

Think Solution

Assignments

Minimum 1. Average 1–3. Maximum 1–3.

Objective

The student can make the following constructions by means of reflections: bisector of an angle, a perpendicular bisector of a segment, a perpendicular to a line through a given point.

PREPARATION

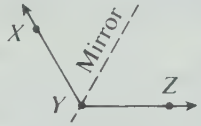
Materials: Plastic mirrors.

Since reflections have been studied earlier in the module, only a brief review of reflections may be needed before the Investigation.

INVESTIGATION

The students should quickly discover how to use their plastic mirror to locate the position of the bisector of the angle.

If $\angle XYZ$ is any angle, the bisector of the angle is located by placing the edge of the mirror on point Y, and rotating the mirror until $\overrightarrow{YX}$ and $\overrightarrow{YZ}$ are reflection images of each other. The edge of the mirror is on the bisector of the angle.



DISCUSSION

Discussion Exercise 3 introduces the *perpendicular bisector* of a segment. Have the students practice this construction with several segments of different length and in various positions. When the mirror is placed on $\overline{AB}$ so that A and B are reflection images of each other, the mirror edge lies along the perpendicular bisector of the segment.

The construction in Exercise 4 is also very simple. For part A the mirror is placed on point R so that the line is reflected onto itself. For part B, the mirror is placed on S so that the line is again reflected onto itself. The edge of the mirror, in each case, is the perpendicular to the line through the given point.

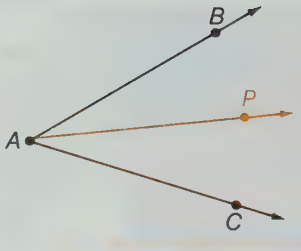
Reflection Constructions

Investigating the Ideas

Ray AP is the **bisector of** $\angle BAC$ because it divides $\angle BAC$ into two congruent angles.

We say: $\overline{AP}$ bisects $\angle BAC$

Draw some angles on your paper.

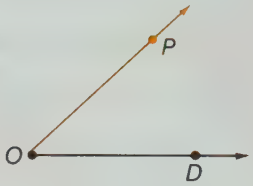


Can you find a way to use a plastic mirror and reflections to bisect the angles?

Discussing the Ideas

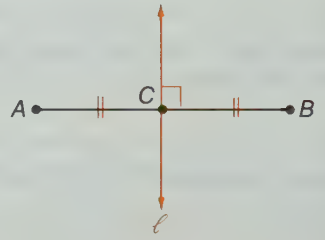
1. Explain how you found where to draw the bisector of your angle?

2. If $\overrightarrow{OP}$ is the bisector of $\angle DOH$, how can you use reflections to find the other ray, $\overrightarrow{OH}$, of $\angle DOH$?

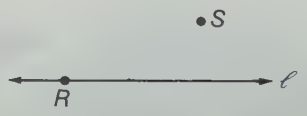


3. Line ℓ is the **perpendicular bisector** ($\perp$ bis) of $\overline{AB}$ at point C because $\ell \perp \overline{AB}$ and $\overline{AC} \cong \overline{BC}$.

   Draw a segment. Then explain how to use reflections to construct the "$\perp$ bis" of the segment.



4. Explain how to use reflections to make these constructions:
   - **A** Construct a line through point R which is perpendicular to ℓ.
   - **B** Construct a line through point S which is perpendicular to ℓ.



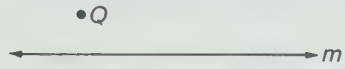
Using the Ideas

1. Draw a segment. Use reflections to construct the perpendicular bisector of the segment.

2. Draw an angle. Find the bisector of the angle by using reflections.

3. Construct a line through point P which is perpendicular to line ℓ.

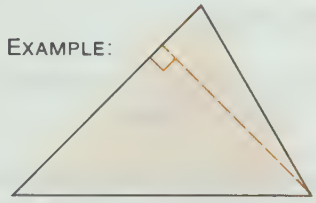
4. Construct a line through point Q which is perpendicular to line m.



5. Draw a triangle. Find the three bisectors of the angles using reflections.

6. Draw a triangle. Construct the perpendicular bisectors of the sides using reflections.

7. Draw a triangle. Construct the three altitudes of the triangle by using reflections. (An **altitude** of a triangle is a segment from one vertex perpendicular to the opposite side of the triangle.)



★ 8. Draw a segment and label it $\overline{AB}$. Then use reflections to construct an equilateral triangle with sides congruent to $\overline{AB}$. (HINT: First construct the $\perp$ bis of $\overline{AB}$.)

think

In what year did Julius Jupiter die? (Half of the date is missing.) How old was Julius when he died?

UTILIZATION

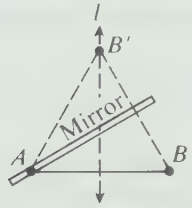
Because of distortions resulting from bending or misalignment of the plastic mirrors, the constructions suggested on this page may not be very accurate. However, the main purpose of the exercises is for the students to see how properties of reflection can be used to make constructions that may be difficult when ruler and compass are used. Stress the ideas involved rather than accuracy of construction.

Solution, Exercise 8

A. Construct the $\perp$ bis of $\overline{AB}$; call it ℓ.

B. Place the edge of the mirror on point A. Turn it until the reflection image of B falls on ℓ. Call the image B^1. $\overline{AB^1} \cong \overline{AB}$.

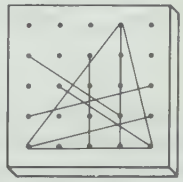
C. Since B^1 is on the $\perp$ bis of $\overline{AB}$, $\overline{B^1B} \cong \overline{AB^1}$. Therefore, $\triangle AB B^1$ is equilateral.



EXTENSION

To provide further practice, assign page 99 in the *Workbook*.

Remedial: Guide students in constructing the altitudes and perpendicular bisectors of the sides of some triangles on the geoboard. Points of concurrency of the segments are shown by the intersecting rubber bands.



Think Solution

The idea of reflections should suggest the missing lower half of the date. The date of death must be DCXXII or 622. Since the birthdate was DLXXI or 571, Julius died at the age of 51.

Assignments

Minimum 1–4. Average 1–6. Maximum 1–8.

Objective

The student can construct an angle congruent to a given angle and can bisect an angle using ruler-and-compass methods.

PREPARATION

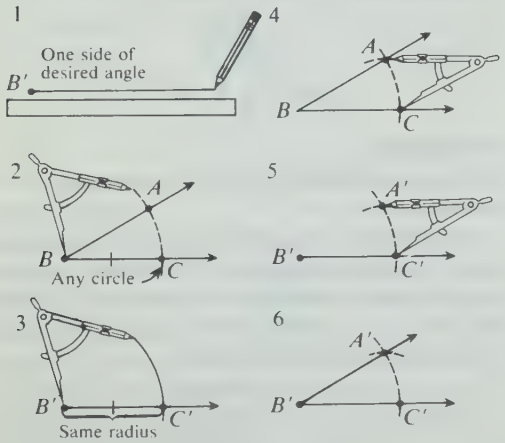
Materials: Rulers and compasses.

The rules governing the use of the compass and straightedge in construction are simple.

1. The straightedge can be used to draw a line through two given points. No marks can be made on the straightedge.
2. The compass can be used to draw a circle through a given point and with a given radius or through another point.

INVESTIGATION

Illustrations 1 through 6 show the sequence of steps of the construction where $\angle B$ is the given angle.



DISCUSSION

In discussing Exercise 2, ask the following questions as an oral activity:

(1) Why is $\overline{BD} \cong \overline{BE}$? (They are both radii of a circle with center B.)

(2) Why is $\overline{BF} \cong \overline{BF}$? (They are the same.)

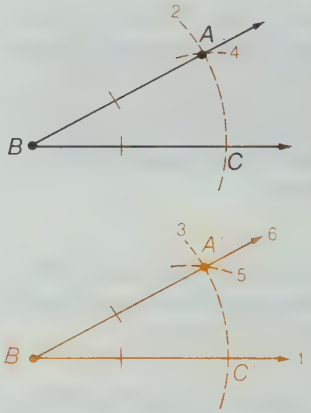
(3) Why is $\overline{DF} \cong \overline{EF}$? (They are both radii of circles with congruent radii. Therefore, since corresponding segments $\overline{DF}$ and $\overline{EF}$ are congruent, $\angle DBF \cong \angle EBF$ so that $\angle ABC$ has been bisected.)

Ruler-and-Compass Constructions of Angles

Investigating the Ideas

Drawings of geometric figures using only a straightedge and compass, and of course, a pencil are called **ruler-and-compass constructions**. A straightedge can be thought of as a ruler without units marked on it.

The numbered steps on the figure at the right shows how $\angle A'B'C'$ was constructed so that it is congruent to $\angle ABC$; that is, $\angle A'B'C' \cong \angle ABC$.

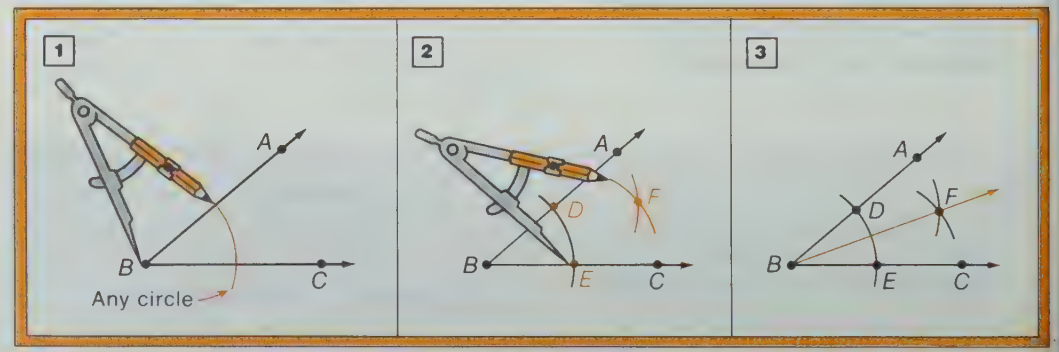


Can you draw an angle on your paper and then construct an angle congruent to it using a straightedge and compass?

Discussing the Ideas

1. Explain the six steps for constructing an angle congruent to a given angle.

2. The illustration below shows three steps needed to bisect $\angle ABC$ using a straightedge and compass. Draw an angle on your paper and then bisect it with your straightedge and compass.

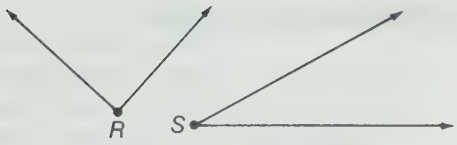


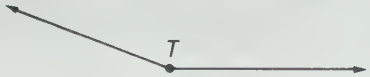
Using the Ideas

1. **A** Draw three angles about the same size as angles R, S, and T.

B Construct angles R', S', and T' congruent to angles R, S, and T respectively.

C Bisect each of the constructed angles R', S', and T'.

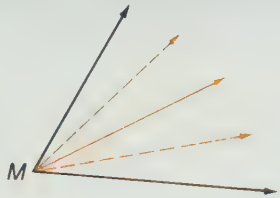




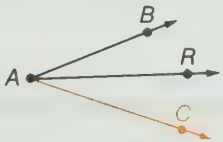
2. Draw an angle M.

A Bisect it.

B Bisect each of the two new angles formed.

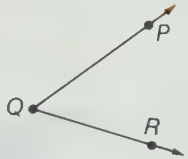


3. Ray AB is one side of $\angle BAC$. Ray AR is the bisector of $\angle BAC$. Show a construction that will give the other side ($\overrightarrow{AC}$) of $\angle BAC$.



4. **A** Draw an angle about the size of $\angle PQR$, and label it $\angle STW$.

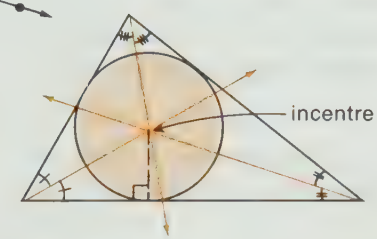
B Construct an angle twice the "size" of $\angle STW$.



5. **A** Draw a large triangle on your paper.

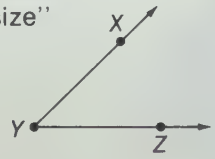
B Construct the bisector of each angle of the triangle.

C If you make the constructions for part **B** carefully, you should find that the three bisectors intersect in one point called the **incentre** of the triangle. The incentre is the centre of a circle that just touches all three sides of the triangle. Draw that circle in your triangle.



6. **A** Draw $\angle XYZ$.

B Construct an angle $1\frac{1}{2}$ times the "size" of $\angle XYZ$.



think

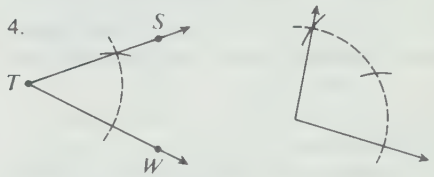
Arrange ten points in five rows with four points in each row.

UTILIZATION

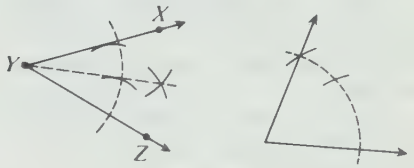
Students should be encouraged to work as accurately and neatly as possible in making the constructions, but mastering the *methods* of construction should be the main goal of this lesson.

Solutions, Exercises 4 and 6

4.



6.



EXTENSION

The exercises on page 67 of the *Duplicator Masters* can be used to provide extra practice with the concepts presented in this lesson.

Enrichment: Have the students draw a large triangle on their papers and then copy the 3 angles so that successive pairs of angles are adjacent to each other. The resulting construction should reinforce their understanding of the fact that the sum of the measures of the angles of a triangle is 180°.

Encourage interested students to undertake Research Project A on page C-99.

Think Solution

The value of this problem is that it encourages students to expand their range of reference. Although it is customary to think of rows in terms of horizontal and vertical lines, this limited interpretation cannot be used to meet the requirements of this problem.

Assignments

Minimum 1–3. Average 1–5. Maximum 1–6.

Objective

The student can construct a triangle congruent to a given triangle by using ruler-and-compass methods.

PREPARATION

Materials: Rulers and compasses.

Ruler-and-compass construction of congruent triangles in this lesson is based on the geometric theorem that if two triangles have their corresponding sides congruent, then the triangles are congruent.

INVESTIGATION

Have the students begin by drawing $\triangle ABC$ on their paper. Now instruct them to study the drawings and try to discover the steps in making the construction. You may need to help some students get started by suggesting that they draw a line and mark off $\overline{A'C'} \cong \overline{AC}$ on the line. The remaining steps then may be found by analyzing the drawings.

DISCUSSION

If feasible, have some students give chalkboard demonstrations and explanations of the various constructions in the Discussion Exercises.

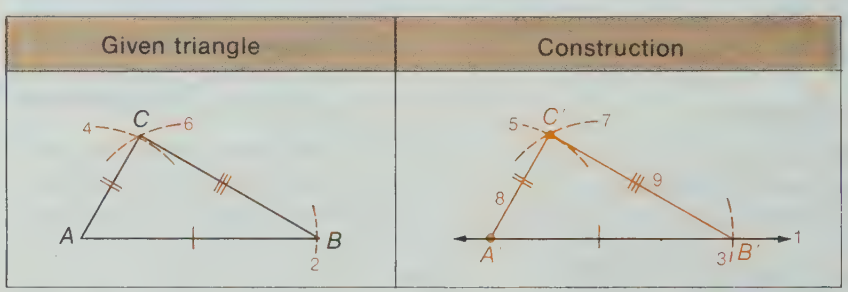
Exercise 2 suggests a method for constructing an equilateral triangle. In Exercise 3, you might draw two segments on the chalkboard and then have some students construct an isosceles triangle having two sides congruent to one of the segments and the third side congruent to the other segment.

Note in Exercise 4 that, since a triangle is a rigid figure, all triangles having sides congruent to the given segments will be congruent triangles.

Ruler-and-Compass Constructions of Triangles

Investigating the Ideas

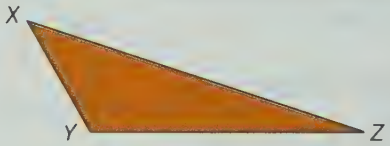
In the construction shown below, $\triangle A'B'C'$ was drawn so that $\triangle A'B'C' \cong \triangle ABC$. Study and follow the steps to see how the construction of $\triangle A'B'C'$ was made.



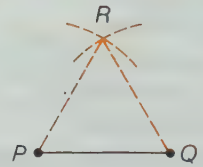
Can you draw a triangle and then construct a triangle congruent to it using a straightedge and compass?

Discussing the Ideas

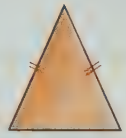
1. Draw a triangle like $\triangle XYZ$. Explain how to construct a triangle congruent to $\triangle XYZ$.



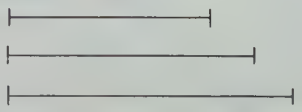
2. Study the construction shown. What kind of triangle is $\triangle PQR$?



3. Show how to draw an isosceles triangle.

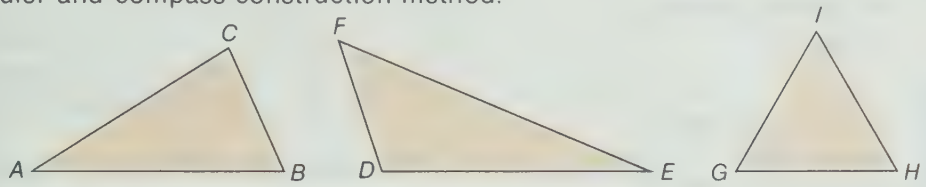


4. Show and explain how to construct a triangle whose sides will be congruent to the three given segments.



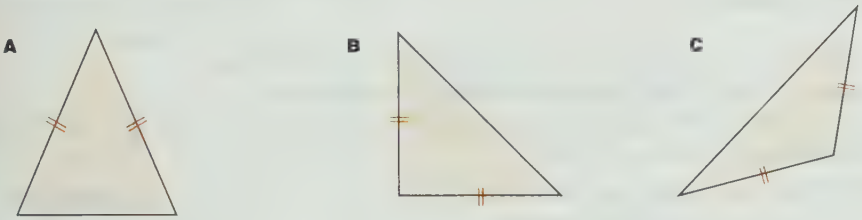
Using the Ideas

1. Draw three different-shaped triangles like those shown below. Then construct a triangle congruent to each of them using the ruler-and-compass construction method.

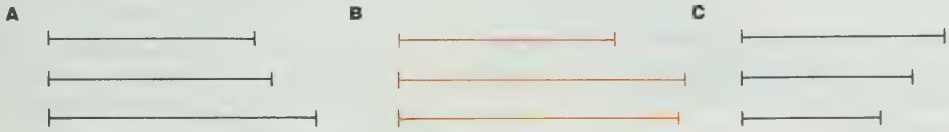


2. Construct an equilateral triangle. Make each side four centimetres long.

3. Construct isoceles triangles congruent to those below.



4. Construct triangle whose sides are congruent to the three given segments.



★ 5. Can you draw three segments that cannot be used as sides of a triangle?

★ 6. The designs below are based on equilateral triangles and circles. Make a copy of at least one of them. Invent some designs of your own.

UTILIZATION

Most students should be able to complete Exercises 1 through 4.

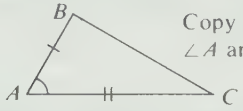
For Exercise 5, students should observe that a triangle cannot be constructed from 3 given segments if the length of the longest segment is greater than the sum of the lengths of the other two segments.

EXTENSION

Further practice with the concepts of this lesson is provided in the *Workbook* on page 100.

Enrichment: There are two other methods of constructing triangles congruent to a given triangle. One is based upon the Side-Angle-Side (SAS) Congruence Theorem, and the other is based upon the Angle-Side-Angle (ASA) Congruence Theorem.

SAS Construction:



Copy $\triangle ABC$ by copying $\angle A$ and sides AB and AC.

ASA Construction:

Copy $\triangle RST$ by copying side ST, $\angle S$, and $\angle T$.

Now would be a good time to recommend Research Project B on page C-99 for students who would find it challenging to explore some further ruler-and-compass constructions.

Assignments

Minimum 1–4. Average 1–4. Maximum 1–6.

Objective

The student can construct perpendicular lines with compass and straightedge.

PREPARATION

Materials: Straightedge and compass.

An earlier lesson explored construction of perpendicular lines by means of reflections. Students may find it interesting to compare that method with the one presented in this lesson.

INVESTIGATION

The purpose of the Investigation is to give students the opportunity to discover for themselves a straightedge-and-compass method of constructing the perpendicular bisector of a segment. Since the construction of perpendicular bisectors has been presented previously, most students should recognize the relationship between $\overline{AB}$ and ℓ.

DISCUSSION

The numerals on the drawings suggest the order of the steps of the constructions in Exercises 2 and 3. Some of your students may be able to understand a verbal argument that the constructed lines must be perpendicular:

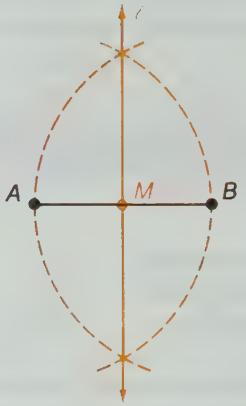
> Call the intersection of steps 3 and 4 point Q. Then $\overline{AP} \cong \overline{BP}$, $\overline{PQ} \cong \overline{PQ}$, and $\overline{AQ} \cong \overline{BQ}$. Therefore $\triangle APQ \cong \triangle BPQ$. Hence the adjacent angles, $\angle APQ$ and $\angle BPQ$, are congruent. The lines are perpendicular because the adjacent angles formed are right angles.

A similar argument can justify the construction for Exercise 3.

Constructing Perpendicular Lines

Investigating the Ideas

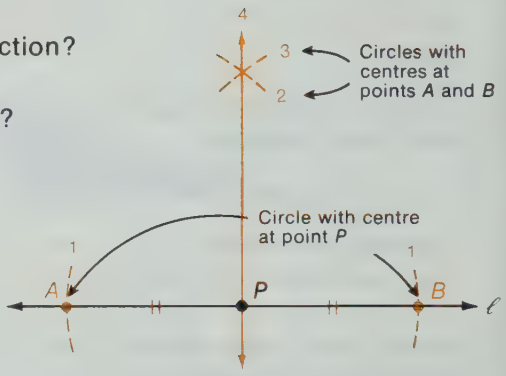
Study the construction at the right.



Can you start with a segment, $\overline{AB}$, and make a construction like this with your straightedge and compass?

Discussing the Ideas

1. The figure in the Investigation suggests a ruler-and-compass construction of a line that is **perpendicular** to a segment. The line also bisects the segment; thus, line ℓ is called the **perpendicular bisector** of $\overline{AB}$.
   - **A** Explain how you made your construction?
   - **B** What kind of angles were formed?
   - **C** What can be said about $\overline{AM}$ and $\overline{MB}$?

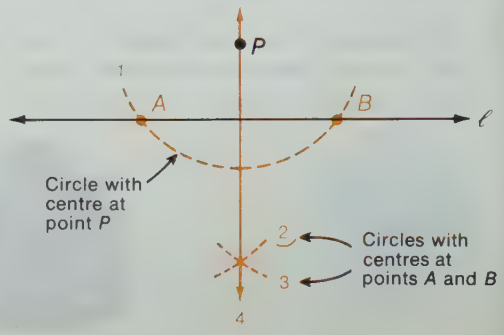


2. Study the construction for drawing "a perpendicular to a line ℓ through a point P *on the line.*"

   Make a construction like this one.

3. Study the construction for drawing "a perpendicular to a line ℓ through a point P which is **not** on the line."

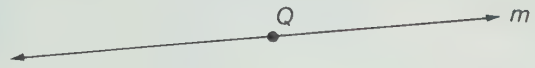
   Make a construction like this one.



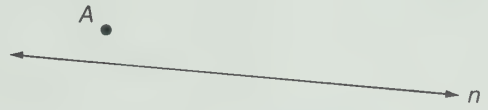
Using the Ideas

1. Draw a segment. Then construct a perpendicular bisector of the segment.

2. Draw a line and call it m. Mark a point Q on it and construct a line through Q that is perpendicular to line m.

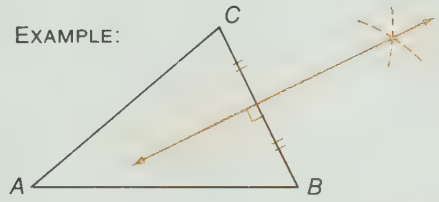


3. Draw a line n and mark a point A not on n. Construct a line through A that is perpendicular to line n.



4. Draw a large triangle. Construct the perpendicular bisectors of each side of the triangle.

EXAMPLE:

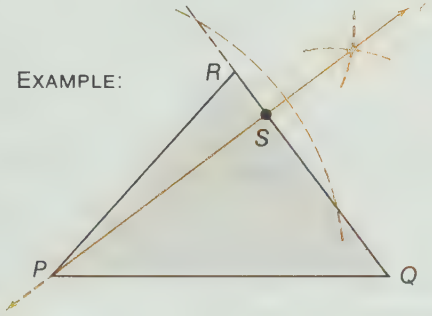


5. **A** Draw a triangle like $\triangle PQR$.

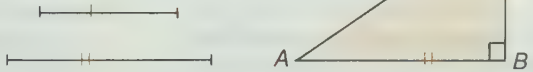
   B Construct a line ℓ through point P such that $\ell \perp \overleftrightarrow{QR}$. Let point S be the point where line ℓ intersects $\overline{QR}$. $\overline{PS}$ is one of the **altitudes of** $\triangle PQR$.

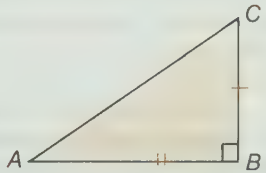
   C Construct the other two altitudes of $\triangle PQR$.

EXAMPLE:

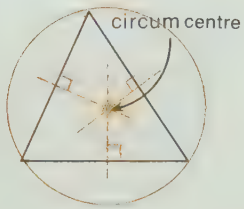


6. Construct a right triangle with legs congruent to these segments.





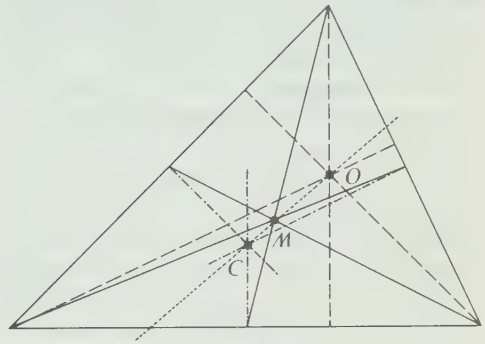
7. The intersection of the perpendicular bisectors of the sides of a triangle (see Exercise 4) is the centre of a circle which will contain the vertices of the triangle. The centre of the circle is the **circumcentre** and the circle is said to **circumscribe** the triangle. Draw a triangle and find the circumcentre. Then construct the circle that circumscribes the triangle.



UTILIZATION

The exercises provide practice in constructing perpendiculars to lines in various situations.

If the construction in Exercise 5 is made carefully, the three altitudes will be *concurrent;* that is, they intersect in a common point called the orthocentre. If the triangle has an obtuse angle, the orthocentre will be in the exterior of the triangle.



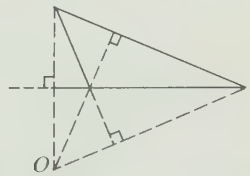
EXTENSION

For more practice, page 97 of the *Workbook* and page 68 in the *Duplicator Masters* may be assigned.

Enrichment: Give students directions for the following construction:

Draw a large triangle on your paper. Carefully construct the three medians to find the centroid of the triangle. Next construct the three altitudes and find the *orthocentre*. Then construct the three perpendicular bisectors of the sides and find the circumcentre.

The three points of concurrency will always be collinear regardless of the shape of the triangle. In an equilateral triangle these points are identical.



Assignments

Minimum 1–4. Average 1–6. Maximum 1–7.

Objective

The student can demonstrate the ability to work with the concepts presented in this module.

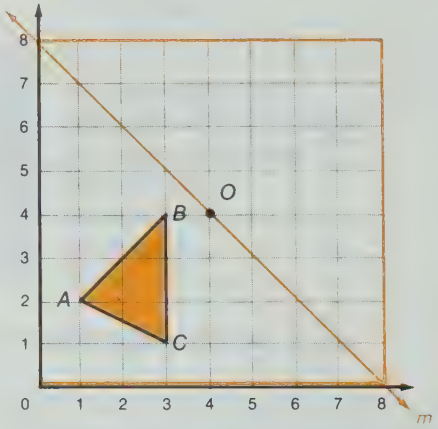
UTILIZATION

Exercises 1 through 7 may be assigned for individual or group review. Have some students demonstrate the constructions called for in Exercises 10, 11, and 12. A discussion of these exercises may strengthen some students' understanding of the ideas of this module.

Further appropriate review exercises are provided on page 102 of the *Workbook*.

REVIEWING THE IDEAS

For each Exercise 1, 2, 3, and 4, refer to the figure on the grid in its original position.



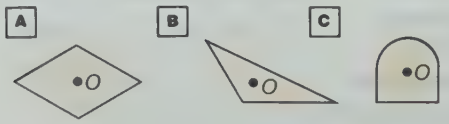
1. Give the co-ordinates of the vertices of $\triangle ABC$.

2. Give the co-ordinates of the images of A, B, and C after a translation of "right 2, up 1."

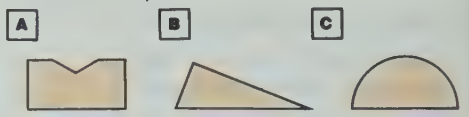
3. Give the co-ordinates of the images of A, B, and C after the square grid has been given a $\frac{1}{4}$ clockwise rotation with center at point O.

4. Give the co-ordinates of the reflection images of A, B, and C of $\triangle ABC$ in line m.

5. Which figure has rotational symmetry?



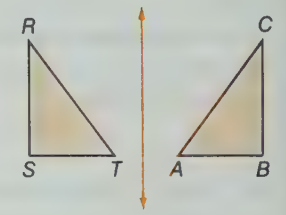
6. Which region could be used to tessellate a plane?



7. Complete the statement:

   If two geometric figures are images of each other by some kind of rigid motion, then the two figures are ?

8. $\triangle RST$ is the reflection image of $\triangle ABC$.

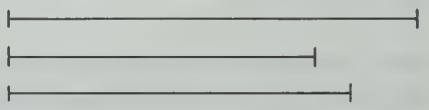


   Write a statement of congruence about the triangles.

9. Suppose $\triangle JKL \cong \triangle GEF$.
   - A What pairs of sides are congruent?
   - B What pairs of angles are congruent?

10. Draw an angle, then find its bisector using
    - A reflections.
    - B ruler-and-compass construction.

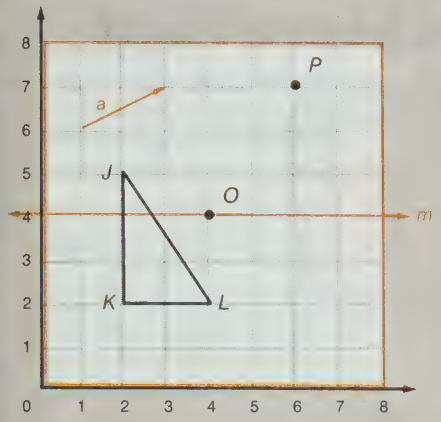
11. Construct a triangle whose sides are congruent to these three segments.



12. Draw a segment. Then construct the perpendicular bisector of the segment using
    - A reflections.
    - B ruler-and-compass construction.

TEST YOURSELF

Refer to the grid for Exercises 1–4.



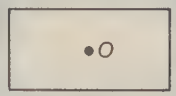
1. Give the co-ordinates of the vertices of $\triangle JKL$.

2. Give the co-ordinates of the images of J, K, and L after a translation by arrow **a**.

3. Give the co-ordinates of the image of point P after a $\frac{1}{2}$ turn with point O as the centre of rotation.

4. Give the co-ordinates of the reflection image of point P in line m.

5. What is the order of rotational symmetry for the rectangle with centre at point O?



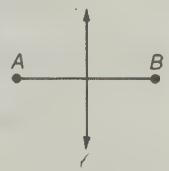
6. **True** or **false**:

   Any triangle can form a tessellation of the plane.

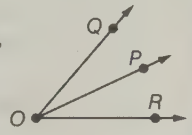
7. If $\triangle XYZ \cong \triangle MOP$, then
   A $\angle Z = \angle$ ▨ B $\overline{XY} \cong$ ▨

Complete the statements for Exercises 8, 9, and 10.

8. If A and B are reflection images of each other in line ℓ, then line ℓ is the ? ? of $\overline{AB}$.



9. If $\overrightarrow{OP}$ bisects $\angle QOR$, then $\overrightarrow{OR}$ is the ? image of $\overrightarrow{OQ}$ in $\overrightarrow{OP}$.



10. Geometric constructions can be made using a ? and straightedge.

RESEARCH PROJECTS

A Any angle can be bisected with a ruler and compass but **trisection** (dividing into three congruent parts) of a general angle using only a ruler and compass has been proved to be impossible. Other devices have been invented that enable us to trisect an angle. Make some models of the devices and report your findings to your class. (See *Mathematical Models* by H. Martyn Cundy and A. P. Rowlett; London: Oxford University Press, 1954.)

B Make ruler-and-compass constructions of as many different-sided regular polygons as you can. (See *Mathematical Recreations and Essays* by E. T. Ball and W. W. Rouse; New York: The Macmillan Company, 1962.)

C Make and color various designs (see C-95, Ex. 6) or tessellations based on geometric figures that have rotational, translational, or reflectional symmetry. Display them on the bulletin board.

TEST YOURSELF

Exercises 1 through 7 deal with motion geometry and congruence. Exercises 8, 9, and 10 are related to constructions, though no actual constructions are required in this self-evaluation test.

After students have had ample time to check their work and review any sources of difficulty, you might want to administer the Module Achievement Test which is available from the publisher as part of the Test Package for this series.

RESEARCH PROJECTS

The reference cited for Research Project A contains information concerning two interesting "angle trisectors." One is a linkage which students can construct from strips of cardboard. Another is called the "Tomahawk Trisector," because of its shape.

Students who undertake Project B will find that many sources provide instructions for constructing regular pentagons. The reference given in the text presents lucid instructions for constructing a regular polygon of 17 sides.

For students who are interested in Project C, you might point out that many examples of tiling and tessellation patterns as well as a discussion of the symmetries evident in the patterns are to be found in *Geometry in the Classroom: New Concepts and Methods*, by H. A. Elliott, J. R. MacLean, and J. M. Jorden (Holt, Rinehart and Winston of Canada, Ltd.), Chapter 7.

MATHEMATICAL RECREATION

The Treasure Hunt gives the student additional practice in using combinations of rigid motions on the geoboard.

Another series of motions such as a student might design are given below.

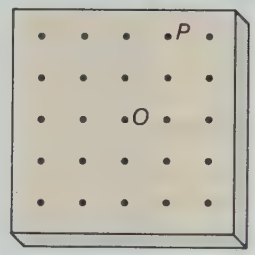
1. Start with the point in the upper left corner of the geoboard.
2. Translate the point "right 4, down 2."
3. Reflect the image of step 2 in the vertical line of symmetry of the geoboard.
4. Rotate the image of step 3 $\frac{1}{4}$-turn clockwise about the centre of the geoboard.
5. Translate the image of step 4 "right 2, down 4."
6. Reflect the image of step 5 in the diagonal of the geoboard that runs from the lower left corner to the upper right corner.

Where is the final image? (*Solution:* The point is in the upper left corner of the geoboard).

MATHEMATICAL RECREATIONS

A Geoboard Treasure Hunt

Try an imaginary treasure hunt on a 5 by 5 geoboard. Follow the steps below in order to locate the treasure.



1. Start at point P.
2. Rotate point P a $\frac{1}{4}$ turn clockwise with point O as the centre of rotation.
3. Translate the image of step 2 "down 1, left 1."
4. Translate the image of step 3 "up 3, left 1."
5. Rotate the image of step 4 using a $\frac{1}{2}$ turn clockwise with point O as the centre of rotation.
6. Translate the image of step 5 "right 2, up 3."
7. Reflect the image of step 6 in the horizontal line of symmetry of the geoboard.
8. The image of step 7 is the location of the hidden treasure. Where is it?

Make up your own treasure hunt on a geoboard. Give your puzzle to a classmate to solve.

MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. $J:(2,5)$, $K:(2,2)$, $L:(4,2)$ **2.** $J:(4,6)$, $K:(4,3)$, $L:(6,3)$
3. (2,1) **4.** (6,1) **5.** 4 **6.** true **7.** ▲ $\angle P$; ■ $\overline{MO}$
8. perpendicular bisector **9.** reflection **10.** compass

UNIT C: GEOMETRY AND MEASUREMENT

Module 4: Circles and Space Figures

General Objectives

To review the parts and properties of circles.
To introduce central and inscribed angles in a circle.
To introduce tangents to circles.
To review and extend work with circumference and area of circles.
To construct models of various polyhedrons and examine the relation between the number of faces, edges, and vertices.
To examine cross sections of space figures.
To find volumes of prisms, pyramids, cylinders, and cones.

	Performance Objectives	Pupil Text	Reteach-Reinforce	Related Activities
112	**RED** The student can identify basic parts of a circle, such as radius, diameter, chord, and arc.	C-102, 103	**WB** 103	
113	**RED** Given the diameter of a circle, the student can find its circumference.	C-108, 109 (C-112, 113)	**WB** 105 **DM** 69	**ASC** AP-5 **PD** b-3
114	**RED** Given the radius of a circle, the student can find its area.	C-110, 111	**WB** 106 **DM** 70	**ASC** AP-6 **PD** e-3 **SWM 2** 221
115	**YELLOW** The student can identify central and inscribed angles of a circle, and can use the fact that an inscribed angles is half the measure of a central angle which subtends the same arc of the circle.	C-104, 105	**WB** 104	**SP** e-3
116	**YELLOW** The student can identify a tangent to a circle.	C-106, 107		
117	**YELLOW** The student can recognize nets for various space figures.	C-116, 117, 118		**SWM 3** 318, 319
118	**YELLOW** Given appropriate figures, the student can identify prisms, pyramids, cylinders, and cones.	C-116, 117		**SWM 3** 304–307
119	**YELLOW** The student can find the volume of prisms and pyramids.	C-120, 121	**WB** 107 **DM** 71	**ASC** AP-13
120	**YELLOW** The student can find the volume of cylinders and cones.	C-120, 121	**WB** 107 **DM** 72	**ASC** AP-14, 15 **PD** e-4
121	***GREEN*** *The student can recognize an ellipse.*	C-114, 115		**SWM 3** 138
122	***GREEN*** *Given a sketch of a plane intersecting a space figure, the student can identify the cross section.*	C-122, 123		**SWM 3** 316, 317
	Reviewing the Ideas	C-124	**WB** 18	

MATHEMATICS

A sizable portion of the material in the module is related to circles. The formal definition of a circle is as follows:

> A circle is the set of all the points in a plane that are at a fixed distance from a given point in the plane.

Some of the early material on circles will be a review, albeit a necessary review, for most students. However, the work with central and inscribed angles in circles as well as tangents to circles will probably be new to most students.

The concepts of circumference and area of circles in the module are approached empirically rather than through rigorous proof. Thus, in exploring circumferences of circles, the student is led to discover that the ratio of the circumference to the diameter is a constant, which we call pi (π).

In finding area of circles, the students work with an intuitive idea of limits when they con-

struct a "crooked parallelogram" from the sectors of a circle and then compute the area of the parallelogram to find an approximation of the area of the circle.

Volumes of prisms and pyramids are compared experimentally before the volume formulas are introduced. Cylinders and cones are treated similarly. An exploration of various polyhedrons and cross sections of space figures rounds out the work of the module.

TEACHING THE MODULE

Materials

Compass, protractor, plastic mirror, small cubes, string, centimetre rulers, circular cone or disks, centimetre graph paper, scissors, index cards, cellophane tape, paste, map pins or small nails, plywood squares, posterboard.

Vocabulary

arc
centre
central angle
chord
circle
circumference
circumscribed polygon
cone
cross section
cube
cylinder
diameter
dodecahedron
ellipse
exterior
focal points
inscribed angle
inscribed polygon
interior
major axis
minor axis
octahedron
pi (π)
polyhedron
prism
pyramid
radius
semicircle
sphere
subtend
tangent
tetrahedron

It is quite likely that many of your students will have had only limited experience with many of the concepts presented in this module. There is a sizable amount of new vocabulary introduced, and some attention should be given to helping the students understand and incorporate the new words into their mathematical vocabulary.

It is vital that students be given the opportunity to construct, handle, and work with the various space figures introduced in the module. If you have classroom sets of wooden models, they can be put to good use in teaching the module. Encourage students to make models of the various space figures.

The formulas for the circumference and area of circles as well as the volume formulas for pyramids, cones, prisms, and cylinders should be learned by students at this level.

Lesson Schedule

You should allow a minimum of 10 days for this module. However, a greater amount of time is advisable if you wish to give students an opportunity to pursue some topics in greater depth. Also, if students make models of many of the solids, more time may be needed. The increase in interest as a result of active student involvement is well worth the extra time.

Evaluation

The Module Achievement Tests that are part of the evaluation program for this series can be very helpful in assessing students' understanding of the ideas presented in the module. If you prefer to design your own evaluation instrument, you will want to be sure that you cover most of the major objectives stated at the beginning of the module and exclude items for which no objectives have been stated.

Use the students' participation in the Investigation and Discussion phases of the daily lessons as well as their work on the exercises as aids in making your evaluations.

RESOURCES FOR ACTIVE LEARNING

Activities

Circles; Cubes; Solid Models, Cuisenaire Co.
Developmental Math Cards, K18, 20, Addison-Wesley
Experiments in Mathematics, Stage 1, pp. 24–35; Stage 2, pp. 34–39; Stage 3, pp. 28–33, 36–41, 44–47, Houghton Mifflin
Mira Activities, Activities 101–126, Creative Publications
Relationshapes Activity Cards, Nos. 64, 84, 86, Addison-Wesley

Manipulative Devices

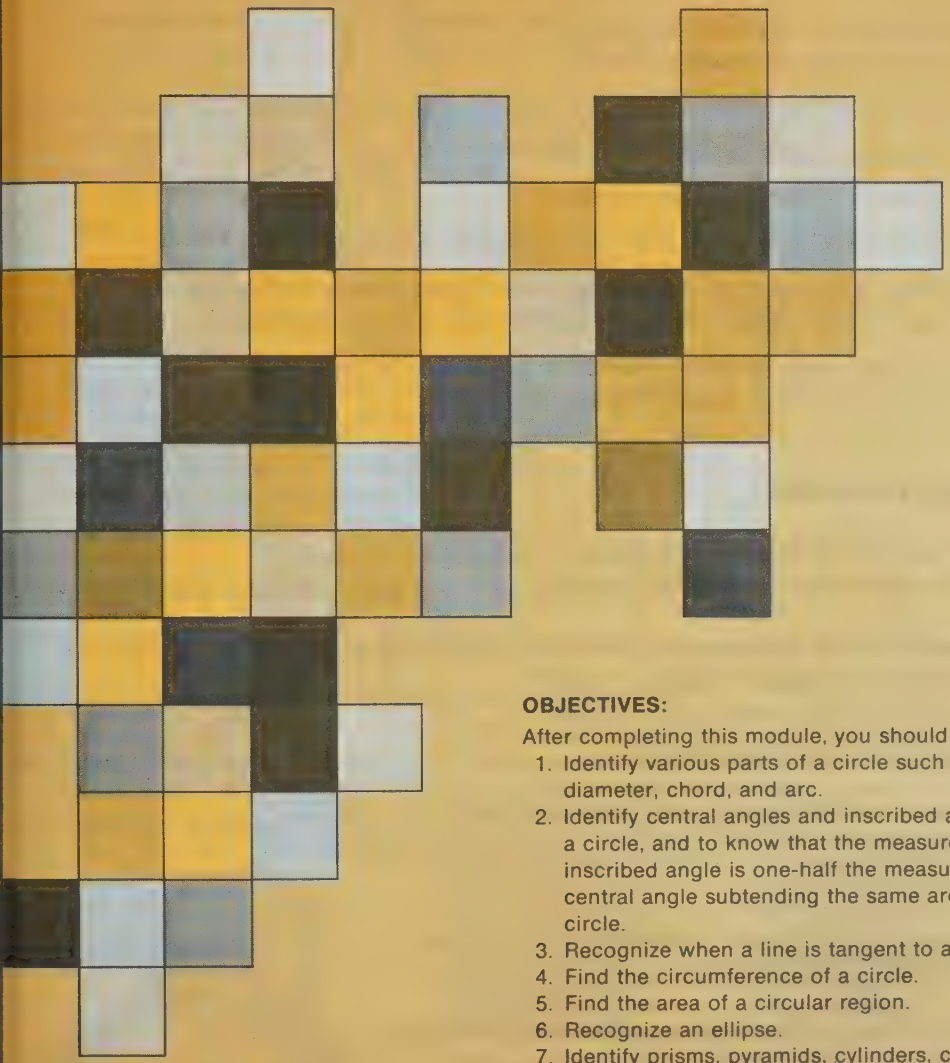
Circle Master Compass (Creative Publications)
Geometric figures and solids (Addison-Wesley; Creative Publications; LaPine)
MIRA (Creative Publications; Cuisenaire Co.)
Polyhedron construction material (Creative Publications; Cuisenaire Co.; Sigma)
Sage Kit (LaPine; Sigma)
Space Spider (Geyer; Mind/Matter)
Thread Sculpture (LaPine; Selective Educational Equipment)

Games and Puzzles

Polyhedron-Rummy (Scott, Foresman)

UNIT C: *Geometry and Measurement*

MODULE 4: *Circles and Space Figures*



OBJECTIVES:

After completing this module, you should be able to:

1. Identify various parts of a circle such as radius, diameter, chord, and arc.
2. Identify central angles and inscribed angles in a circle, and to know that the measure of an inscribed angle is one-half the measure of a central angle subtending the same arc of a circle.
3. Recognize when a line is tangent to a circle.
4. Find the circumference of a circle.
5. Find the area of a circular region.
6. Recognize an ellipse.
7. Identify prisms, pyramids, cylinders, cones, and spheres.
8. Make nets and models of various space figures.
9. Find the volume of prisms, pyramids, cylinders, and cones.
10. Sketch a cross section of a space figure given a picture of a plane intersecting a space figure.

Objective

The student can construct and name various parts of circles.

PREPARATION

Materials: Rulers and compasses.

This first lesson of the module is exploratory and is meant to review the basic ideas of circles and parts of circles. Therefore you can have the students begin immediately with the Investigation.

INVESTIGATION

The Investigation gives students the experience of drawing a circle and a radius, diameter, and chord of a circle. The purpose of the Investigation is to provide a model to aid in the discussion of the various parts of the circle.

DISCUSSION

Students' definitions of a circle will vary greatly. Some will be extremely intuitive, such as "a closed round figure." Others will be more formal. Formally, a circle is the set of all points in a plane that are at a fixed distance from one point in the plane.

Observe that if we do not specify that the points are coplanar we have the definition of a sphere. Exercises 2, 3, and 4 introduce the terms *radius, diameter,* and *chord* and relate these terms to the drawing made for the Investigation.

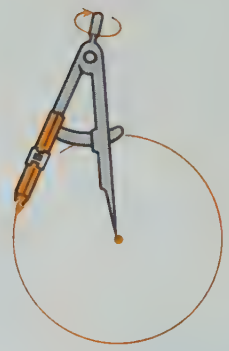
The *interior* of a circle (Exercise 5) is the set of all points in the plane whose distance from the centre of the circle is less than the length of the radius. Points whose distance from the centre is greater than the length of the radius are in the *exterior* of the circle.

Treat the lesson with a light touch, emphasizing the discovery aspects of the exercises rather than the mathematical terms that are introduced.

Circles

Investigating the Ideas

Can you make the construction following the directions below?



1. Use your compass to draw a **circle**.
2. Label the **centre** of the circle O.
3. Choose a point P on the circle. Draw and label segment $\overline{OP}$.
4. Draw a segment that contains point O and whose endpoints are on the circle. Label this segment $\overline{XY}$.
5. Choose two points on the circle. Connect these points with a segment. Label this segment $\overline{AB}$.

Discussing the Ideas

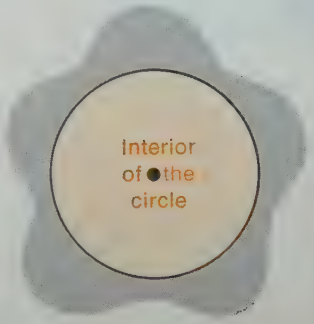
1. Give your own definition of a circle. Check your definition with the definition given in the glossary.

2. The segment $\overline{OP}$ that you drew in step 3 above was a **radius** of the circle. (The plural of radius is **radii**.)
   - A How many radii does a circle have?
   - B Are they all the same length?

3. Segments like $\overline{XY}$ (step 4) are called **diameters** of a circle. How does the length of a diameter compare to the length of a radius of a circle?

4. A segment like $\overline{AB}$ (step 5) is a **chord** of a circle. Is a diameter also a chord of a circle? Explain.

5. Suppose that a circle has a radius of 4 centimetres.
   - A If a point is in the **interior** of the circle, what can you say about its distance from the centre of the circle?
   - B If a point is 6.1 centimetres from the centre of the circle, is it in the interior or the **exterior** of the circle?



Using the Ideas

1. **A** Use a compass to draw a circle that has a radius of 3 centimetres.
   B How long is the diameter of the circle?
   C Draw a chord of the circle that is 4.5 centimetres long.
   D Mark a point X that is 2.5 centimetres from the centre of the circle. Is X in the interior of the circle?

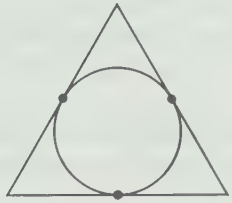
2. Draw a circle and two diameters that are not perpendicular. Label their endpoints E, F, G, and H. Connect the endpoints. What kind of figure is $EFGH$?

3. What kind of figure would you have had in Exercise 2 if the diameters had been perpendicular? Is it a regular polygon?

4. A polygon is **inscribed** in a circle if each of its vertices is on the circle. The figure suggests a way of inscribing a regular hexagon in a circle.

   Experiment with your ruler and compass to see if you can construct a regular inscribed hexagon.

5. A polygon is **circumscribed** about a circle if each of its sides touches the circle in just one point.



   Construct a square which circumscribes a circle.

★6. By experimenting with your ruler and compass, you can invent many interesting designs. The figures below illustrate a few of them. Try to copy one of these and then invent some of your own.

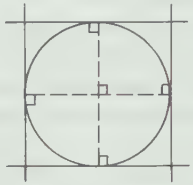
UTILIZATION

Students will need compass and centimetre rulers for the exercises.

Inscribed and circumscribed polygons are introduced in Exercises 4 and 5. You may want to discuss these terms before assigning the exercises. Afterwards, have some students demonstrate their constructions at the chalkboard.

Solution, Exercise 5

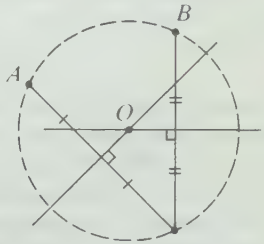
Construct two perpendicular diameters. Then construct perpendicular segments at the extremities of the two diameters.



EXTENSION

See page 103 of the *Workbook* for additional practice exercises.

Enrichment: A method constructing a circle through three noncollinear points will be interesting to some students. If A, B, and C are three noncollinear points, construct the $\perp$ bisector of two segments, say $\overline{AC}$ and $\overline{BC}$. Call the point of intersection O. Then O is the centre of a circle that contains points A, B, and C.



Assignments

Minimum 1–3. Average 1–3, 6. Maximum 1–6.

Objective

The student can identify central and inscribed angles in a circle.

PREPARATION

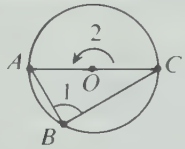
Materials: Rulers, compasses, and protractors.

Display a figure like the one shown. Have the students identify the chords, radii, and diameter shown. Also have them name all the angles in the figure.

INVESTIGATION

Have the students follow the flow chart instructions and measure $\angle 1$ and $\angle 2$ of their drawing. Have each student make at least 2 different drawings with the three points spaced differently on the circles. In each drawing the student should find that $m\angle 2$ is twice $m\angle 1$.

You might suggest that some students choose as points A and C the endpoints of a diameter. Then $m\angle 2 = m\angle AOC = 180°$ and $m\angle 1 = 90°$.



DISCUSSION

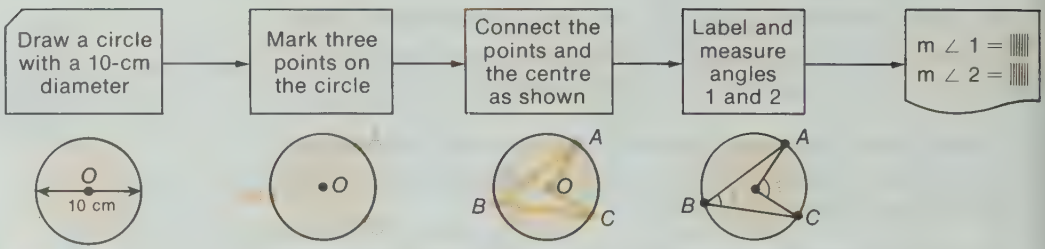
There is a considerable amount of new terminology introduced in the Discussion questions. Proceed carefully and provide additional examples of central angles, inscribed angles, and arcs.

In discussing vocabulary, the intuitive aspects of the words should be explored whenever reasonable. For example, the word "subtends" might bring to mind various ideas such as those suggested by "encloses" or "includes." These terms are very descriptive of the relationships shared by an angle and the arc that subtends it.

Understanding the relationships stated in Exercise 5 is a secondary goal of the lesson. Primary emphasis should be upon naming angles and arcs in circles.

Arcs and Angles of Circles

Investigating the Ideas



Can you find a relationship between the measures of the two angles?

Try this experiment again using three other points on the circle.

Discussing the Ideas

1. Study the definitions at the right.

- **A** Which angles in the Investigation are central angles?
- **B** Which angles in the Investigation are inscribed angles?

Central angle

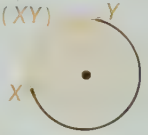
An angle with its vertex at the centre of the circle

Inscribed angle

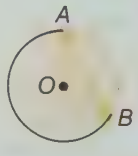
An angle that contains three points of the circle and has its vertex on the circle

2. The portion of the circle shown in color is an **arc**. How would you define an arc of a circle?

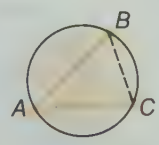
Arc XY (XY)



3. Central angle $\angle AOB$, chord $\overline{AB}$, and arc $\overset{\frown}{AB}$ are shown in the figure at the right. Each of these figures is said to **subtend** the other. For example: $\overset{\frown}{AB}$ subtends $\angle AOB$ or $\angle AOB$ is subtended by $\overset{\frown}{AB}$.



- **A** What kind of angle is $\angle BAC$ with respect to the circle?
- **B** Which arc subtends $\angle BAC$?
- **C** Which chord subtends $\angle BAC$?



4. An arc that is exactly half of a circle is called a **semicircle**. What special chord of a circle does a semicircle subtend?

5. The Investigation suggests the following statement:
If a central angle and an inscribed angle in a circle subtend the same arc, then the measure of the central angle is ? the measure of the inscribed angle.

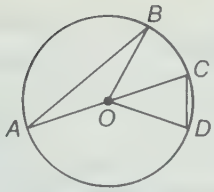
Using the Ideas

1. **A** Point O is the centre of the circle. Name four central angles of circle.

B Name two inscribed angles.

C Name two angles that subtend $\overset{\frown}{BC}$.

D What angle subtends $\overline{CD}$?



2. **A** Name the central angle subtended by $\overset{\frown}{AE}$.

B Name the central angle subtended by $\overset{\frown}{BC}$.

C Name the inscribed angle subtended by $\overset{\frown}{BC}$.

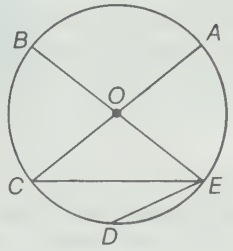
D Name the central angle subtended by $\overset{\frown}{CDE}$.

E Name the central angle subtended by $\overset{\frown}{CE}$.

F Name the arc subtended by $\angle BOA$.

G Name the arc subtended by $\angle DEB$.

H Name an inscribed angle that does not have one side as a diameter.



3. Draw a circle and its diameter.

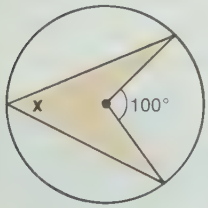
A Inscribe an angle that subtends the diameter. What is the measure of the angle?

B Inscribe another angle subtending the same diameter. What is the measure of this angle?

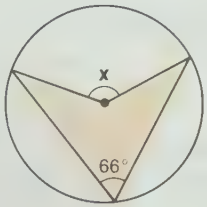
C If an inscribed angle subtends a diameter, what kind of an angle is it?

4. For each circle, give the degree measure for **x**.

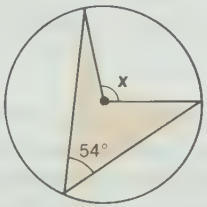
A



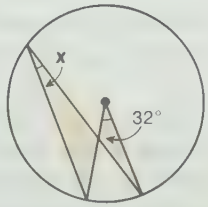
B



C



D



think

Can you unscramble each of these "words" to form a mathematical term that is in this lesson?

car glean recent dreamite dorch cleric undebts biscinder

UTILIZATION

Note that in naming the arc in Exercise 2D, a middle letter is given. This is necessary only when the possibility of ambiguity exists, and in such a case the middle letter indicates which of two possible arcs is meant.

The answers to Exercise 4 can be found without measuring. Each central angle is twice as large as the inscribed angle that subtends the same arc.

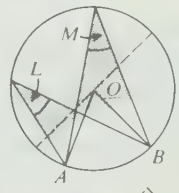
EXTENSION

To provide more practice with these concepts, page 104 in the *Workbook* may be assigned.

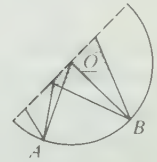
Remedial: Ask students to do the following: Draw a figure with two inscribed angles and one central angle all subtending the same arc. Then cut your paper along the dotted line. Cut out $\angle L$ and $\angle M$. How do they compare? ($\angle L \cong \angle M$)

Fit $\angle L$ and $\angle M$ into $\angle AOB$. How do they fit? ($m\angle L + m\angle M = m\angle AOB$)

1



2



3

Think Solution

This Think problem can be tried by all students. The problem focuses attention on some of the terms introduced in the lesson and may aid students in learning the correct spelling of the words.

Assignments

Minimum 1, 2. Average 1–3. Maximum 1–4.

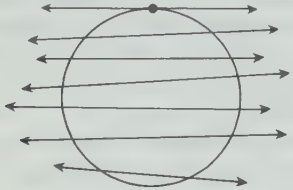
Objective

The student can identify lines that are tangents to circles and construct lines tangent to circles.

PREPARATION

Materials: Rulers, compasses, and plastic mirrors.

Draw a circle on the chalkboard. Have students draw several lines that intersect the circle in two points (secant lines). Ask if someone can draw a line that will touch the circle in just one point. While this is difficult to do physically, we can readily think about such lines in geometry.



INVESTIGATION

The purpose of the Investigation is to set the stage for a discussion of lines tangent to circles. Some students may think there are only two common tangents to the circles instead of 4.

DISCUSSION

The two most important ideas in the Discussion Exercises are the definition of a tangent to a circle and the observation that a tangent is perpendicular to the radius at the point of tangency. The latter idea can be utilized to construct tangents to circles through a given point.

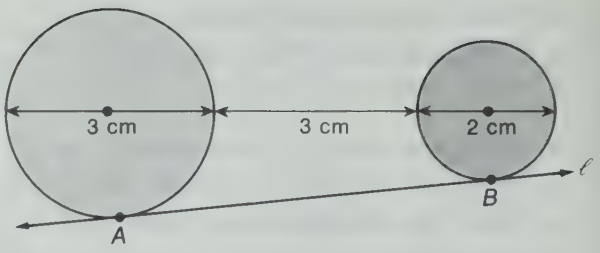
For Exercise 3, students should place the plastic mirror along $\overline{OP}$ and note whether either line a or b is reflected onto itself. Since line a is the one that is reflected onto itself, $\overline{OP} \perp a$ and a is tangent to the circle at point P.

In Exercise 4, any circle which has its center on the line perpendicular to line r at point Q will be tangent to line r. The perpendicular can be constructed either by means of the reflection or by the ruler-and-compass method.

Tangents to Circles

Investigating the Ideas

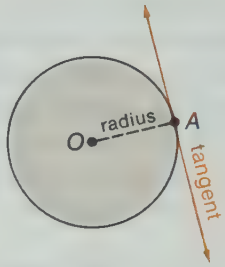
Draw two circles like these about 3 centimetres apart and a line ℓ that touches both circles at one point on each circle



How many other lines like line ℓ can you draw so that each line touches both circles at one point on each circle?

Discussing the Ideas

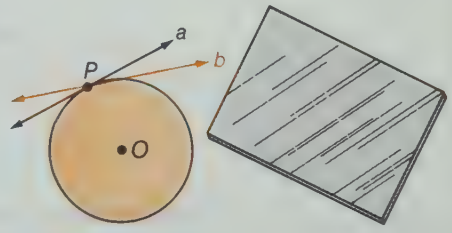
1. A line is **tangent** to a circle if it intersects the circle in just one point. In the figure, the line is tangent to the circle at point A. Point A is called the **point of tangency**.



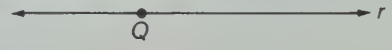
   - **A** How many lines are tangent to a circle at a point on the circle?
   - **B** What relationship do you think there is between a tangent to a circle and the radius of the circle at the point of tangency?

2. How many tangents to both circles did you find in the Investigation?

3. Lines a and b both intersect at point P on the circle. How can you use reflections and a plastic mirror to decide which line, if either, is a tangent to the circle?



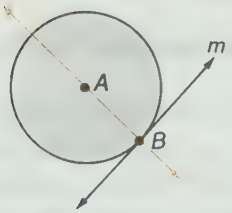
4. **A** How would you construct a circle tangent to a given line r at a given point Q on the line?



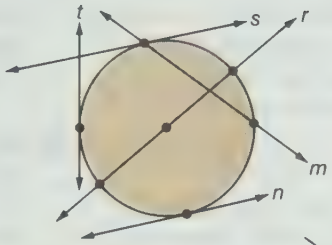
   - **B** Is there more than one circle tangent to the line at point Q?

Using the Ideas

1. Line m is tangent to the circle with centre A.
   - A How are the two lines m and $\overleftrightarrow{AB}$ related?
   - B A line passing through the point of tangency and the centre of the circle is ? to the tangent line.



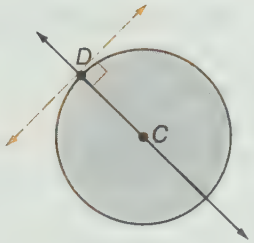
2. Which of the lines are tangent to the circle? All points of intersection are shown.



3. Draw a line k.
   Mark a point B on k.
   Construct the perpendicular to k at point B.
   Now draw three circles tangent to k at point B.

4. Draw a circle.
   Mark a point D on the circle.
   Label the centre C.
   Draw a line through C and D.
   Construct the perpendicular to $\overleftrightarrow{CD}$ at point D.

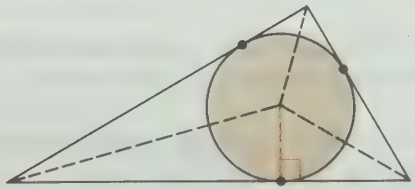
   Is the line perpendicular to $\overleftrightarrow{CD}$ tangent to the circle at point D?



5. Draw two parallel lines.
   Construct a circle that is tangent to both lines.

6. Draw an acute angle.
   Construct a circle that is tangent to both sides of the angle.

7. Draw a triangle.
   Bisect each angle.
   Then complete the construction for the circle inscribed in this triangle.

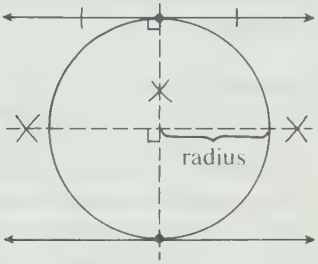


UTILIZATION

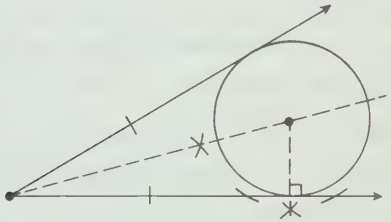
The exercises provide experiences in constructing lines tangent to circles. Ruler and compass constructions can be used, but encourage students to use their plastic mirrors, if they wish, to help construct perpendicular lines.

Solutions, Exercises 5 and 6

5.



6. The centre of the circle will be on the bisector of the angle.



EXTENSION

Enrichment: Supply the students with sheets of waxed paper or thin tracing paper. Have them draw a circle about 12 cm in diameter. They should then carefully crease the paper so that a point of the circle falls directly on the centre of the circle, and repeat the folds many times. The result will be an "envelope" of tangents to a circle whose centre is the same as the original circle but whose radius is $\frac{1}{2}$ the original circle. If the folds are made carefully, an interesting design will result.

Assignments

Minimum 1–3. Average 1–5. Maximum 1–7.

Objective

The student can find the circumference of a circle by using the formula for the circumference of a circle.

PREPARATION

Materials: Circular cans or disks of various sizes, string, centimetre rulers.

The preparation period should be devoted to seeing that sufficient material is available so that small groups of students can work together on the Investigation.

INVESTIGATION

Suggest to the students that they measure the circumference of the circles to the nearest tenth of a centimetre. If careful measurements are made, the ratio of the circumference to the diameter for each circle will not vary greatly. Students should find the circumference of at least three different circles in order to compare the ratios.

DISCUSSION

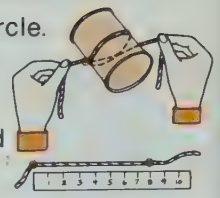
The most important idea to bring out is that the ratio $\frac{C}{d}$ is the same for every circle. This ratio is symbolized by the Greek letter π (pi). From the relation $\frac{C}{d} = \pi$, we get the formula for the circumference of a circle: $C = \pi \cdot d$. In computing circumferences of circles, we must use approximations for π. In most practical applications $3\frac{1}{7}$ or 3.14 is a sufficiently close approximation.

Circumference of Circles

Investigating the Ideas

The distance around a circle is called the **circumference** of the circle.

Choose several different-sized circular objects. Then find a way to measure the circumference of each of them. The figures at the right suggest some ways to accomplish this.



A Find the length of a piece of string needed to go around this circular object.

B Roll a circular disk along a line for one full turn. Measure the distance along the line.

Can you find the ratio of the circumference to the diameter of each circle to the nearest hundredth?

Discussing the Ideas

1. Diane made this table to show her results for the Investigation.

circle	circumference (c)	diameter (d)	c ÷ d
A	20.5 cm	6.5 cm	3.15
B	33.6 cm	10.7 cm	3.14
C	9 cm	2.8 cm	3.10

   A Are her results similar to yours?

   B If you find the average of the ratios $\boldsymbol{C \div d}$, what is this number to the nearest hundredth?

2. Since the actual measurement of physical objects is only an approximation, the ratio $\frac{c}{d}$ that you found was slightly different for each object measured. However, mathematicians have proved that the ratio $\frac{c}{d}$ is the same for all circles.
   This number, **the ratio of the circumference to the diameter**, is named by the Greek letter π (pronounced "pie").
   The decimal for π is endless and nonrepeating.
   The first fifteen places are shown here: $\pi = 3.141592653589793\ldots$

$$\frac{C}{d} = \pi$$

$$C = \pi \cdot d$$

   A Find a decimal for $\frac{22}{7} = 3\frac{1}{7}$ to the nearest hundredth.

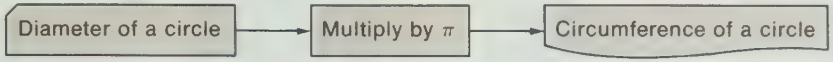
   B Is $3\frac{1}{7}$ a good approximation of π?

3. If you know the diameter of a circle, what would you do to compute the circumference of the circle?

4. If you knew the circumference of a circle, how would you find the diameter of the circle?

Using the Ideas

1. The flow chart can help you better understand the formula $C = \pi \cdot d$ for finding the circumference of a circle.



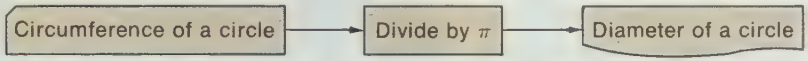
Use the fact $\pi \approx 3.14$ to find the approximate circumference of circles with the following diameters:

A 30 cm **B** 5 cm **C** 1.9 m **D** 16 mm **E** 10.8 cm

2. Use the fact $\pi \approx 3\frac{1}{7}$ to find the approximate circumference of each circle whose diameter is given below.

A 14 **B** $3\frac{1}{2}$ **C** $10\frac{1}{2}$ **D** $1\frac{3}{4}$ **E** $2\frac{1}{10}$ **F** $4\frac{1}{5}$

3. This flow chart is the inverse of the one shown in Exercise 1.



Use the inverse flow chart and the fact $\pi \approx 3.14$ to find the diameter (to the nearest tenth) of circles with the following circumferences:

A 15.7 cm **B** 38 mm **C** 8.4 m **D** 1.27 m **E** 0.19 m

4. Bill measured the distance around his head and found that it was 56 centimetres. If his head were perfectly round, what would be the diameter of his head?

5. A bicycle wheel has a diameter of about 66 centimetres.
   - **A** About how many metres will it travel in one revolution?
   - **B** About how many metres will it travel in ten revolutions?
   - **C** About how many metres will it travel in 120 revolutions?
   - **D** About how many revolutions will it make travelling one kilometre?

6. The radius of the circle is 10 millimetres.

   - **A** Use your centimetre ruler to find the perimeter of the small square.
   - **B** What is the perimeter of the large square?
   - **C** What is the average of the perimeters of the two squares?
   - **D** What is the circumference of the circle?
   - **E** What is the difference between your answers for parts **C** and **D**?

More practice, page S-27, Set 46

UTILIZATION

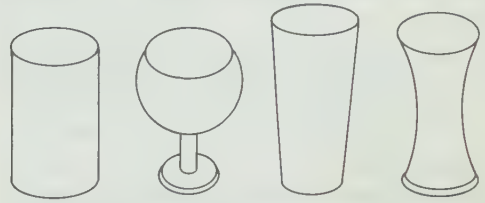
The exercises provide practice in using the formula $C = \pi d$ and in finding diameters of circles of given circumference.

Exercise 6 suggests another approach to finding the circumference of circles. The circumference of a circle could be defined as the limit of the perimeters of inscribed polygons with an increasingly great number of sides. This idea can be approached at this level by having the students observe the fact that the circumference of a circle is somewhere between the perimeter of an inscribed polygon and the perimeter of a circumscribed polygon. In this case the polygons are squares.

EXTENSION

Additional practice is provided by Supplementary Exercise Set 46, page S-27; page 105 of the *Workbook;* page 69 in the *Duplicator Masters;* and *Arithmetic Skill Card* AP-5.

Enrichment: Display a variety of drinking glasses, goblets, etc. Have the students guess whether the height of each glass is greater or smaller than the circumference of its rim.



Students will be surprised by the results of actual measurement. The circumference is usually quite a bit more than the height unless the glass is exceptionally tall and narrow. Note that the height must be more than π times the diameter of the rim of the glass in order to exceed the circumference.

Assignments

Minimum 1, 2. Average 1–4. Maximum 1–6.

Objective

The student can find the area of a circular region by using the area formula, $A = \pi \cdot r^2$.

PREPARATION

Materials: Centimetre graph paper (*Duplicator Masters,* page 83), compass, scissors, protractor, transparent tape or paste.

Review the concept of area and ask students to estimate the area of a few odd-shaped regions on a grid before beginning the Investigation.

INVESTIGATION

The purpose of the Investigation is to provide two methods of approximating the area of a circle. Student estimates of the area of the circle will vary. One way to arrive at a good estimate is to observe that the circle covers all of the 10 by 10 grid except for about $5\frac{1}{2}$ squares in each of the four corners. Therefore about 22 squares out of the 100 squares are uncovered, so the area should be about 78 cm^2.

Encourage the students to work as carefully as possible in cutting the sectors to form the "parallelogram." Suggest that they paste or tape the sectors on graph paper or between parallel lines 5 cm apart as an aid in estimating the area. The resulting "parallelogram" should have a height of 5 cm and a length of about 16 cm, and therefore an area of about 80 cm^2.

DISCUSSION

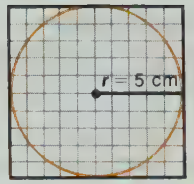
A rigorous proof of the formula for finding the area of a circle involves a knowledge of *limits* and, hence, is not appropriate at this level; the essential ideas for finding the area of a circular region is simply *suggested* by the text.

The important thing to observe with respect to this "crooked" parallelogram is the fact that the height of the parallelogram is approximately equal to the radius of the circle, and the base of the parallelogram is approximately equal to one half the circumference. Hence, the area of the parallelogram is one half the circumference times the radius. This gives the familiar area formula for a circle: $A = \pi r^2$.

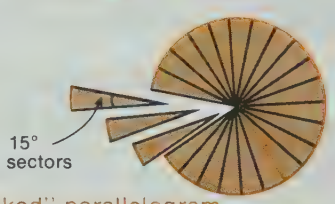
Areas of Circular Regions

Investigating the Ideas

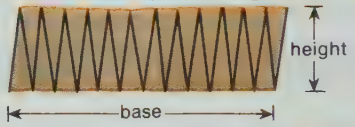
Draw a circle with a radius of five centimetres on graph paper. Then try each of these two methods for finding the area of the circular region.



1. Estimate the number of squares that the circular region covers.
2. Cut the circular region into sectors with angles of 15°. Tape or paste the sectors down as shown to form a "crooked" parallelogram. Measure the base and height, and compute the area of the parallelogram.



A "crooked" parallelogram



Which method do you think gives the best approximation of the area of a circular region?

Discussing the Ideas

1. A Is the height of the "crooked" parallelogram the same as the radius of the circle?

   B Explain why the base of the "crooked" parallelogram is $\frac{1}{2} \cdot C$, where C is the circumference of the circle.

2. Think about cutting a circular region into a very large number of sectors, then explain this statement:

   The area A of a circle with radius r and circumference C is given by the formula $A = \frac{1}{2} \cdot C \cdot r$.

3. Since $C = \pi \cdot d = 2 \cdot \pi \cdot r$, then $\frac{1}{2} \cdot C = \pi \cdot r$.

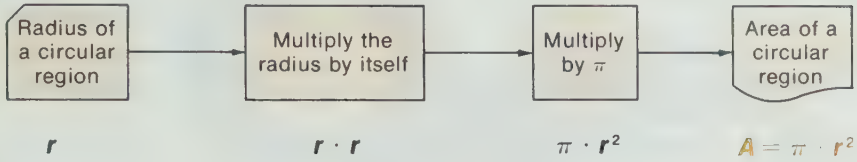
   Therefore, $A = \frac{1}{2} \cdot (\pi \cdot r) \cdot r$ or

   Explain how to use this formula to find the area of a circular region whose radius is 2 centimetres.

4. Check your estimates of the area of the circular region in the Investigation using the area formula $A = \pi \cdot r^2$.

Using the Ideas

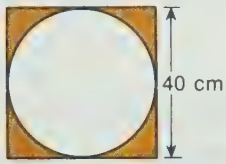
1. Use the flow chart to help you find the area of circular regions with the following radii (r) or diameters (d). Use $\pi \approx 3.14$.



A $r = 3$ cm **C** $d = 2$ m **E** $d = 10$ cm **G** $r = 24.1$ mm

B $r = 9$ m **D** $r = 60$ mm **F** $r = 2.3$ m **H** $d = 39$ cm

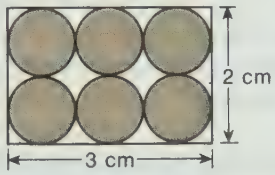
2. A circular region whose diameter is 40 centimetres is cut out of a square piece of tin 40 centimetres on a side. What is the area of the piece of tin which is left over? (Use $\pi \approx 3\frac{1}{7}$.)



3. A circular flower bed is surrounded by a walk two metres wide. The diameter of the flower bed is 20 metres. What is the surface area of the walk?

4. A cheese pizza with a diameter of 24 centimetres costs $1.55.
   - **A** About how much per square centimetre does the pizza cost?
   - **B** At this rate, about how much would a 30-centimetre round pizza cost?

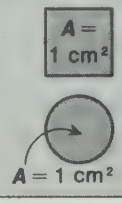
★ 5. The figure shows six circular disks, each of which has a diameter of one centimetre, placed on a rectangular region that is 3 by 2 centimetres. What percent of the rectangular region is covered by the six circular disks?



★ 6. If the rectangular region of Exercise 5 is covered by circular disks one millimetre in diameter and placed in a similar pattern, what percent is covered?

think

Which is larger, the perimeter of the square or the circumference of the circle?



More practice, page S-27, Set 47

UTILIZATION

The flow chart in Exercise 1 should help students to apply the area formula. Exercises 2, 3, and 4 call for application of the area formula to the solution of word problems.

Answers, Exercises 5 and 6

5. Area of circles: $6 \times 3.14 \times (0.5)^2 = 18.84 \times (0.25) = 4.71$

   Area of rectangle: $2 \times 3 = 6$

   $$\frac{4.71}{6.00} = 78.5\%$$

   The circles cover 78.5% of the rectangle.

6. Each circle takes up the same amount of area relative to the square circumscribed about it. There are $20 \times 30 = 600$ circles with diameter 0.1 cm, or 1 mm. Area of all circles:

   $600 \cdot \pi \cdot r^2$
   $= 600 \cdot 3.14 \cdot (0.05)^2$
   $= 600 \cdot 3.14 \cdot (0.0025)$
   $= 4.7100$ cm²

   Area of rectangle: $2 \times 3 = 6$ cm²

   $$\frac{4.71}{6.00} = 78.5\%$$

   The circles still cover 78.5% of the rectangle.

EXTENSION

For more practice, suggest Supplementary Exercise Set 47, page S-27; page 106 in the *Workbook;* page 70 in the *Duplicator Masters;* and *Arithmetic Skill Card* AP-6.

Think Solution

If the area of the square is 1 cm², then letting s be the length of one side of the square, we have $s^2 = 1$ cm², and $s = 1$ cm. Thus, the perimeter of the square is $4 \cdot s = 4$ cm.

If the area of the circle is 1 cm², then $\pi r^2 = 1$, $r^2 = \frac{1}{\pi} \approx 0.318$, and $r \approx 0.56$ cm. Thus the circumference of the circle is

$$2\pi r \approx 2 \cdot (3.14) \cdot (0.56)$$
$$\approx 3.5$$

Since $4 > 3.5$, the perimeter of the square is greater than the circumference of the circle.

Assignments

Minimum 1, 2. Average 1–4. Maximum 1–6.

Objective

The student can use the circumference and area formulas for circles to solve word problems.

PREPARATION

Review the formulas for circumferences and area of circles. The circumference formula should be presented both as

$$C = \pi \cdot d \text{ and } C = 2 \cdot \pi \cdot r.$$

The area formula is $A = \pi \cdot r^2$, which could also be expressed as $A = \frac{\pi \cdot d^2}{4}$. Since $r = \frac{d}{2}$, $r^2 = \left(\frac{d}{2}\right)^2 = \frac{d^2}{4}$, so the two formulas are equivalent.

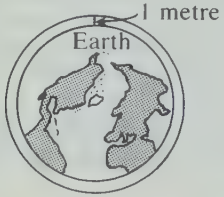
UTILIZATION

Assign the problems according to the abilities of your class. Problems 1 through 4 can be assigned to all students.

The drawing for Exercises 5 and 6 shows a 400-metre track. Distances around a track are ordinarily measured around the inside edge. In short races around a curved portion of a track in which each runner must stay in an assigned lane, the starting positions are staggered so that at the end all will have run the same distance.

EXTENSION

Pose this problem for some of your more able students: The circumference of the Earth around the equator is about 40 000 kilometres. Suppose that a steel ring is wrapped tightly around the Earth at the equator. Then a second steel ring is constructed so that it is 1 metre above the surface of the earth all the way around the equator. How much longer must this second ring be? (Answer: $2 \cdot \pi$ metres, or about 6.28 metres longer)

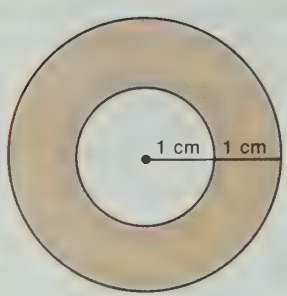


Assignments

Minimum 1–4. Average 1–7. Maximum 1–10.

Circle Problems

Refer to the figure at the right for Exercises 1 through 4.



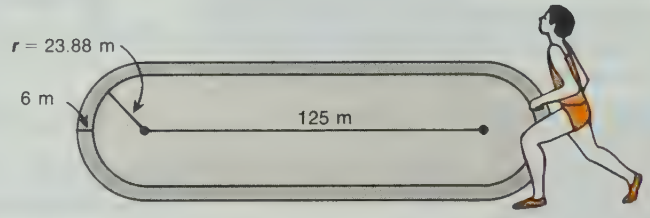
1. What is the circumference of the small circle?

2. What is the circumference of the large circle?

3. What is the area of the inner circular region?

4. What is the area of the shaded part?

The figure at the right represents an oval race track. The running track is 6 metres wide.



5. How far is it around the inside of the track? (Round your answer to the nearest metre.)

6. How much farther is it around the outside edge of the track than the inside?

7. During the 1968 Olympics in Mexico City, Lee Evans set a world record by running the distance of the track in 43.8 seconds. About how many metres per second (to the nearest tenth) did he run?

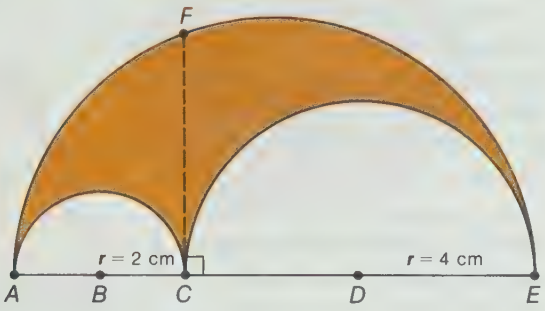
8. On July 30, 1970, Marilyn Neufville of Canada ran the distance of the track in 51.0 seconds to set a world record for women. What was her speed in metres per second (to the nearest tenth)?

9. What is the total area covered by the track and the infield?

10. How much greater is the area of the infield than the area of the running track?

THE SHOEMAKER'S KNIFE

The shaded region of the figure is called a "shoemaker's knife."



1. Make a full size drawing of the shoemaker's knife on your paper.

2. What is the distance around the large semicircle from A to E?

3. What is the distance from A to E going around the two smaller semicircles?

4. What is the area of the large semicircular region?

5. Find the area of the small semicircular region.

6. Find the area of the middle-sized semicircular region.

7. What is the area of the shoemaker's knife?

8. Draw $FC \perp AE$. Draw $\overline{AF}$ and $\overline{FE}$. Label the point where $\overline{AF}$ intersects the small semicircle as G and the point where $\overline{FE}$ intersects the middle-sized circle as H. Draw the quadrilateral $GFHC$. What kind of quadrilateral does it seem to be?

9. Using different-sized semicircles but keeping the length $\overline{AE}$ the same as before, draw other shoemaker's knives.
   - A Repeat Exercise 8. Do you get the same answer?
   - B If the two smaller semicircles are the same size, what is your answer?

Objective

The student can solve word problems related to the geometric shape called the "shoemaker's knife."

PREPARATION

Materials: Rulers and compasses.

Review methods of computing area and circumference of circles. You might also review the term *semicircle*. A semicircle is one of the two arcs of a circle determined by the end points of a diameter. When we speak of the "area of a semicircle," we mean the area of the region bounded by the semicircle arc and the diameter it subtends.

UTILIZATION

Exercises 1, 2, and 3 will enable the student to discover that the length $\overset{\frown}{AFE}$ = length $\overset{\frown}{AC}$ + length $\overset{\frown}{CE}$. This possibly surprising relationship holds true regardless of the size of the two smaller semicircles as long as the sum of their diameters is equal to the larger diameter.

In Exercise 8, quadrilateral $GFHC$ will always be a rectangle because $\angle AFE$ is an angle inscribed in a semicircle and therefore a right angle. Similarly, $\angle AGC$ and $\angle CHE$ are also right angles, for they are inscribed in the smaller semicircles. $\angle GCH$ must also be right because the sum of the measures of the angles of a quadrilateral is 360°. All the angles of $GFHC$ are right angles; therefore it is a rectangle. If the two smaller semicircles are the same size, the rectangle will be a square.

Assignments

Minimum 1–6. Average 1–8. Maximum 1–9.

Objective

The student can identify an ellipse and some of the properties of an ellipse.

PREPARATION

Materials: String, map pins or nails, small squares of plywood.

Since the Investigation will probably take several minutes, you should provide materials and have the students begin work on it at once.

INVESTIGATION

Considerable time might be saved if small pieces of plywood with two small-headed nails already in place could be prepared in advance of this lesson. The students could then mount their sheets of paper simply by punching holes through the paper so that it lies flat on the board. Several students can work together and take turns mounting their sheets on the same piece of plywood to draw the ellipses.

DISCUSSION

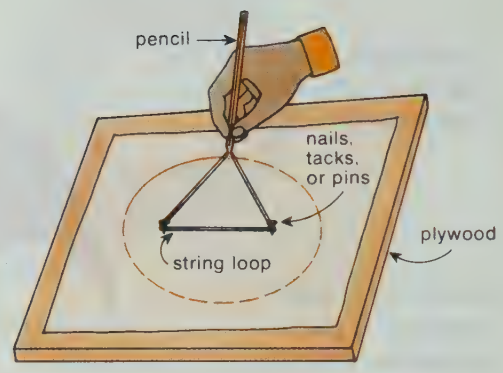
An ellipse is sometimes defined as the set of all points in a plane, the sum of whose distances from two fixed points is a constant. The two fixed points are the *focal points*. The string construction provides a method for satisfying this definition. That is if P is any point on the ellipse and F_1 and F_2 are the focal points, then distance $\overline{F_1P}$ + distance $\overline{F_2P}$ = a constant. The constant is the length of the major axis of the ellipse.

The reflecting property of the ellipse mentioned in Exercise 4 can be used to explain "whispering galleries." Another way to illustrate this reflecting property is to think of an elliptical billiard table. Then if a ball is placed at one of the focal points and is hit, when it hits the boundary, it will rebound so that it will go through the other focal point.

Ellipses

Investigating the Ideas

A method is shown at the right for drawing an interesting geometric figure. It is drawn by keeping the loop stretched tightly against the two nails and drawing completely around the nails.



Can you make the drawing by studying and following the the illustration at the right?

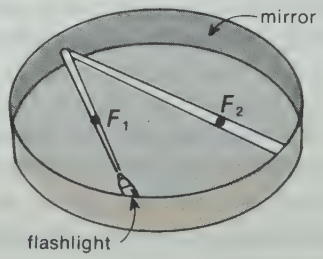
Discussing the Ideas

1. The figure that you drew for the Investigation is called an **ellipse**. What are some objects that are elliptical in shape?

2. The two points marked by the nails (pins, tacks) are called **focal points** of the ellipse.
   - A What happens when you draw another ellipse using the same loop of string in the Investigation but move the two focal points closer together?
   - B What happens if you move the focal points farther apart?

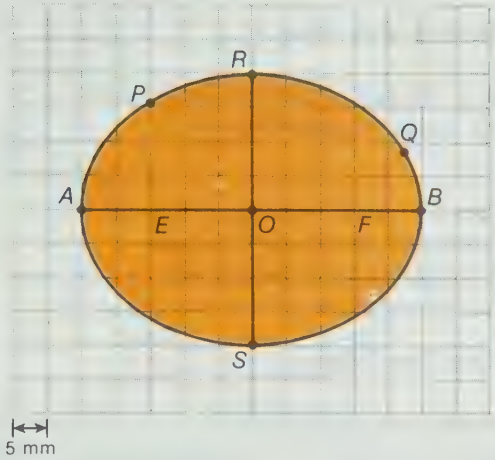
3. Does an ellipse have any lines of symmetry? If so, describe them.

4. An ellipse has an interesting reflecting property. Suppose an elliptical region were bounded by a curved mirror. Then any ray of light from one focal point would be reflected through the other focal point. Show several of these rays on the ellipse that you drew for the Investigation.



Using the Ideas

Refer to the ellipse shown for Exercise 1 through 8 E and F are the focal points of the ellipse.



1. Find the length of the polygonal path from E to P to F.

2. Find the length of the polygonal path from E to Q to F.

3. Find the length of the polygonal path from E to S to F.

4. The **major axis** of the ellipse is $\overline{AB}$. What is the length of $\overline{AB}$?

5. How does the length of the major axis $\overline{AB}$ compare with the length of a polygonal path from one focal point to a point on the ellipse to the other focal point?

6. The **minor axis** of the ellipse is $\overline{RS}$. What is the length of $\overline{RS}$?

7. A Estimate the area of the ellipse. (Each small square is 25 mm^2.)
   B Compute the area using the following formula:

   Area = (length $\overline{OB}$) · (length $\overline{OS}$) · π

   Compare this answer with your estimate.

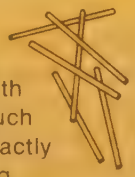
8. The **eccentricity** of an ellipse is defined to be the ratio:

$$\frac{\text{length of } \overline{OF}}{\text{length of } \overline{OB}}$$

   What is the eccentricity of the ellipse above?

think

Arrange six sticks all of the same length (or toothpicks) in such a way as to form exactly four triangles having the same size and shape.



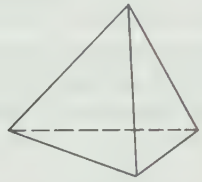
UTILIZATION

Students should measure the paths in Exercises 1 through 4 to the nearest millimetre in order to arrive at the correct conclusion for the question in Exercise 5. Every polygonal path from focal point E to the ellipse and back to focal point F has the same length, which is the same as the length of the major axis of the ellipse.

EXTENSION

Think Solution

Students will probably try to place the sticks together to form four triangles in one plane. The solution, however, is to be found by considering the triangles in different planes so that each triangle shares each side with an adjoining triangle; that is, each match forms the edge of a regular tetrahedron.



Assignments

Minimum 1–5. Average 1–6. Maximum 1–8.

Objective

The student can make models of some space figures, given a pattern for the figures.

PREPARATION

Materials: Thin posterboard, scissors, rulers and compasses, transparent tape, models of space figures (if available).

INVESTIGATION

The pattern given in this Investigation is for a regular octahedron. Since this polyhedron may be unfamiliar to most students, its assembly may offer them a mild challenge and its final shape may be somewhat surprising. The edges should be taped together with transparent tape. You might want to suggest that tabs be left on some edges to provide extra support.

DISCUSSION

Although considerable new terminology is introduced in the Discussion Exercises, the main purpose of the lesson is to help students develop spatial intuition by making and handling models of space figures.

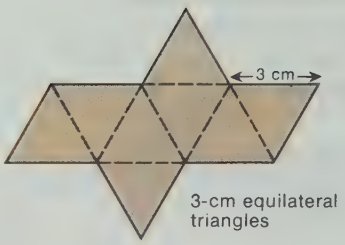
The students should not be expected to learn the names of all the special space figures that are introduced in this section. Only the names of the more common ones, such as *sphere, cylinder, cone,* and *prism,* should be memorized. A special effort should be made to see that students find their experience in this section pleasant and successful.

Try to provide models of as many as possible of the figures illustrated in the Discussion Exercises. A model of an oblique cylinder can be made by cutting the ends of a mailing tube at an angle which is not perpendicular to the axis of the tube. The cross sections formed will be ellipses.

Some Patterns for Space Figures

Investigating the Ideas

Here is the **net** or **pattern** for a space figure called an **octahedron**.



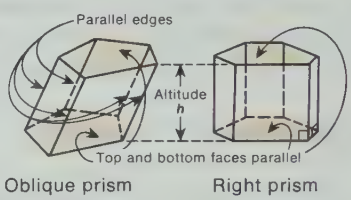
Can you make a copy of the net, cut it out, and fold it to form a model of an octahedron?

Discussing the Ideas

1. A **polyhedron** is the surface of a solid whose faces are polygonal regions. The word polyhedron means "many faces." Why do you think that the polyhedron in the Investigation is called an octahedron?

2. How many faces, edges, and vertices does an octahedron have?

3. The illustration at the right shows an **oblique prism** and a **right prism**.

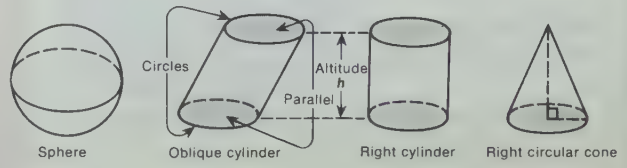


   - **A** In what way does an oblique prism differ from a right prism?
   - **B** What are some ways in which a prism differs from an octahedron?
   - **C** Explain why the oblique prism could be called a **pentagonal prism**.

4. The pyramid shown is a **pentagonal pyramid** because the base is a pentagon. Describe or draw the following:

   - **A** a square pyramid
   - **B** a triangular pyramid
   - **C** a hexagonal pyramid
   - **D** an octagonal pyramid

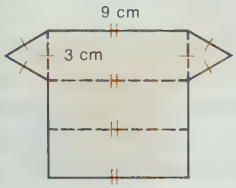
5. Some other space figures have some surfaces which are not flat like polyhedrons. Can you give some examples of physical objects like the four figures at the right?



Using the Ideas

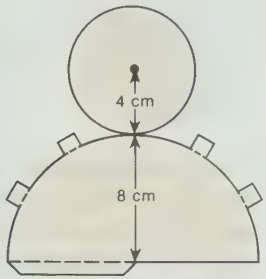
Make a net of each space figure below on light posterboard or tagboard. Fold along the dotted lines and tape the edges together. Congruent segments are marked by the same number of slashes.

1.



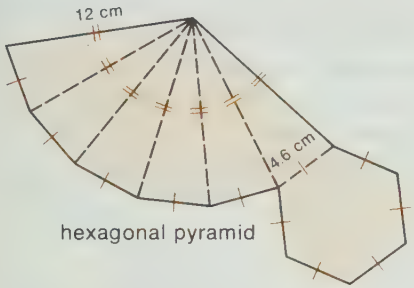
triangular prism

2.



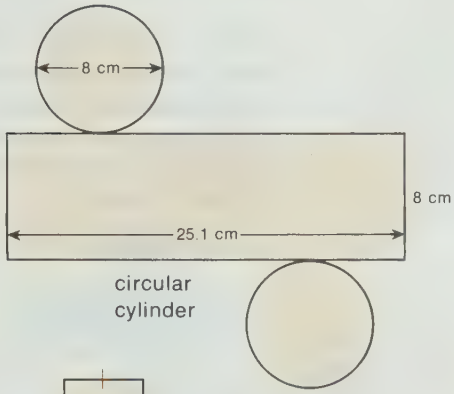
cone

3.



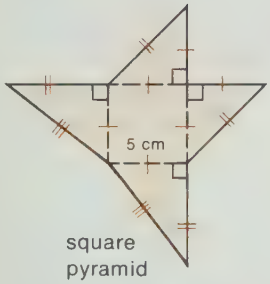
hexagonal pyramid

4.



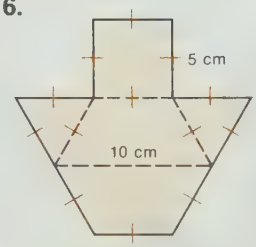
circular cylinder

5.



square pyramid

6.



pentahedron

★ **7.** Show how three of the square pyramids (Exercise 5) can be placed together to form a cube.

★ **8.** Place two of the pentahedrons (Exercise 6) together to form a **regular tetrahedron** (triangular pyramid).

UTILIZATION

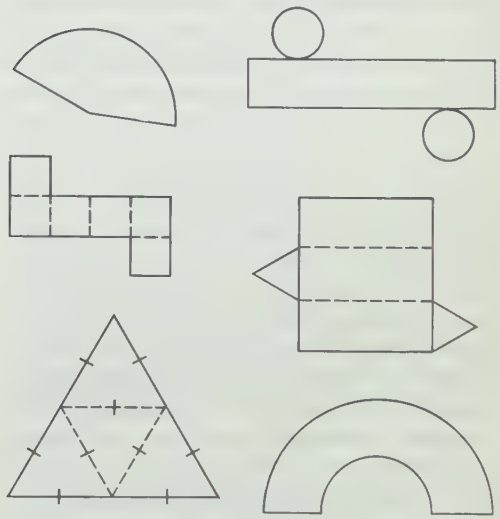
You may prefer to assign these exercises selectively, having each student make only one or two of the models.

The space figures constructed for Exercises 5 and 6 can form the basis of interesting puzzles, as suggested in Exercises 7 and 8. The puzzle suggested in Exercise 7 could be enhanced by construction of an open-top cube whose edges are slightly longer than 5 cm. Then the three pyramids could be arranged so that they fit snugly inside the cube.

The puzzle suggested in Exercise 8 is deceptive in that the two pieces must first be positioned in such a way that the square faces meet; then the pieces can be rotated until a regular tetrahedron is formed.

EXTENSION

Enrichment: Provide some pictures of patterns of space figures like those below. Ask students whether they can correctly name the figures that would result if the edges of the figures were joined.



Assignments

Minimum 1–4. Average 1–6. Maximum 1–8.

Objective

The student can make patterns and models of the five regular polyhedrons.

PREPARATION

Materials: Light posterboard, ruler, compass, transparent tape, protractor.

This lesson can be considered as an extension of the previous lesson.

UTILIZATION

If templates are made for an equilateral triangle and a pentagon, the nets for the space figures can be made much more quickly and accurately. Templates can be made by cutting out the polygonal region from an index card.

(Each angle 108°)

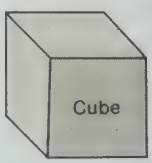
EXTENSION

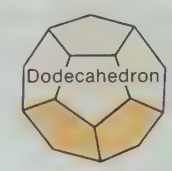
Discuss with the class why there are only 5 regular polyhedrons. The sum of the measures of the angles at each vertex must be less than 360° and must consist of at least 3 angles. Considering faces which are equilateral triangles, whose angles are 60° each, we can have polyhedrons with 3 equilateral triangles at each vertex ($3 \times 60 = 180°$, the tetrahedron), 4 equilateral triangles ($4 \times 60° = 240°$, the octahedron), and 5 equilateral triangles ($5 \times 60° = 300°$, the icosahedron). The cube is the only regular solid composed of squares, since each angle of a square is 90° and $3 \times 90° = 270°$, but $4 \times 90° = 360°$ so no polyhedron exists with 4 squares at each vertex. The dodecahedron has 3 regular pentagons meeting at each vertex ($3 \times 108° = 324°$). No regular polygon with a greater number of sides can be used because the sum of their angles will equal or exceed 360°.

Assignments

Minimum Any 1 model.
Average Any 2 models.
Maximum Any 3 models.

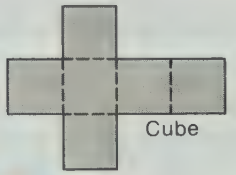
Patterns for Regular Polyhedrons

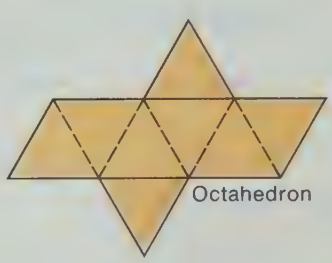


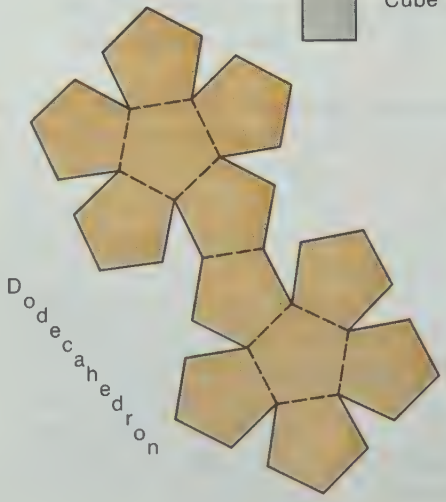


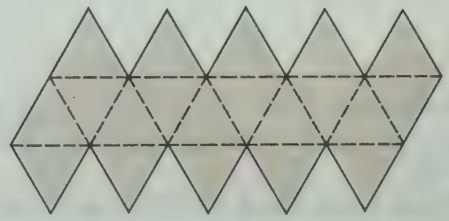
The five **regular polyhedrons** are shown above. The faces of each polyhedron are congruent regular polygonal regions. These polyhedrons are sometimes called the **Platonic solids** named after the Greek philosopher Plato. The ancient Egyptians were probably familiar with all of the regular polyhedrons except the dodecahedron.

A net for each of the regular polyhedrons is given on this page. Make a larger net for each polyhedron using light weight posterboard. Cut out the net, fold along the dotted lines, and tape the edges together to make a model of each of them.

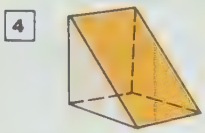






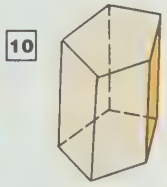


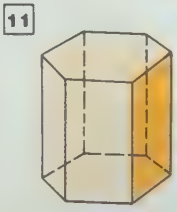
FACES, EDGES, AND VERTICES



There is an interesting relationship between the number of faces, edges, and vertices of polyhedrons such as the ones pictured. You can discover this relationship called **Euler's Theorem** (named after Leonhard Euler, 1707–1783) by copying and completing the table.

Polyhedron	V	F	E	$V + F$	$(V + F) - E$
1	8	6	12	14	
2	10				
3					
4					
5					
6					
7					
8					
9					
10					
11					
12					





13 In general, for polyhedrons such as these, $(V + F) - E =$?.

Objective

The student can count the faces, edges, and vertices of a polyhedron and show that $(V + F) - E = 2$.

PREPARATION

Materials: Polyhedron models of various kinds.

Use the regular polyhedrons constructed on the previous page to initiate a discussion and identification of vertices, faces, and edges.

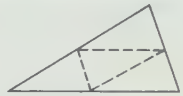
UTILIZATION

Assign the exercises as independent work. If possible, provide models of some of the figures for students who have difficulty.

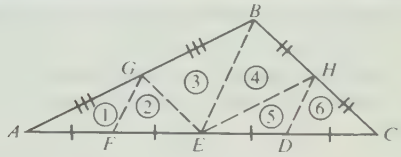
The Euler formula is not especially important at this level, but students may find it interesting and it may be useful as background for later experiences with geometry.

EXTENSION

Enrichment: Challenge students to show that a tetrahedron can be constructed from any triangular region whose angles are all *acute*. This can be done relatively easily by finding the midpoints of the sides and folding along segments joining the midpoints.



If one angle of the region is *obtuse*, a tetrahedron can still be formed but the construction is much more complex. Find the midpoints of the sides and points F and D as shown. Then fold on the dotted lines. Regions 2 and 5 come together to form one face and regions 1 and 6 come together to form a second face. Regions 3 and 4 provide the remaining two faces.



Assignments

Minimum 1–4. Average 1–8, 13.
Maximum 1–13.

Objective

The student can compute volumes of prisms, cylinders, pyramids, and cones by using the appropriate volume formula.

PREPARATION

Materials: Posterboard; scissors; compasses; protractors; centimetre rulers; models of cube, cylinder, square pyramid, and cone.

If possible, display models of the figures shown in the Investigation. Point out that this lesson will help the student to see how the volumes of these figures are related and how to compute the volumes.

INVESTIGATION

Most students will be able to reason that the volume of the cube is greater than the volume of the cylinder because the cylinder will "fit inside" the cube with space at each corner left over. Similarly, the cone will "fit inside" the square pyramid.

Unless students have had previous experience with the Investigation question they are likely to guess that the cube has about twice the volume of the pyramid instead of three times the volume. Similarly, they may be surprised to learn that the cylinder has a volume three times that of the cone.

DISCUSSION

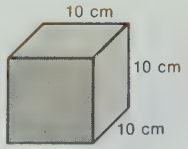
The Discussion period should be devoted to construction of the four solids. This work can be divided so that not every student needs to make all four models.

In any case, it is important to conduct a demonstration that compares the volume of the square pyramid with that of the cube, and the volume of the cylinder with that of the cone. The comparisons can be made by using sand, rice, or beans to fill each container. Commercially made containers of appropriate dimensions could also be used to compare the volumes.

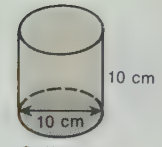
Comparing Volumes of Space Figures

Investigating the Ideas

1. Which has the greater volume, the cube or the cylinder?

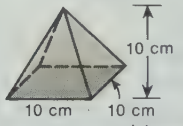


Cube

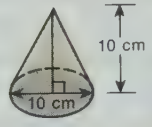


Cylinder

2. Is the volume of the pyramid more or less than the volume of a cone



Square pyramid



Cone

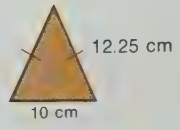
Can you guess how the volumes of the cube and the pyramid compare? Can you guess how the volumes of the cylinder and the cone compare?

Find a way to check your guesses.

Discussing the Ideas

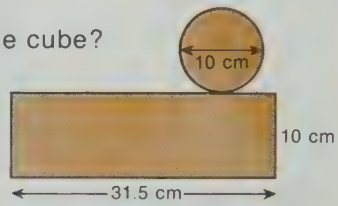
1. Make a cardboard model of the cube pictured above. Leave one face (the top) open.

2. Make a model of the pyramid without its base. It has the same height as the cube. You will need four isoceles triangles with the dimensions shown at the right.

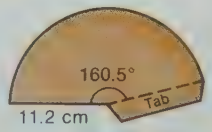


3. Fill the pyramid with rice, beans, or sand. How many pyramidfuls does it take to fill the cube?

4. Make a model of the cylinder omitting one of its bases.



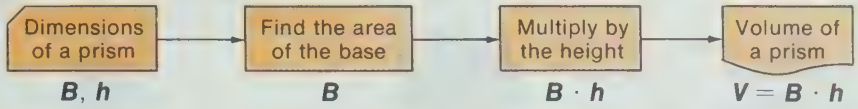
5. Make a model of the cone that has the same height as the cylinder. Omit the base. Use a pattern like this one.

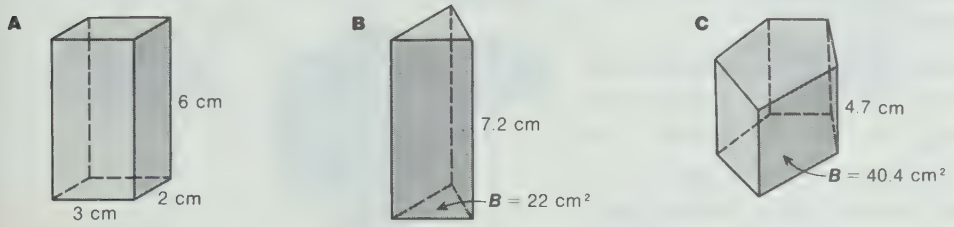


6. How many conefuls of rice, beans, or sand does it take to fill the cylinder?

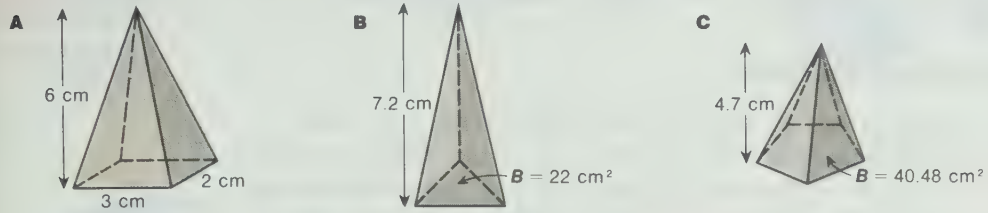
Using the Ideas

1. Find the volume of each prism. The flow chart will help you.

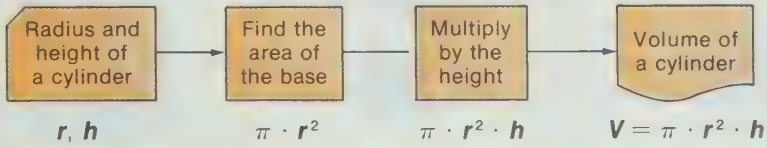


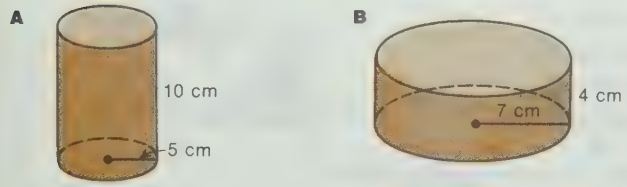


2. The volume of a pyramid is one-third the volume of a prism having the same base and height. Use the formula $V = \frac{1}{3} \cdot B \cdot h$ to find the volume of each pyramid.



3. Find the volume of each cylinder with radius r and height h. The flow chart will help you. Use $\pi \approx 3.14$.





C $r = 0.8$ cm, $h = 3.2$ cm
D $r = 1.3$ m, $h = 3.8$ m
E $r = 4.2$ cm, $h = 4.2$ cm
F $r = 305$ mm, $h = 460$ mm

4. The volume of a cone is one-third the volume of a cylinder having the same base and height. Use the formula $V = \frac{1}{3} \cdot \pi \cdot r^2 \cdot h$ to find the volume.

A $r = 5$ cm, $h = 10$ cm
B $r = 1$ cm, $h = 1$ cm
C $r = 2.5$ cm, $h = 4$ cm
D $r = 0.4$ cm, $h = 2.1$ cm
E $r = 2$ m, $h = 3.9$ m
F $r = 18$ mm, $h = 42$ mm

More practice, page S-28, Sets 48 and 49

UTILIZATION

The Exercises employ flow charts to present the formulas for the volumes of space figures. It is not essential that students memorize the formulas at this time.

Point out that the volumes of prisms and cylinders are computed by the same basic formula: area of base · height. In the case of the circular cylinders the area of the base is πr^2, hence base · height = $\pi r^2 h$.

EXTENSION

See Supplementary Exercise Sets 48 and 49, page S-28; *Workbook* page 107; pages 71 and 72 of the *Duplicator Masters;* and *Arithmetic Skill Cards* AP-12 through AP-15 for additional practice with the concepts in this lesson.

Enrichment: The volume of irregular objects can be determined by measuring the amount of water they displace. Place a fixed amount of water, say 600 ml, in a graduated container.

Drop some irregular object such as a small rock in the water. Suppose the water level then rises to the 650 ml mark. The stone has displaced 50 ml of water, hence the volume of the stone must be 50 cm^3 since 1 ml of water has a volume of 1 cm^3.

Assignments

Minimum 1A; 2A; 3A, B; 4A, B.
Average 1A, B; 2A, B; 3A–D; 4A–D.
Maximum 1–4.

Objective

The student can identify a cross section of a space figure, given a picture of the space figure and the plane of intersection.

PREPARATION

Materials: Small cubes, index cards, scissors.

You might wish to have students make models of the cube with 4-cm edges, although any small cubes could be utilized in the Investigation.

INVESTIGATION

The Investigation provides a concrete model of a plane intersecting a cube. The intersection of the plane and the cube is called a *cross section* of the cube.

Many different "holes" will fit on the cube. Any acute triangular hole will fit. Some students may discover trapezoidal or rectangular "holes" as well as triangular ones.

DISCUSSION

Give students the opportunity to demonstrate the cross sections of a cube that they found for the Investigation as they discuss these exercises. The cross section illustrated in Exercise 2 is a rectangle. If the card is tilted slightly, an isosceles trapezoid will be formed.

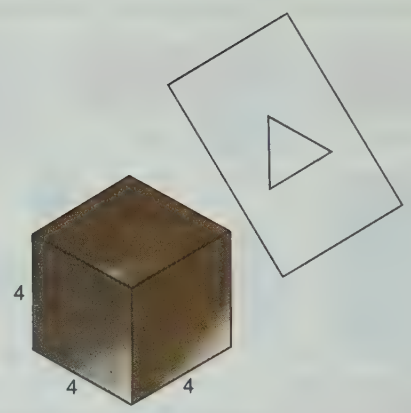
Cross Sections

Investigating the Ideas

You will need a small cube with 4-centimetre edges and some index cards to carry out this investigation.

Draw an equilateral triangle on the index card and cut out this region to form a triangular hole.

Find a way to place the card on the cube so that the triangular hole "fits" on the cube, without bending the card.

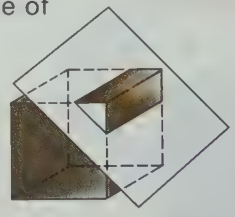


? Can you find other differently-shaped holes that "fit" on the cube?

Discussing the Ideas

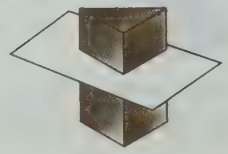
1. When the index card with the triangular hole is placed on the cube so that it just "fits," you have made a model of a plane intersecting a cube. This particular intersection of the plane and the cube is an equilateral triangle. The intersection of a cube and a plane is called a **cross section** of the cube.
   - **A** Show a cross section of the cube using a triangular hole which is not equilateral.
   - **B** Do you think that there are some triangles which cannot be cross sections of the cube? Show this with one of your index cards.

2. **A** What is the cross section of the cube illustrated in the drawing?
   - **B** What other shaped cross sections of a cube can you find? Show them.

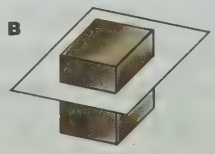


3. Other space figures also have cross sections. What is the shape of the cross section in each figure below?

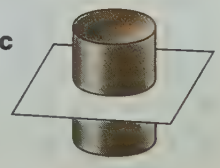
A



B



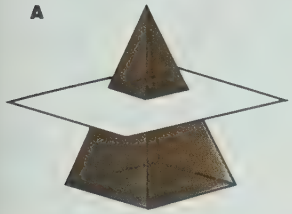
C



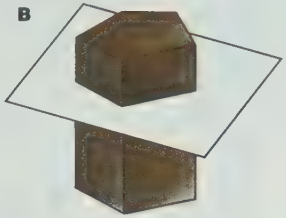
Using the Ideas

1. Make a sketch of the cross section formed by the intersection of the plane and each space figure.

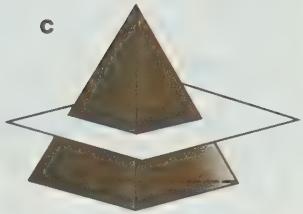
A



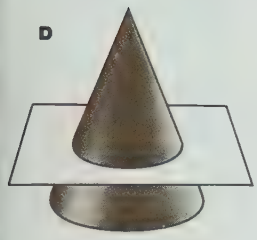
B



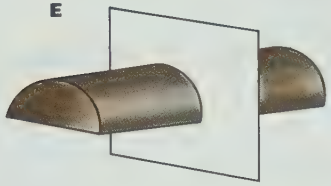
C



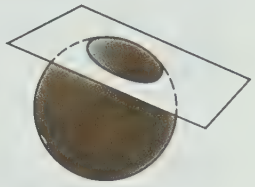
D



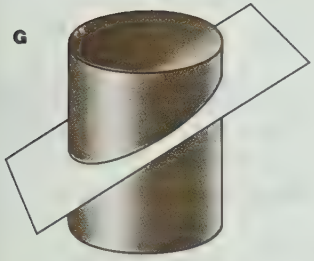
E



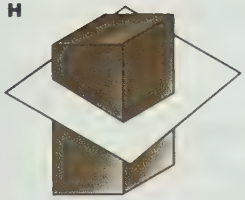
F



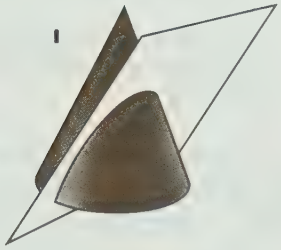
G



H

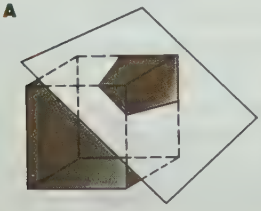


I

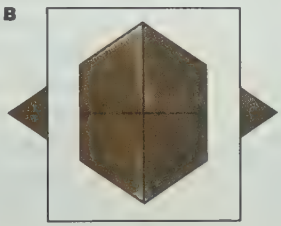


★ 2. Draw the approximate shape of each cross section for each figure.

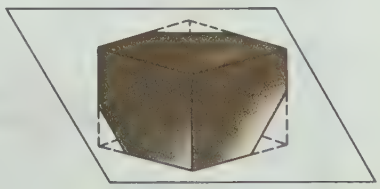
A



B



C



UTILIZATION

Ask your students to sketch the cross section for each figure presented in Exercises 1 and 2. These exercises may stimulate interesting discussions. For example, your students may disagree on what some of the cross sections would actually look like. You may want to find methods of demonstrating some of these and suggest variations that would result from changing the angle of intersection.

EXTENSION

Collect a variety of familiar objects and have students try to visualize and then draw approximate cross sections of the objects. Use objects such as a football, a piece of metal pipe, pencils, containers of all kinds. This kind of practice will aid the development of the students' spatial visualization.

Assignments

Minimum 1A–F. Average 1A–I.
Maximum 1, 2.

Objective

The student can demonstrate the ability to work with the concepts presented in this module.

UTILIZATION

The review exercises cover the major ideas and concepts developed in the chapter. If you have omitted some of the lessons of this module, be careful not to assign any review problems related to material presented in such lessons.

Plan to allow sufficient time for discussion and checking of the exercises after students have completed them.

Page 124 of the *Workbook* may be assigned to provide further review of the ideas presented in this lesson.

REVIEWING THE IDEAS

1. A Name the diameter.

B $\overline{AO}$ is a ? of the circle.

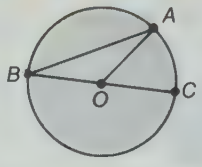
C Name a chord that is not a diameter.

D Name the arc subtended by $\angle AOC$.

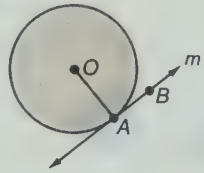
E Name the central angle subtended by $\widehat{AC}$.

F Name the inscribed angle subtended by $\widehat{AC}$.

G If $m\angle CBA = 25°$, then $m\angle COB =$?.



2. Line m is tangent to circle O at point A. What can you say about $\angle OAB$?



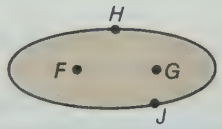
3. If a diameter of a circle is 10.6 cm, how long is a radius?

4. What is the decimal approximation for π to the nearest hundredth?

5. What is the circumference of a circle having a diameter of 21 cm? (Use $\pi \approx \frac{22}{7}$.)

6. What is the area of a circular region having a radius of 3.5 cm? (Use $\pi \approx 3.14$.)

7. If F and G are the focal points of an ellipse, how do the lengths of the polygonal paths FHG and FJG compare?



8. This is a net for a

A tetrahedron
B octahedron
C icosahedron
D dodecahedron

9. Identify each figure using the words listed.

A

B

C

D

E

F

G

H

circular cylinder
cone
cube
hexagonal prism
sphere
square pyramid
tetrahedron
triangular prism

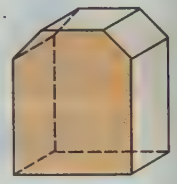
10. Look at the polyhedron.

A How many vertices?

B How many faces?

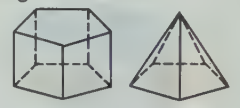
C How many edges?

D What is the $(V + F) - E$?



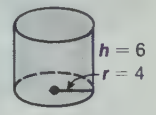
11. The base of a prism has an area of 42 cm^2 and a height of 9 cm.

A What is the volume of the prism?

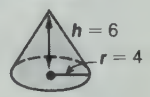


B What is the volume of a pyramid whose base area and height are the same as the prism?

12. A What is the volume of the cylinder?



B What is the volume of the cone?



13. The cross section formed by the intersection of the plane and the cube is a

A square

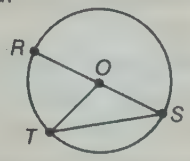
B rectangle

C parallelogram

TEST YOURSELF

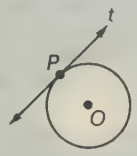
1. Give the name for each part of the circle listed.

   A $\overline{RS}$
   B $\overline{ST}$
   C $\overline{OS}$
   D $\overset{\frown}{RT}$



2. Using the figure in Exercise 1, if $m\angle ROT = 80°$, what is $m\angle RST$?

3. Line t intersects the circle only at point P. Line t is a ? of the circle.

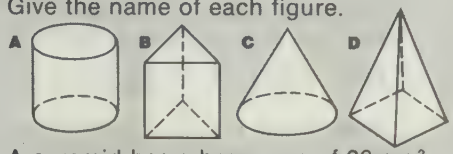


4. Find the circumference of a circle whose diameter is 6.2 cm. (Use $\pi \approx 3.14$.)

5. Find the area of a circle whose radius is 1.6 m. (Use $\pi \approx 3.14$.)

6. This is a net for a
   A tetrahedron
   B hexahedron
   C octahedron

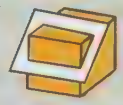
7. Give the name of each figure.



8. A pyramid has a base area of 36 cm^2 and a height of 10 cm. What is the volume of the pyramid?

9. A cone has a volume of 120 cm^3. If a cylinder has the same base and height as the cone, what is the volume of the cylinder?

10. What is the shape of the cross section shown by this figure?



RESEARCH PROJECTS

A Find the sizes and prices of pizzas. Compare the area of the pizzas and decide which size is the best buy for the money.

B A cube can be used to tessellate space. That is, a stack of cubes can be arranged so that no gaps occur in the stack. What are some other polyhedra that could be used to tessellate space? (See *Geometry in the Classroom: New Concepts and Methods* by H. A. Elliot, James R. McLean, and Janet M. Jorden; Holt, Rinehart and Winston of Canada Limited, 1968.)

C Find as many different sided polygons that can be cross sections of a cube. Make models to show these cross sections. You should be able to find triangles, squares, rectangles, trapezoids, a pentagon, and a hexagon.

D There are 13 semiregular polyhedra called the **Archimedian solids**. Find some patterns for these solids and make models of some of them. (See *Polyhedra Models for the Classroom* by Magnus J. Winniger; Washington, D.C.: National Council of Teachers of Mathematics, 1966.)

TEST YOURSELF

Assign the self-evaluation problems for independent student work. Call attention to the corresponding objective for each question (see annotated cross references). If time permits, discuss the problems that individual students had difficulty with and refer the students to the appropriate lessons of the module for a review of the pertinent ideas.

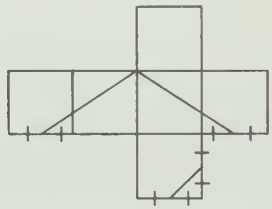
You might want to follow up this self-test by administering a module achievement test of your own devising or the test for this module that is available from the publisher.

RESEARCH PROJECTS

Research Project A involves problems related to areas of circles and could easily be expanded to include experiences in comparison shopping by asking students to gather data from several pizza parlors to determine which offers the most for the money.

Tessellating space (Project B) is also called "stacking." Any regular prism can be used to tessellate space. Other tessellations can be made by using combinations of polyhedrons. For example, regular tetrahedrons and octahedrons can be used for stacking. The cited reference provides other examples.

Students who undertake Project C will recall that in the lesson on cross sections they found cross sections that were triangles and cross sections that were quadrilaterals. Exercise 2C on page C-123 depicts how a hexagon can be a cross section of a cube. It is formed by connecting points of the six faces as shown below.



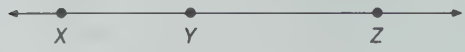
CUMULATIVE REVIEW

Plan to spend at least one day with the students working on and discussing the Cumulative Review Exercises, especially if you plan to give a test covering the complete learning unit. (Achievement Tests for each unit are available from the publisher as part of evaluation program designed for this series.)

Because this review set is relatively lengthy, you may wish to assign the exercises selectively. If so, it will be helpful to note that Exercises 1 through 12 pertain to Module 1; Exercises 13 through 19, to Module 2; Exercises 20 through 26, to Module 3; and Exercises 27 through 32, to Module 4.

CUMULATIVE REVIEW

1. Name three segments using the points given.

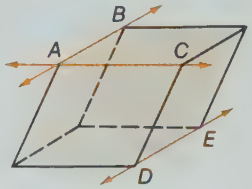


2. Using the drawing in Exercise 1, Which ray does not contain point Z?

[A] $\overrightarrow{XY}$ [B] $\overrightarrow{YX}$ [C] $\overrightarrow{ZX}$

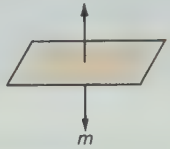
3. Complete each sentence with the words **intersecting**, **parallel**, or **skew**.

A $\overline{AB}$ and $\overline{DE}$ are ? lines.
B $\overline{AB}$ and $\overline{AC}$ are ? lines.
C $\overline{AC}$ and $\overline{DE}$ are ? lines.



4. Which phrase best describes the picture?

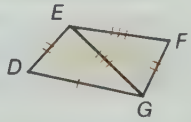
[A] a plane parallel to line m
[B] a plane containing line m
[C] a line intersecting a plane



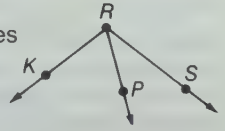
5. Complete the sentence:

All the points in a plane that are on one side of a line in the plane is called a ? .

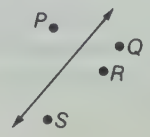
6. Name two pairs of congruent segments in the figure.



7. Name three angles shown by the drawing.



8. Which point Q, R, or S is the reflection image of point P in the line?



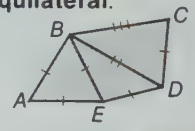
9. Use the words listed below to complete each sentence about the figure.

A $\angle DBE$ is a(n) ? angle.
B $\angle ABC$ and $\angle FBE$ are ? angles.
C $\angle ABE$ is a(n) ? angle.
D $\angle FBD$ is a(n) ? angle.
E $\angle CBD$ and $\angle DBF$ are ? angles.
F $\overrightarrow{AE}$ is ? to $\overrightarrow{FC}$.
G $\angle ABC$ is a(n) ? angle.

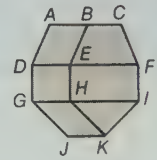
acute right perpendicular obtuse straight vertical supplementary

10. Tell whether each triangle is **scalene**, **isoceles**, or **equilateral**.

A $\triangle ABE$
B $\triangle BCD$
C $\triangle BDE$



11. Name the types of polygons using the letters shown in the figure.



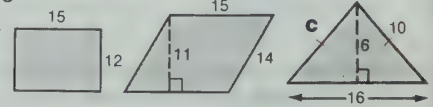
A $ABED$ C $BCFE$ E $BCFIHE$
B $DEHG$ D $HEFI$ F HIK

12. Is this a convex polygon?

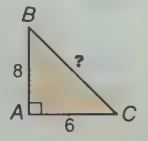
13. If a segment is 38 cm long, then it is ? mm long or ? m long.

14. Find the perimeter and area of each figure.

A

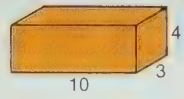


15. Using the Pythagorean Theorem, find the length of $\overline{BC}$.



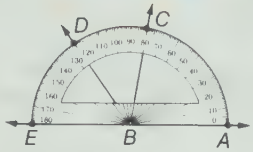
16. **A** Find the volume of the figure.

B Find the surface area.



17. **A** $m\angle ABD$ is ? .

B $m\angle$? is 45°.



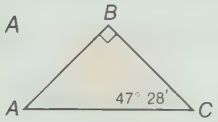
18. **A** Find the sum.

$$\begin{array}{r} 124°58'19'' \\ +\ 55°23'26'' \\ \hline \end{array}$$

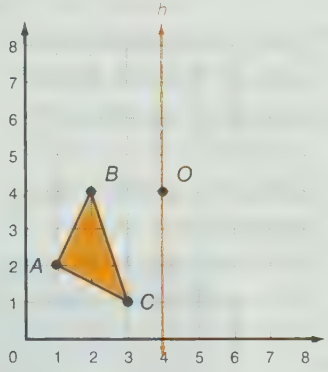
B Find the difference

$$\begin{array}{r} 83°46'37'' \\ -\ 45°18'49'' \\ \hline \end{array}$$

19. What is $m\angle A$ of $\triangle ABC$?



Refer to the grid for Exercises 20 through 23.



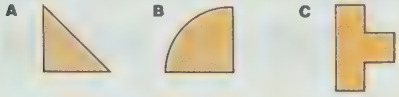
20. Give the co-ordinates of the vertices of $\triangle ABC$.

21. Give the co-ordinates of the images of A, B, and C after a translation of "right 4, up 3."

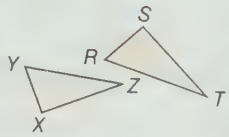
22. Give the co-ordinates of the images of A, B, and C after a $\frac{1}{2}$ turn with point O as the centre of rotation.

23. Give the reflection image of $\triangle ABC$ in line n.

24. Which region(s) could tessellate a plane?



25. If $\triangle RST \cong \triangle XYZ$, name the pairs of angles and sides that are congruent to each other.

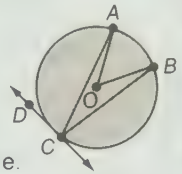


26. If $\overrightarrow{OB}$ is the bisector of $\angle AOC$, how do $\angle AOB$ and $\angle BOC$ compare?

27. **A** Name an inscribed angle in the figure.

B Name a central angle.

C If $m\angle AOB = 40°$, then $m\angle ACB$ = ? .



28. In the figure in Exercise 27, CD is a ? to the circle.

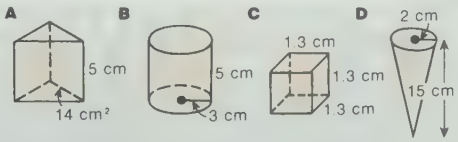
29. How many faces, edges, and vertices does this figure have?

30. A small table has a circular top with a diameter of 60 cm.

A What is the circumference of the table top?

B What is the area of the table top?

31. Find the volume of each figure.



32. Which space figure cannot have a circular cross section?

[A] sphere [B] pyramid [C] cone

MATHEMATICAL RECREATION

This page is devoted to some unusual problems concerning nickels. If feasible, provide 5 or 6 nickels for the students to use as they work on these problems.

Capable problem-solvers will especially enjoy these problems, but other students may also find them challenging and interesting.

MATHEMATICAL RECREATIONS

NICKEL PROBLEMS

1. Which name best describes the shape of a nickel?
   - A circle
   - B sphere
   - C circular region
   - D solid cylinder

2. What is the diameter of a nickel to the nearest millimetre?

3. What is the thickness of a nickel to the nearest millimetre?

4. About how many nickels would be in a stack in order to be as tall as one standing on edge?

5. You can find the volume of a nickel by finding the product of the area of its base and its height. That is,

   Volume = area of base · height

   $V = \pi \cdot r^2 \cdot h$

   - A What is the area of the base?
   - B What is the height?
   - C What is the volume?

6. What is the volume, to the nearest cubic centimetre, of a dollar's worth of nickels?

7. A nickel is composed of 75% copper and 25% nickel. About how many cubic centimetres of copper are in a dollar's worth of nickels (see Exercise 6)?

8. A If the smallest possible square were drawn around four adjacent (touching) nickels, what would be the area of the square in square centimetres?

   B About what part of the square would be covered by the nickels?

MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. A diameter; B chord; C radius; D arc **2.** 40° **3.** tangent **4.** 19.468 cm **5.** 8.0384 m^2 **6.** A **7.** A circular cylinder; B triangular prism; C cone; D square pyramid **8.** 120 cm^3 **9.** 360 cm^3 **10.** rectangle

C–128

UNIT D: INTEGERS, PROBABILITY, AND STATISTICS

Module 1: Integers

General Objectives

To introduce the set of integers.
To show how the basic principles can be used to develop the computational rules for integers.
To introduce graphs of integers on the number line and graphs of ordered pairs of integers in the co-ordinate plane.
To provide experience with functions with integer domains.

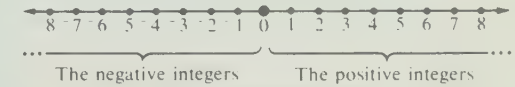
	Performance Objectives	Pupil Text	Reteach-Reinforce	Related Activities	
123	**RED** Given an integer, the student can name its opposite.	D-2, 3		**PD** d-1	
124	**RED** Given any two integers, the student can find the sum.	D-4 through 9	**WB** 109, 110 **DM** 73	**PD** d-2, 3 **DS** c-1, 2	**SWM 3** 70-73
125	**RED** Given any two integers, the student can find the difference.	(D-10, 11) D-12, 13	**WB** 111, 112 **DM** 74	**PD** d-4 **DS** c-1, 2	
126	**RED** Given any two integers, the student can find the product.	D-16, 17	**WB** 113, 114 **DM** 75	**DS** c-3, 4	
127	**RED** Given any two integers, the student can find the quotient.	D-18, 19	**WB** 114 **DM** 76	**DS** c-3, 4	
128	**RED** Given addition and multiplication equations, the student can solve them by using the basic principles.	D-4, 5 D-14, 15	**WB** 109, 113		
129	**YELLOW** Given two integers, the students can compare them by naming the larger.	D-20, 21		**PD** d-1	
130	**YELLOW** The student can give the co-ordinates of a point using integer pairs.	D-22, 23	**WB** 115 **DM** 77		**SWM 3** 152, 153
131	**YELLOW** The student can identify the rule needed to graph a set of integer pairs.	D-22, 23	**WB** 115 **DM** 77		
132	***GREEN*** *The student can demonstrate an understanding of functions involving pairs of integers.*	D-24, 25	**DM** 77		
	Reviewing the Ideas	D-26	**WB** 116		

MATHEMATICS

In many instances in both applied and theoretical mathematics, the set of whole numbers and fractional numbers are inadequate. There are times when it is helpful to have numbers that answer not only the question "How many?" but also "In what direction?" Such numbers are desirable for describing the direction in which electric current flows, net change in stock prices, bookkeeping debits and credits, temperature fluctuations, locations in space with reference to a central point, and so on. Moreover, theoretical and applied aspects of mathematics require a system in which addition, multiplication, and subtraction are always possible.

These considerations have led to the creation of the set of integers by means of the principle called the *opposites principle*. For each whole number a, there is a corresponding number ^{-}a, which has the property $^{-}a + a = 0$.

The numbers indicated by ^{-}a are called the *negative integers*. On the number line these are opposite the whole numbers, which are called the *positive integers*. The set containing the positive integers, the negative integers, and zero is called the set of *integers*.



The Set of integers

The following basic principles, which hold true for the addition and multiplication of whole numbers, are now assumed to hold true for the integers. For any integers a, b, and c:

Commutative Principle
(Add.) $a + b = b + a$
(Mult.) $a \cdot b = b \cdot a$
Associative Principle
(Add.) $a + (b + c) = (a + b) + c$
(Mult.) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
Distributive Principle
$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
Zero Principle
$a + 0 = a$; $a \cdot 0 = 0$
One Principle
$a \cdot 1 = a$

The three examples below show how the basic principles for addition can be used to find sums of integers.

A. $10 + {}^{-}4 = (6 + 4) + {}^{-}4$
$= 6 + (4 + {}^{-}4)$ Associative prin.
$= 6 + 0$ Opposites Prin.
$= 6$ Zero Principle

B. $({}^{-}4 + {}^{-}2) + 6 = ({}^{-}4 + {}^{-}2) + (4 + 2)$
$= ({}^{-}4 + 4) + ({}^{-}2 + 2)$
Commutative and Associative Principles
$= 0 + 0$ Opposites Principle
$= 0$

Now, $^{-}6$ is the only number that gives 0 when added to 6. Since $^{-}4 + {}^{-}2$ gives 0 when added to 6, we have $^{-}4 + {}^{-}2 = {}^{-}6$

C. $^{-}8 + 3 = (^{-}5 + {}^{-}3) + 3$
$= {}^{-}5 + (^{-}3 + 3)$ Associative Prin.
$0\ {}^{-}5 + 0$ Opposites Prin.
$= {}^{-}5$ Zero Principle

We define the subtraction of integers in much the same way that we defined the subtraction of whole numbers. Hence, in order to find a difference, we think of finding a missing addend. Thus, for integers a and b, the difference $a - b$ is the integer that added to b gives a.

S	A	A		A	A	S
a	$-$	b	$= n$	n	$+$	$b = a$

Like the sum of two integers, the product of two integers can be determined by relying on the basic principles.

We define quotients by defining the division of integers as we defined the division of whole numbers. That is, to find a quotient, we think about finding a missing factor:

For integers a and b, where b is a nonzero factor of a,
$a \div b = n$ if and only if $n \cdot b = a$.

TEACHING THE MODULE

Materials
Graph paper, co-ordinate grid paper.

Vocabulary

axis	integer pair
co-ordinate axes	negative integer
co-ordinates	opposite
function	opposites principle
graph	origin
integer	positive integer

The computational rules for integers are developed through physical situations such as rise and fall of temperatures, paying bills and receiving cheques, and the number line examples as well as through application of the basic principles.

The material of this module can be exciting for most of your students, but do not expect them to master all of the factual material. The mastery of the essential facts can best be fostered in an atmosphere in which students become stimulated by active involvement in the discovery of new ideas in mathematics.

Lesson Schedule
It is likely that much of the material in the module will be new to most students, so you will probably need to cover it at a fairly moderate pace. Allow a minimum of 10–12 days for complete coverage. A longer amount of time may be needed by many students. However, you should be careful not to spend too much time on any one aspect of the module.

Evaluation
It is relatively simple to evaluate the students' ability to compute with integers. However, you should not overemphasize computation in this module. Your students will review the integers again in Book 2 of *School Mathematics*, and will no doubt receive further work with integers in high school. Hence, complete mastery is not important at this stage.

Attempt to maintain an air of exploration and discovery in working with a new set of numbers in this module. The attitudes toward this kind of material that you can develop in students now will be extremely important as the student begins a study of algebra in later school years.

RESOURCES FOR ACTIVE LEARNING

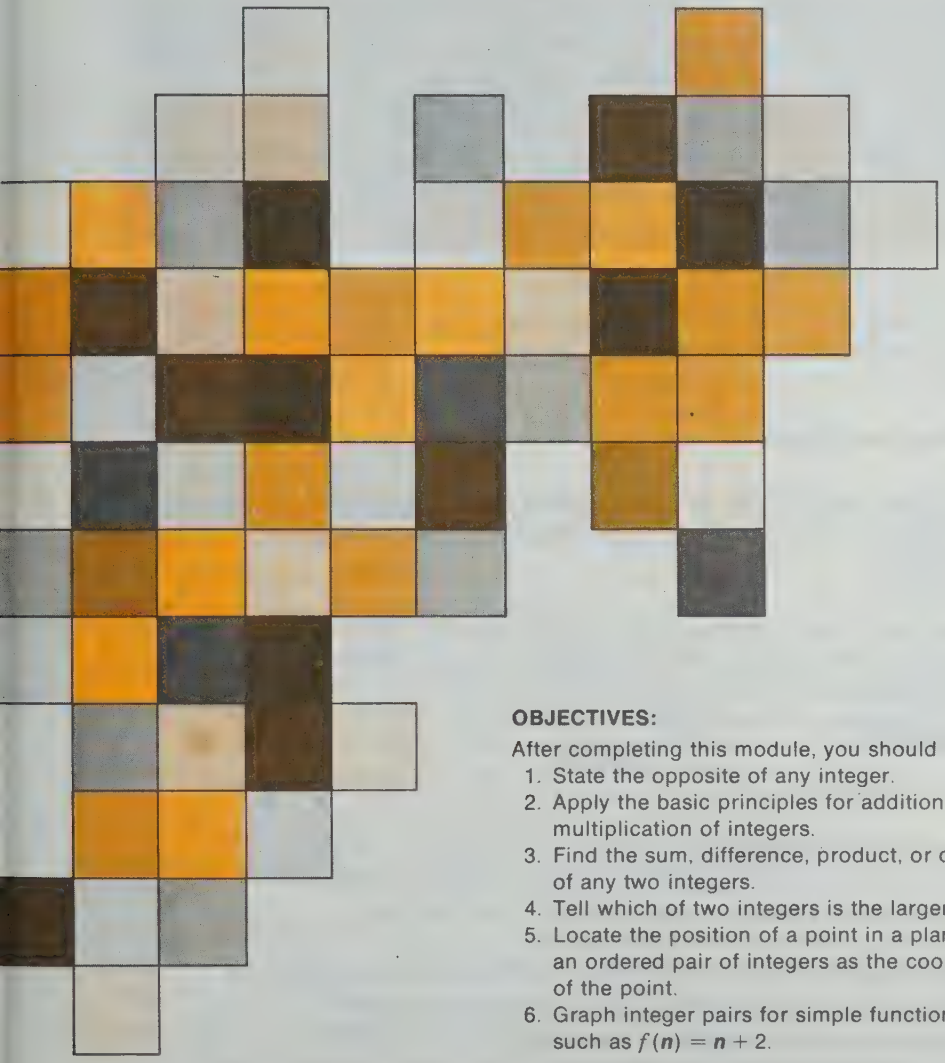
Activities
Developmental Math Cards, L16, 17, Addison-Wesley
Graph Gallery, Creative Publications
Inquiry in Mathematics via the Geoboard, "Tic-Tac-Toe," Card Set 24, Walker
Let's Play Games in Mathematics, Vol. 7, "Shake a Square," pp. 72–73, National Textbook Company
Madison Project: *Discovery*, "The Point-set Game," Chapter 7; *Explorations*, Part 1, Addison-Wesley
Making and Using Graphs and Nomographs (Integers and graphs), pp. 39–70, Rand McNally
Nuffield Project: *Computation and Structure 4* (Integers), pp. 22–49, Wiley
Sourcebook for Substitutes, pp. 71–72, 80, 94–96, Addison-Wesley

Manipulative Devices
Integers Kit (General Learning Corp.)
Spinner Kit (Houghton Mifflin)

Games and Puzzles
Battleship (retail market)
Vectors (Selective Educational Equipment)
Counter-Trek (Sigma)

UNIT D: *Integers, Probability, and Statistics*

MODULE 1: *Integers*



OBJECTIVES:

After completing this module, you should be able to:

1. State the opposite of any integer.
2. Apply the basic principles for addition and multiplication of integers.
3. Find the sum, difference, product, or quotient of any two integers.
4. Tell which of two integers is the larger.
5. Locate the position of a point in a plane given an ordered pair of integers as the coordinates of the point.
6. Graph integer pairs for simple function rules such as $f(n) = n + 2$.

Objective

Given an integer, the student can give the opposite of the integer.

PREPARATION

Since this lesson introduces a new topic, the Investigation can be undertaken without review or other preparation.

INVESTIGATION

The Investigation is designed to introduce the student to thinking about opposites. Ask students to write the word they think has a meaning that is opposite each listed word. When all students are finished, have them give the opposites they have found and discuss their ideas. For some of the words, students may suggest more than one word that is opposite in meaning. Here are sample responses.

1. down, up
2. forward, backward (reverse)
3. below, above (over)
4. north, south
5. left, right
6. under, over (above)
7. raise, lower
8. win, lose
9. east, west
10. positive, negative
11. decrease, increase
12. empty, full
13. faster, slower
14. gain, loss (lose)
15. plus, minus (subtract)

DISCUSSION

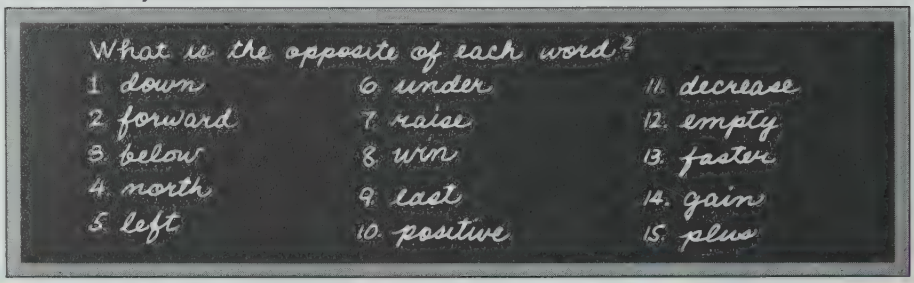
The Discussion leads students from the consideration of words that are opposites in meaning toward the idea of numbers that are opposites.

The negative integers are introduced by displaying them on the number line and by matching them one to one with the set of nonzero whole numbers. Note that the nonnegative whole numbers are now called *positive integers*, and the newly introduced negative numbers are called *negative integers*. The full set of integers consists of the positive integers, the negative integers, and zero. The phrase, "is the opposite of," is used to describe the relationship between such pairs of numbers as 1 and $^{-}1$, 2 and $^{-}2$, and so on.

What are Opposites?

Investigating the Ideas

Study the list of words shown on the chalkboard.

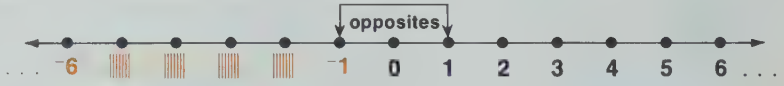


Can you write a word that has an "opposite" meaning for each of the words above?

Discussing the Ideas

1. Think of two words that are opposite in meaning. Tell a classmate one of the words and then see if he can name its opposite.

2. Just as some words are opposites, so are some numbers opposites of each other. We can think of the numbers for the points on opposite sides of zero on the number line as **opposites** of each other.



For the point **opposite 1**, we write $^{-}1$ and call it negative 1
We say that 1 is the opposite of $^{-}1$ or $^{-}1$ is the opposite of 1.
Give the other missing numbers on the number line.

3. Each nonzero whole number (positive integer) has an opposite. The opposite of a positive integer is a negative integer. Positive and negative integers, together with zero, form a set of numbers called the **integers**.

The set of integers = {0, 1, $^{-}1$, 2, $^{-}2$, 3, $^{-}3$, 4, $^{-}4$, 5, $^{-}5$, . . .}

Give the opposite of each integer.

A 7 **B** 21 **C** $^{-}2$ **D** $^{-}19$ **E** 47 **F** 0

Using the Ideas

1. Give the opposite of each integer.

A 9	**D** $^{-}4$	**G** 25	**J** 586	**M** $^{-}5936$
B 7	**E** $^{-}9$	**H** 59	**K** $^{-}1079$	**N** 100
C 5	**F** $^{-}11$	**I** $^{-}86$	**L** 1 687 943	**O** $^{-}863\,650$

2. Think about the integers on the number line in order to give the integers described in each part.

- **A** The opposite of 5
- **B** The opposite of $^{-}3$.
- **C** The integer 2 units left of zero
- **D** The integer 4 units right of zero
- **E** The integer 3 units left of $^{-}1$
- **F** The integer 3 units right of $^{-}5$
- **G** The integer 3 units right of $^{-}3$
- **H** The integer 3 units left of 3
- **I** The integer 10 units right of $^{-}5$
- **J** The integer 6 units left of $^{-}2$
- **K** The opposite of the opposite of 4
- **L** The opposite of zero

3. **A** If 17° represents "17 degrees **above** zero," what integer represents "17 degrees **below** zero?"

B If $^{-}8$ represents a loss of 8 points in a game, what integer represents a win of 8 points?

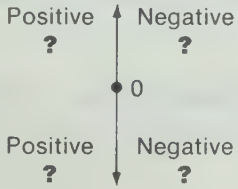
C If $^{-}15$ represents a discharge of 15 amperes of electrical current, what integer represents a charge of 15 amperes?

D If 6 denotes 6 seconds after blastoff for a space craft, what integer denotes 6 seconds before blastoff?

4. You may wonder why the positive numbers "go to the right" and the negative numbers "go to the left." It is simply a matter of choice.

A Draw a horizontal picture of the number line. Choose a zero point. Choose the positive number direction to the left. Mark points 1 cm apart. Label the number line.

B Draw a vertical picture of the number line. Choose a zero point. Choose a positive number direction either up or down and mark points 1 cm apart. Label the number line.



UTILIZATION

Assign the exercises as independent work. Exercises 1 and 2 could be discussed orally. Exercise 3 is important because it illustrates a practical application of negative integers.

EXTENSION

Now, while the idea of negative numbers has the special appeal of novelty, would be an opportune time to encourage some students to explore the history of negative numbers and present their findings to the class. (See Research Project B, page D-27.)

Assignments

Minimum 1–3. Average 1–3. Maximum 1–4.

Objective

The student can apply the opposites principle and the commutative, associative, and zero principles of addition to integers.

PREPARATION

Review the associative, commutative, and zero principles for whole numbers and fractional numbers. Then tell your students that in this lesson they are going to look at an important new principle for the addition of integers. Ask them to read the material at the beginning of the lesson before proceeding to the Discussion Exercises.

DISCUSSION

The Discussion Exercises are designed to focus attention on the basic principles for addition of integers, which are given in the table. The commutative, associative, and zero principles will be familiar to the student because they have been previously applied to the set of positive integers. Now these principles are accepted to hold for all integers—positive, negative, or zero. The "new" principle is the opposites principle. The importance of this principle in developing an understanding of addition of integers cannot be overestimated. Exercises 2 and 3 utilize the number line in helping to develop an understanding of this principle. Notice that use of the number line involves an agreement on the direction to jump for negative integers in addition. Since addition of positive integers has always been illustrated by jumping to the right on a horizontal number line, there should be little difficulty in getting students to accept the idea that jumps in the opposite direction, that is, to the left, should indicate addition of negative integers.

You may want to give additional examples like those in Exercises 5 and 6 to strengthen understanding of the basic principles of addition of integers.

Basic Principles for Addition of Integers

Discussing the Ideas

The table shows the basic principles that we accept for addition of integers.

Opposites principle	**Zero principle**
The sum of each integer and its opposite is zero.	The sum of any integer and zero is that integer. $a + 0 = a$
Commutative principle	**Associative principle**
When finding the sum of any two integers, you can change the order of the addends and the sum is the same For integers a and b, $a + b = b + a$	When finding the sum of any three integers, you can change the grouping of the addends and the sum is the same. For integers a, b, and c, $(a + b) + c = a + (b + c)$
Since you can change both the **order** and the **grouping** of the addends, you can rearrange addends in any way that is convenient.	

1. Figure A reviews whole number addition on the number line. Explain each jump and give the sum.

A

$1 + 2 = n$

2. Figure B suggests a way to think about the sum $^-1 + 1$. Explain each jump and give the sum.

B

$^-1 + 1 = n$

3. Explain each jump in Figure C and give the sum.

C

$^-2 + 2 = n$

4. Which basic principles do Figures B and C illustrate?

5. What is the sum in each part? Which basic principle must you use?

   A $2 + 0$ B $^-5 + 0$ C $0 + ^-7$ D $0 + 8$

6. Which basic principles do Examples A, B, and C illustrate?

EXAMPLES: A $^-4 + ^-3 = ^-3 + ^-4$
B $5 + ^-5 = 0$
C $(^-2 + 5) + ^-5 = ^-2 + (5 + ^-5)$

Using the Ideas

1. Find the sums.

   A $^{-}5 + 5$ C $^{-}23 + 23$ E $^{-}3 + 0$ G $^{-}18 + 0$
   B $^{-}8 + 8$ D $^{-}967 + 967$ F $^{-}9 + 0$ H $^{-}76 + 0$

2. Find the integer for *a*. Then find the integer for *b*. The opposites and zero principles will help you complete these exercises.

   A $^{-}1 + 1 = a$ — $1 + {}^{-}1 = b$ D $^{-}8 + 0 = a$ — $0 + {}^{-}8 = b$
   B $^{-}7 + 7 = a$ — $7 + {}^{-}7 = b$ E $^{-}72 + 0 = a$ — $0 + {}^{-}72 = b$
   C $^{-}56 + 56 = a$ — $56 + {}^{-}56 = b$ F $^{-}963 + 0 = a$ — $0 + {}^{-}963 = b$

3. Find the missing integer.

   A ▒ is the integer that adds to 8 to give the sum 0.
   B ▒ is the integer that adds to $^{-}17$ to give the sum 0.
   C ▒ is the integer that adds to 58 to give the sum 0.
   D ▒ is the integer that adds to $^{-}867$ to give the sum 0.
   E ▒ is the integer that adds to 1083 to give the sum 0.

4. Find the integer for *a*. Then find the integer for *b*.

   A $2 + (3 + {}^{-}3) = a$ — $(2 + 3) + {}^{-}3 = b$
   B $6 + (4 + {}^{-}4) = a$ — $(6 + 4) + {}^{-}4 = b$
   C $^{-}3 + (^{-}5 + 5) = a$ — $(^{-}3 + {}^{-}5) + 5 = b$
   D $^{-}26 + (^{-}8 + 8) = a$ — $(^{-}26 + {}^{-}8) + 8 = b$

5. Solve the equations. The variables represent integers.

   A $^{-}4 + x = {}^{-}4$ D $t + {}^{-}12 = 0$ G $n + (^{-}5 + {}^{-}8) = (6 + {}^{-}5) + {}^{-}8$
   B $7 + n = 0$ E $9 + {}^{-}9 = n + 9$ H $^{-}13 + (t + 4) = (^{-}13 + {}^{-}8) + 4$
   C $r + 0 = {}^{-}66$ F $^{-}56 + k = 0 + {}^{-}56$ I $(^{-}72 + 4) + {}^{-}45 = {}^{-}72 + (4 + s)$

6. Find the integer for *a*. Then find the integer for *b*. Remember that the commutative and associative principles together suggest that you can rearrange addends in any way you choose.

   A Since $(4 + {}^{-}4) + (5 + {}^{-}5) = a$, then $(4 + 5) + (^{-}4 + {}^{-}5) = b$.
   B Since $(6 + {}^{-}6) + {}^{-}8 = a$, then $(6 + {}^{-}8) + {}^{-}6 = b$.
   C Since $17 + (9 + {}^{-}9) = a$, then $(^{-}9 + 17) + 9 = b$.

★7. Solve the equations.

   A $(6 + {}^{-}7) + {}^{-}6 = n$ C $(n + {}^{-}8) + {}^{-}4 = {}^{-}8$ E $(n + {}^{-}2) + {}^{-}4 = 0$
   B $3 + (^{-}7 + 4) = n$ D $(^{-}7 + n) + 7 = 15$ F $5 + (^{-}3 + {}^{-}2) = n$

UTILIZATION

The main purpose of the exercises is to focus attention upon the role of the basic principles in addition. Little in the way of computing sums is expected at this stage other than sums found by applying the zero principle or the opposites principle. The general problem of adding any two integers will be presented in the lesson that follows this one.

EXTENSION

Page 109 of the *Workbook* provides exercises that may be used to give further practice with the ideas presented in this lesson.

Remedial: Ask students to draw a number line and explain how to use "jumps" on the number line to find each of these sums.

A. $^{-}3 + 3 = n$ C. $^{-}6 + 6 = n$
B. $^{-}4 + 4 = n$ D. $^{-}10 + 10 = n$

Enrichment: Have students use commutativity and associativity to show as many different ways as they can of arranging and grouping the integers $^{-}6$, 4, and 2 to form a sum. What is the sum in each case? *Solution:* 12 ways. Each sum is 0.

$^{-}6 + (4 + 2)$	$(^{-}6 + 4) + 2$
$^{-}6 + (2 + 4)$	$(^{-}6 + 2) + 4$
$4 + (^{-}6 + 2)$	$(4 + {}^{-}6) + 2$
$4 + (2 + {}^{-}6)$	$(4 + 2) + {}^{-}6$
$2 + (4 + {}^{-}6)$	$(2 + 4) + {}^{-}6$
$2 + (^{-}6 + 4)$	$(2 + {}^{-}6) + 4$

Assignments

Minimum 1–4. Average 1–6. Maximum 1–7.

Objective

Given two integers, the student can find their sum by thinking of physical examples or by using the basic principles for addition of integers.

PREPARATION

Review the opposites principle and the zero principle for integers; these two principles play important roles in developing methods of finding sums of integers.

INVESTIGATION

Investigation equations A and B can be solved by the students on the basis of their experiences with money and temperatures. The number line is suggested as a way of finding the sum in C, while the associative, opposites, and zero principles are suggested for equation D. Not every student will be successful in solving the equations, but the Investigation results should help you judge what points need to be emphasized in the Discussion.

DISCUSSION

All of the examples in the Investigation dealt with cases in which the two integers' sum was a positive integer. Slower students may need to continue to rely on physical models for addition, such as those in A and B in the Investigation, while other students can use the number line or the more deductive method illustrated in equation D. The steps for this method are as follows:

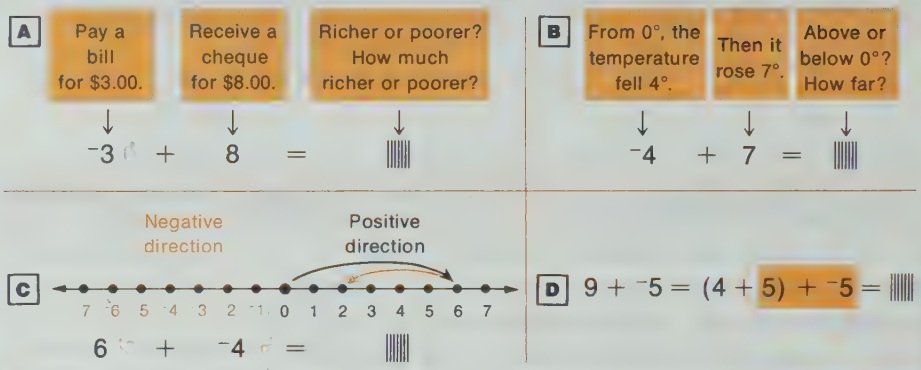
$9 + {}^{-}5 = (4 + 5) + {}^{-}5$	Addition fact $(9 = 4 + 5)$
$= 4 + (5 + {}^{-}5)$	Associative principle
$= 4 + 0$	Opposites principle
$= 4$	Zero principle

Exercise 6 introduces the addition of two negative integers by means of the number line. Students should be able quickly to generalize that the sum of any two negative integers is a negative integer.

Adding Integers

Investigating the Ideas

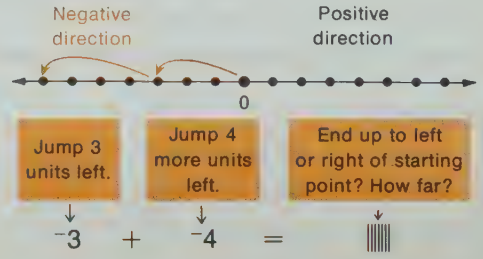
You can find the sum of two integers by using physical examples and logical reasoning.


A
Pay a bill for $3.00.
Receive a cheque for $8.00.
Richer or poorer? How much richer or poorer?
⁻3 + 8 =
B
From 0°, the temperature fell 4°.
Then it rose 7°.
Above or below 0°? How far?
⁻4 + 7 =
C
Negative direction
Positive direction
7 6 5 4 3 2 1 0 1 2 3 4 5 6 7
6 + ⁻4 =
D
9 + ⁻5 = (4 + 5) + ⁻5 =

How many of the sums in A, B, C, and D can you find?

Discussing the Ideas

1. Explain how you found the sum for A.
2. What is the sum for B? Explain your answer.
3. Explain how to use the number line in order to find sum for C.
4. Can you show and explain some other integer sums using the number line?
5. How can the basic principles be used to find the sum for D?
6. The number line suggests a way of finding the sum of two negative integers. What is the sum?


Negative direction
Positive direction
0
Jump 3 units left.
Jump 4 more units left.
End up to left or right of starting point? How far?
⁻3 + ⁻4 =

Using the Ideas

1. Think about the cheques and bills in order to find the sums.

	Pay a bill for $4.	Receive cheque for $10.	How much richer or poorer?		Pay a bill for $3.	Pay a bill for $6.	How much richer or poorer?		Pay a bill for $7.	Receive cheque for $10.	How much richer or poorer?
	↓	↓	↓		↓	↓	↓		↓	↓	↓
A	$^{-}4$	+ 10	= *n*	**B**	$^{-}3$	+ $^{-}6$	= *n*	**C**	$^{-}7$	+ 10	= *n*

2. Think about a thermometer in order to find the sums.

	20° above 0°.	Temperature falls 5°.	New temperature?		12° below 0°.	Temperature falls 7°.	New temperature?		10° above 0°.	Temperature falls 6°.	New temperature?
	↓	↓	↓		↓	↓	↓		↓	↓	↓
A	20	+ $^{-}5$	= *n*	**B**	$^{-}12$	+ $^{-}7$	= *n*	**C**	10	+ $^{-}6$	= *n*

3. Think about jumps on the number line starting at zero.

	Jump 4 units right.	Jump 2 units left.	How far left or right of zero?		Jump 5 units left.	Jump 3 more units left.	How far left or right of zero?		Jump 8 units right.	Jump 5 units left.	How far left or right of zero?
	↓	↓	↓		↓	↓	↓		↓	↓	↓
A	4	+ $^{-}2$	= *n*	**B**	$^{-}5$	+ $^{-}3$	= *n*	**C**	8	+ $^{-}5$	= *n*

4. Solve the equation to find the sum of the addends in the colored screen. Because of the basic principles, you can add any two addends first.

A **4 + $^{-}3$**
$1 + 3 + {}^{-}3 = n$

B **8 + $^{-}6$**
$2 + 6 + {}^{-}6 = n$

C **5 + $^{-}2$**
$3 + 2 + {}^{-}2 = n$

D **$^{-}3 + 7$**
$^{-}3 + 3 + 4 = n$

E **$^{-}9 + 15$**
$^{-}9 + 9 + 6 = n$

F **28 + $^{-}12$**
$16 + 12 + {}^{-}12 = n$

5. Find the sums. Use any method you choose.

A	$3 + {}^{-}3$	**H**	$^{-}2 + {}^{-}3$	**O**	$7 + {}^{-}8$	**V**	$({}^{-}6 + {}^{-}3) + {}^{-}10$		
B	$10 + {}^{-}7$	**I**	$^{-}10 + {}^{-}5$	**P**	$10 + {}^{-}17$	**W**	$85 + {}^{-}37$		
C	$12 + {}^{-}6$	**J**	$^{-}7 + {}^{-}3$	**Q**	$({}^{-}8 + {}^{-}5) + {}^{-}7$	**X**	$82 + {}^{-}75$		
D	$12 + {}^{-}4$	**K**	$8 + {}^{-}2$	**R**	$({}^{-}4 + {}^{-}6) + 7$	**Y**	$71 + {}^{-}49$		
E	$5 + {}^{-}3$	**L**	$^{-}2 + 1$	**S**	$5 + ({}^{-}8 + {}^{-}2)$	**Z**	$^{-}1000 + {}^{-}712$		
F	$^{-}7 + {}^{-}9$	**M**	$^{-}7 + 5$	**T**	$^{-}6 + (2 + 4)$				
G	$21 + {}^{-}11$	**N**	$4 + 2$	**U**	$^{-}8 + 2 + {}^{-}3$				

UTILIZATION

The exercise set is designed to help students discover on their own the rules for addition of integers, but you will probably need to give some of them additional help. Insofar as possible, however, avoid stating formal rules for computing with integers. It is more important for the students to understand the ideas involved in adding integers than it is for them to memorize specific and complicated rules for this process. Developing skill in adding integers should be a continuing objective.

EXTENSION

Further practice exercises appropriate for this lesson are provided on page 110 of the *Workbook*.

Enrichment: Label the faces of a small cube with positive integers 1, 2, 3, 4, 5, and 6. Label the faces of a second cube with negative integers $^{-}1$, $^{-}2$, $^{-}3$, $^{-}4$, $^{-}5$, and $^{-}6$. Two or more students take turns tossing the cubes and recording their sums. The first player to get a cumulative score of 10 or more, or $^{-}10$ or less, is the winner. A pair of dice can be used for the game if one die is designated to show negative integers. Preferably the dice should be different in color.

Assignments

Minimum 1–3. Average 1–4, 5A–N.
Maximum 1–5.

Objective

Given two integers, the student can find their sum.

PREPARATION

Review the two cases of addition covered in the previous lesson: (1) adding a positive and a negative integer in which the sum is a positive integer; (2) adding two negative integers.

DISCUSSION

This lesson introduces the student to the case of addition of integers in which one integer is positive and one is negative and the sum is a negative integer. Again the student is presented with physical models to illustrate addition of integers. Receiving a cheque and paying a bill, rising and falling temperatures, and making jumps on the number line are considered before the rules for addition are justified by means of the basic principles.

You may want to justify some of the steps in finding the sums in Exercise 5. For example in 5A the reasoning is as follows:

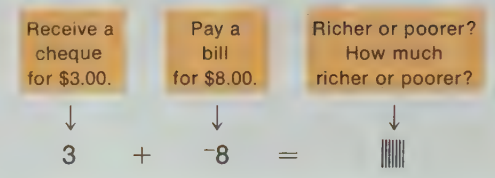
$^{-}9 + 2 = (^{-}7 + ^{-}2) + 2$	Addition fact for integers previously established ($^{-}9 = ^{-}7 + ^{-}2$)
$= ^{-}7 + (^{-}2 + 2)$	Associativity (Add.)
$= ^{-}7 + 0$	Opposites prin.
$= ^{-}7$	Zero prin.

You should not expect students to be able to present a proof such as the one above. Allow students to use a variety of methods and develop their own shortcuts and rules to find sums. Work toward understanding rather than overemphasizing rules for adding integers.

More About Adding Integers

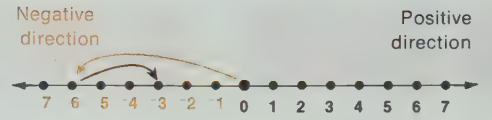
Discussing the Ideas

1. Think about cheques and bills. What is this sum?



2. Explain how the number line can help you find this sum.

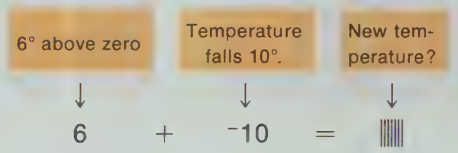
$^{-}6 + 3 =$ ▮



3. Think about the number line to find these sums.

A $5 + ^{-}9$ **B** $^{-}7 + 2$ **C** $2 + ^{-}6$ **D** $^{-}8 + 7$

4. Think about a thermometer to find this sum.



5. Solve the equations in order to find the sum of the addends in the colored screen. Because of the basic principles, you can add any two addends first.

A $^{-}9 + 2$ **B** $5 + ^{-}6$ **C** $^{-}10 + 4$

$^{-}7 + ^{-}2 + 2 = n$ $5 + ^{-}5 + ^{-}1 = n$ $^{-}6 + ^{-}4 + 4 = n$

6. If two integers are both positive, is their sum positive or negative?

7. If two integers are both negative, is their sum positive or negative?

8. If one integer is positive and one is negative, how can you decide whether the sum is a positive integer, a negative integer, or zero?

9. If the sum of two integers is zero, what can you say about the integers?

10. If one of the two integers is zero, what is the sum?

Using the Ideas

1. Solve the equations. Think of cheques and bills.

	Receive a cheque for $5.	Pay a bill for $8.	How much richer or poorer?		Pay a bill for $6.	Receive a cheque for $2.	How much richer or poorer?		Pay a bill for $7.	Receive a cheque for $5.	How much richer or poorer?
	↓	↓	↓		↓	↓	↓		↓	↓	↓
A	5	+ $^{-}8$	= *n*	B	$^{-}6$	+ 2	= *n*	C	$^{-}7$	+ 5	= *n*

2. Solve the equations. Think about jumps on the number line starting at zero.

	Jump 4 units right.	Jump 7 units left.	How far left or right of zero?		Jump 6 units left.	Jump 1 unit right.	How far left or right of zero?		Jump 8 units left.	Jump 5 more units left.	How far left or right of zero?
	↓	↓	↓		↓	↓	↓		↓	↓	↓
A	4	+ $^{-}7$	= *n*	B	$^{-}6$	+ 1	= *n*	C	$^{-}8$	+ $^{-}5$	= *n*

3. Solve the equations. Think about a thermometer.

	5° above 0°.	Temperature falls 7°.	New temperature?		10° below 0°.	Temperature rises 7°.	New temperature?		10° above 0°.	Temperature falls 15°.	New temperature?
	↓	↓	↓		↓	↓	↓		↓	↓	↓
A	5	+ $^{-}7$	= *n*	B	$^{-}10$	+ 7	= *n*	C	10	+ $^{-}15$	= *n*

4. When you find the number for *n* in the cloud, you have found the sum of the two integers in the gray screen. Copy the equation, giving the correct sum.

	n + $^{-}2$		*n* + $^{-}8$		$^{-}7$ + *n*		$^{-}2$ + *n*	
A	$^{-}6 + 2 = n$	C	$^{-}12 + 8 = n$	E	$7 + {}^{-}10 = n$	G	$2 + {}^{-}7 = n$	
	n + $^{-}8$		*n* + $^{-}6$		$^{-}6$ + *n*		*n* + $^{-}38$	
B	$^{-}9 + 8 = n$	D	$^{-}20 + 6 = n$	F	$6 + {}^{-}8 = n$	H	$^{-}42 + 38 = n$	

5. Find the sums. Use any method you choose.

A	$^{-}4 + 3$	F	$2 + {}^{-}8$	K	$^{-}2 + {}^{-}3$	P	$^{-}1 + 6$	U	$15 + {}^{-}9$
B	$^{-}7 + 6$	G	$7 + {}^{-}10$	L	$7 + {}^{-}7$	Q	$^{-}5 + 5$	V	$4 + {}^{-}13$
C	$^{-}9 + 2$	H	$^{-}8 + 7$	M	$8 + {}^{-}1$	R	$6 + {}^{-}4$	W	$^{-}8 + {}^{-}7$
D	$^{-}8 + 4$	I	$^{-}12 + 9$	N	$4 + {}^{-}9$	S	$^{-}7 + 0$	X	$17 + {}^{-}9$
E	$6 + {}^{-}10$	J	$4 + {}^{-}11$	O	$^{-}4 + 9$	T	$^{-}1 + {}^{-}9$	Y	$^{-}14 + 7$

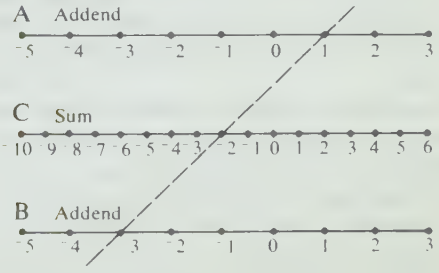
UTILIZATION

Notice the gradual sophistication of the exercises on this page. Exercises 1, 2, and 3 rely upon physical interpretations of integers. Exercise 4 relies on basic mathematical principles. Exercise 5 allows the student to use any method he wishes. Observe that the last exercise also reviews previous cases of addition.

EXTENSION

To provide additional practice, make selective assignments from Supplementary Exercise Set 50 on page S-29 and *Duplicator Masters* page 73.

Enrichment: Students might be interested in learning how a *nomograph* can be constructed to show sums or differences of integers. A portion of such a nomograph is shown below. The A and B scales are identical. The unit for the C scale in the centre is $\frac{1}{2}$ that for A and B. If a straightedge is placed across the addends on the A and B scales, the sum is found where the straightedge intersects the C scale. The dashed line represents the straightedge and shows that $^{-}3 + 1 = {}^{-}2$.



Assignments

Minimum 1–3. Average 1–4, 5A–J. Maximum 1–5.

Objective

Given an addition equation with a missing addend, the student can solve the equation by finding the missing addend.

PREPARATION

Conduct an oral review of addition of integers. You might include a few questions of this type: "5 plus what number is 2?" "What must you add to $^-1$ to get 3 as a sum?" "I am thinking of a number that adds to $^-3$ and gives $^-7$ as the sum. What is the number?"

INVESTIGATION

The Investigation should require only a short time to complete. You may give additional flow charts similar to the ones shown if needed. The purpose of the lesson is to familiarize students with missing addend problems so that, when subtraction of integers is introduced in the following lesson, they can find differences by thinking about missing addends.

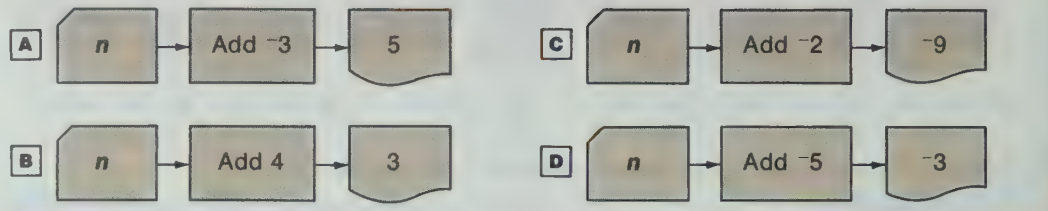
DISCUSSION

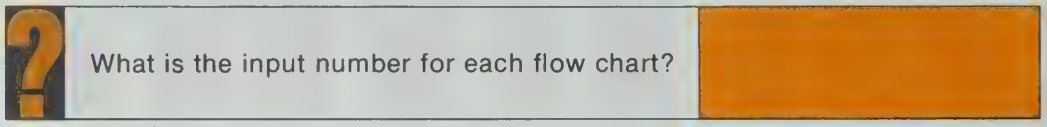
Exercise 1 illustrates how an equation can be written for a flow chart. This technique was used in Module 3 of Unit A with whole numbers. Now it is applied to flow charts using integers.

Exercise 4 introduces equations for word problems. Supply additional examples as needed.

Finding Missing Addends

Investigating the Ideas





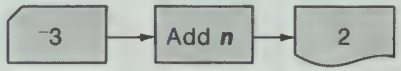
Discussing the Ideas

1. Each input number in the flow chart above is a **missing addend** in an addition equation.

   An equation for flow chart A is $n + ^-3 = 5$ (Addends: n, $^-3$; Sum: 5)

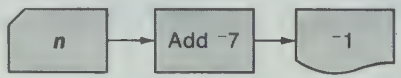
   Write an equation for each of the other flow charts.

2. A What is the number for n in the flow chart?



   B Write an equation for the flow chart.

3. A Write an equation for this flow chart.



   B Solve the equation for n.

4. A What equation is suggested by this word problem?
   When $^-2$ is added to a certain number, the sum is 8. What is the number?

   B Solve the equation for part A.

Using the Ideas

1. Find the number for n in each flow chart.

A n → Add $^{-}6$ → $^{-}2$

B n → Add 7 → $^{-}4$

C $^{-}10$ → Add n → $^{-}1$

D $^{-}7$ → Add n → $^{-}13$

2. Write and solve an equation for each flow chart.

A n → Add $^{-}5$ → 4

B n → Add $^{-}9$ → $^{-}4$

C 8 → Add n → $^{-}8$

D $^{-}6$ → Add n → 1

3. Solve the equations by finding the missing addends.

A $n + {}^{-}1 = {}^{-}5$	**F** $n + 6 = 3$	**K** $n + {}^{-}3 = {}^{-}5$	**P** $12 = n + {}^{-}3$
B $n + {}^{-}3 = {}^{-}4$	**G** $^{-}2 + n = 5$	**L** $8 + n = {}^{-}2$	**Q** $8 = {}^{-}6 + n$
C $n + 7 = 0$	**H** $n + {}^{-}4 = 2$	**M** $^{-}3 + n = {}^{-}7$	**R** $^{-}3 = n + 2$
D $5 + n = 3$	**I** $n + {}^{-}3 = 5$	**N** $n + {}^{-}6 = {}^{-}10$	**S** $^{-}7 = {}^{-}4 + n$
E $n + 2 = 1$	**J** $n + 4 = {}^{-}1$	**O** $9 + n = 2$	**T** $^{-}9 = 9 + n$

4. Write and solve an equation for each part.

A When $^{-}9$ is added to a certain number, the sum is 3. What is the number?

B During a certain winter day, the temperature rose 12° to reach a high of 8° above zero. What was the low temperature of the day?

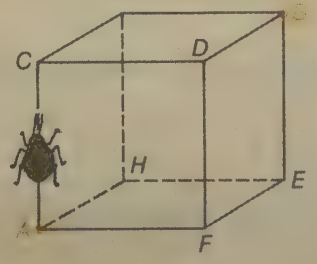
C What number added to $^{-}19$ will give a sum of 7?

think

The symbol *ACDB* describes a "trip" from *A* to *B* along just three edges of the cube. We agree that in a trip no edge is to be traversed twice.

Tell how many different trips there are from *A* to *B* along just

1. three edges of the cube
2. four edges of the cube
3. five edges of the cube
4. seven edges of the cube



More practice, S-29, Set 51

UTILIZATION

Assign the exercises as needed for your students. Plan to allow time for discussion of the exercises. It is vital that students be able to solve addition problems by finding missing addends before encountering subtraction of integers. Extra time spent on this lesson will be helpful in the work with subtraction.

EXTENSION

Further practice exercises are provided by Supplementary Exercise Set 51 on page S-29 and *Workbook* page 111.

Remedial: Ask students to find the sums.

A. $^{-}8 + 3$	F. $0 + {}^{-}4$	K. $^{-}3 + 2$
B. $5 + {}^{-}6$	G. $1 + {}^{-}1$	L. $^{-}6 + 9$
C. $^{-}6 + 8$	H. $16 + {}^{-}8$	M. $^{-}2 + {}^{-}2 + {}^{-}2$
D. $^{-}4 + {}^{-}3$	I. $^{-}10 + {}^{-}1$	N. $^{-}3 + 4 + {}^{-}6$
E. $^{-}9 + 19$	J. $^{-}10 + 1$	O. $^{-}8 + 4 + 4$

Enrichment: Students can construct flow charts like those in Exercise 1 and then exchange them with other students to see if they can find the missing numbers.

Think Solution

1. Six trips: *ACDB, ACGB, AFDB, AHEB, AHGB, AFEB*
2. No trips
3. Six trips: *ACDFEB, ACGHEB, AFEHGB, AFDCGB, AHEFDB, AHGCDB*
4. Twelve trips: *AHEFACGB, AHEFACDB, AFEHACGB, AHGCAFDB, AHGCAFEB, ACDFAHGB, AFEHACDB, ACGHAFDB, ACGHAFEB, ACDFAHEB, AFDCAHGB, AFDCAHEB*

Assignments

Minimum 1, 2, 3A–J. Average 1–3, 4A. Maximum 1–4.

Objective

The student can find the difference of two integers by using the inverse relationship between addition and subtraction.

PREPARATION

Write several integer addition equations with missing addends on the chalkboard. Ask the students to give the missing addends.

INVESTIGATION

The students should read and study the Investigation material carefully. Most students will be familiar with the inverse relationship of addition and subtraction from their work with whole and fractional numbers. The only new idea is that the same relationship exists for addition and subtraction of integers.

Allow as many students as possible to show their subtraction equation on the chalkboard and explain their solution.

DISCUSSION

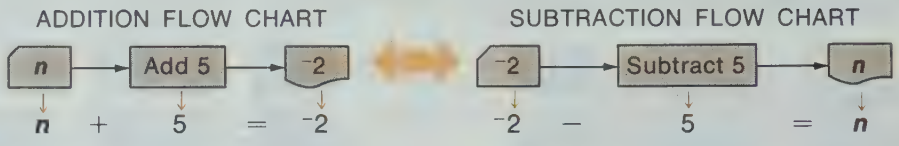
The inverse relationship between addition and subtraction was introduced in *Investigating School Mathematics* in Book 1 and was developed very carefully through all the grades. This lesson demonstrates how this important idea can be used to simplify a concept that might otherwise be quite difficult.

Students often have difficulty when faced with such problems as $5 - {}^{-}5 = n$. A student might well say, "Yes, I see that the missing addend must be 10, but I don't understand why." As an illustration, you might tell your students to imagine that they have \$10. The postman delivers a bill for \$5.00. The problem is essentially $10 + {}^{-}5$, which is 5. The postman then finds that he has delivered the bill to the wrong house, and so he takes the bill away. Now the problem is $5 - {}^{-}5$, since the postman's taking the bill away indicates a subtraction. The original \$10 is restored. Thus, $5 - {}^{-}5 = 10$. Some of your students may complain that this example is more complicated than the mathematics. Explain that this is why you will not encourage them to use such models.

Subtracting Integers

Investigating the Ideas

You can think of subtraction of integers as finding a missing addend. If you can find the missing addend in the addition equation below, then you have found the difference for the subtraction equation. Study the flow charts and equations.



Can you find the number for n in either flow chart?

Write five more subtraction equations and then solve them by finding missing addends.

Discussing the Ideas

1. Explain how you found the differences in the subtraction equations that you wrote for the Investigation?

2. Carol checked her first subtraction equation this way:

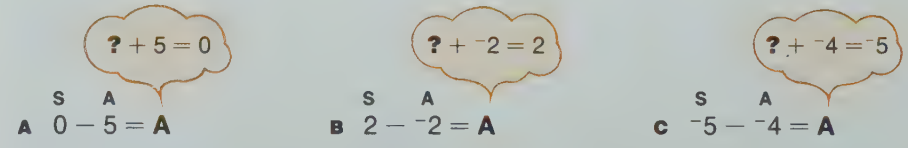
   S A A A A S
   $4 - {}^{-}2 = 6$ because $6 + {}^{-}2 = 4$

   Check the rest of her problems. Did Carol make any mistakes? Can you correct the mistakes?

Carol

1. $4 - {}^{-}2 = n$, $n = 6$
2. ${}^{-}5 - {}^{-}1 = n$, $n = {}^{-}4$
3. ${}^{-}7 - 2 = n$, $n = 5$
4. $3 - 9 = n$, $n = {}^{-}6$
5. $1 - {}^{-}3 = n$, $n = {}^{-}2$

3. Explain how to find each difference. The equation in the clouds may help you.

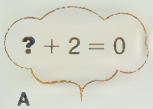


4. Find the differences. Explain your reasoning.

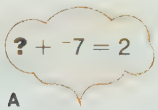
A	B	C	D	E	F	G
5	-5	-3	9	4	5	-2
−3	−6	−4	−6	$-{}^{-}1$	$-{}^{-}8$	$-{}^{-}3$

Using the Ideas

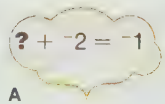
1. Find the differences. Think about missing addends.



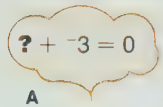
S A
A $0 - 2 = A$



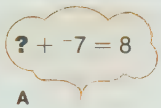
S A
C $2 - {}^{-}7 = A$



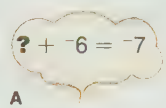
S A
E ${}^{-}1 - {}^{-}2 = A$



S A
B $0 - {}^{-}3 = A$



S A
D $8 - {}^{-}7 = A$



S A
F ${}^{-}7 - {}^{-}6 = A$

2. Solve the equations.

	S	A	A		S	A	A		S	A	A		S	A	A
A	2	−	⁻4 = *n*	**G**	⁻4	−	⁻2 = *n*	**M**	⁻5	−	⁻1 = *n*	**S**	1	−	3 = *n*
B	3	−	⁻2 = *n*	**H**	⁻7	−	⁻4 = *n*	**N**	⁻3	−	⁻2 = *n*	**T**	4	−	9 = *n*
C	⁻5	−	⁻2 = *n*	**I**	⁻2	−	⁻3 = *n*	**O**	8	−	⁻7 = *n*	**U**	⁻13	−	⁻5 = *n*
D	⁻4	−	⁻3 = *n*	**J**	⁻1	−	⁻5 = *n*	**P**	⁻9	−	⁻4 = *n*	**V**	⁻11	−	⁻4 = *n*
E	⁻1	−	⁻1 = *n*	**K**	⁻8	−	⁻7 = *n*	**Q**	14	−	⁻6 = *n*	**W**	74	−	⁻3 = *n*
F	⁻4	−	5 = *n*	**L**	⁻4	−	⁻3 = *n*	**R**	9	−	⁻5 = *n*	**X**	⁻16	−	⁻14 = *n*

3. Find the differences.

A	⁻3 − 7	**F**	2 − 8	**K**	⁻3 − ⁻8	**P**	⁻8 − 12
B	3 − 7	**G**	⁻2 − ⁻8	**L**	7 − ⁻3	**Q**	15 − ⁻3
C	⁻8 − 2	**H**	8 − ⁻2	**M**	3 − 5	**R**	⁻6 − 9
D	⁻8 − ⁻2	**I**	⁻5 − 3	**N**	⁻6 − 3	**S**	⁻8 − 7
E	⁻2 − 8	**J**	5 − ⁻3	**O**	⁻2 − ⁻1	**T**	⁻2 − ⁻3

★**4.** Solve the equations. It may help you to think about sums and addends.

A	$x - 3 = {}^{-}5$	**E**	$y = 3 - {}^{-}10$	**I**	$5 - x = 10$	**M**	$^{-}5 - c = 5$
B	$x - {}^{-}3 = {}^{-}5$	**F**	$y - 3 = {}^{-}10$	**J**	$c - 6 = {}^{-}6$	**N**	$y + y = {}^{-}8$
C	$3 - x = {}^{-}5$	**G**	$3 + y = {}^{-}10$	**K**	$d - {}^{-}7 = {}^{-}7$	**O**	$^{-}8 - y = y$
D	$3 - x = 5$	**H**	$3 - y = {}^{-}10$	**L**	${}^{-}7 - d = 1$	**P**	$^{-}10 - y = y$

More practice, page S-30, Set 52

UTILIZATION

Assign the exercises as independent written work. The equations in Exercise 4 are more difficult and should be assigned only to more able students.

Remind students that if the same letter appears more than once in an equation as a variable, it must be replaced by the same number in each case.

EXTENSION

To provide additional appropriate practice, make selective assignments from Supplementary Exercise Set 52 on page S-30, *Workbook* page 112 and *Duplicator Masters* page 74.

Enrichment: The nomograph shown in the Extension section on page D-9 can also be used to find differences of integers. Their sum is located in the centre scale and the given addend located either on the A or B scale. If the straightedge is placed across these two points, the missing addend is found on the third scale. Using the nomograph as pictured on page D-9, we can write these differences.

S A A
$^{-}2 - 1 = {}^{-}3$ or S A A
$^{-}2 - {}^{-}3 = 1$

Note that the sum (S) is always located on the centre scale.

Assignments

Minimum 1, 2. Average 1–3. Maximum 1–4.

Objective

The student can find products of integers by applying the basic principles for multiplication.

PREPARATION

As an introductory activity for this lesson, review with the students the basic principles for adding integers. Follow this by a review of the distributive principle for the set of whole numbers.

DISCUSSION

In presenting addition of integers, considerable emphasis was placed on first introducing the basic principles and then using them to develop the rules for adding any two integers. This background enables you now to show your students that the basic principles governing the multiplication of integers parallel those for addition. The distributive principle must be introduced when both addition and multiplication have been developed.

Be sure to stress that in addition equations and in multiplication equations the addends and the factors can be rearranged in any way that is convenient because of the commutative and associative principles for addition and multiplication. However, when both addition and multiplication are involved in the same equation, the addends and factors cannot be rearranged at will. Students must observe carefully how the distributive principle is applied.

Basic Principles for Multiplication of Integers

Discussing the Ideas

The basic principles for multiplication of integers are suggested by the basic principles for multiplication of whole numbers.

1. What is the product in each part?

A	$6 \cdot 1$	**D**	$^-8 \cdot 1$
B	$^-2 \cdot 1$	**E**	$19 \cdot 1$
C	$^-10 \cdot 1$	**F**	$^-23 \cdot 1$

One principle

If you choose an integer and multiply by 1, the product is the integer that you choose.

In general, for any integer a,

$$a \cdot 1 = a$$

2.

Commutative principle

When finding the product of two integers, you can change the order of the factors and the product is the same.

In general, for integers a and b,

$$a \cdot b = b \cdot a$$

Associative principle

When finding the product of three integers, you can change the grouping of the factors and the product is the same.

In general, for integers a, b, and c,

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

Since you can change both the **order** and the **grouping** of the factors, you can rearrange factors in any way that is convenient.

- **A** Using the integers $^-3$ and $^-4$, give an example of the **commutative principle** for multiplication.
- **B** Using the integers 13, 7, and $^-5$, give an example of the **associative principle** for multiplication.
- **C** Is the product $(^-6 \cdot 4) \cdot ^-9$ the same as $(4 \cdot ^-9) \cdot ^-6$? Why?

3. How would you complete this equation to illustrate the **distributive principle**?

$^-2 \cdot (3 + ^-4) = \underline{?}$

Distributive principle

For all integers a, b, and c,

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

Using the Ideas

Find the integer for ***a***. Then find the integer for ***b***.

A $^{-}7 \cdot 1 = a \rightarrow 1 \cdot {}^{-}7 = b$

B $^{-}12 \cdot 1 = a \rightarrow 1 \cdot {}^{-}12 = b$

C $5 \cdot 0 = a \rightarrow 0 \cdot 5 = b$

D $^{-}5 \cdot 0 = a \rightarrow 0 \cdot {}^{-}5 = b$

Study the examples. $^{-}3 + {}^{-}3 = 2 \cdot {}^{-}3 = {}^{-}6$

$^{-}5 + {}^{-}5 + {}^{-}5 = 3 \cdot {}^{-}5 = {}^{-}15$

Find the integer for ***a***. Then find the integer for ***b***.

A $^{-}3 + {}^{-}3 = a \rightarrow 2 \cdot {}^{-}3 = b$

B $^{-}4 + {}^{-}4 = a \rightarrow 2 \cdot {}^{-}4 = b$

C $^{-}5 + {}^{-}5 = a \rightarrow 2 \cdot {}^{-}5 = b$

D $^{-}7 + {}^{-}7 = a \rightarrow 2 \cdot {}^{-}7 = b$

E $^{-}58 + {}^{-}58 = a \rightarrow 2 \cdot {}^{-}58 = b$

F $^{-}4 + {}^{-}4 + {}^{-}4 = a \rightarrow 3 \cdot {}^{-}4 = b$

G $^{-}5 + {}^{-}5 + {}^{-}5 + {}^{-}5 = a \rightarrow 4 \cdot {}^{-}5 = b$

H $^{-}8 + {}^{-}8 + {}^{-}8 + {}^{-}8 = a \rightarrow 4 \cdot {}^{-}8 = b$

Find the integer for ***a***. Then find the integer for ***b***.

A $2 \cdot {}^{-}3 = a \rightarrow {}^{-}3 \cdot 2 = b$

B $2 \cdot {}^{-}5 = a \rightarrow {}^{-}5 \cdot 2 = b$

C $3 \cdot {}^{-}7 = a \rightarrow {}^{-}7 \cdot 3 = b$

D $4 \cdot {}^{-}6 = a \rightarrow {}^{-}6 \cdot 4 = b$

Copy the equations and write the correct number for each ▒.
Each equation can be completed by thinking about the equation above it and a basic principle or multiplication fact.

A
$$2 + {}^{-}2 = \text{▒}$$
$$3 \cdot (2 + {}^{-}2) = \text{▒}$$
$$(3 \cdot 2) + (3 \cdot {}^{-}2) = \text{▒}$$
$$6 + 3 \cdot {}^{-}2 = \text{▒}$$
$$3 \cdot {}^{-}2 = \text{▒}$$

This number adds to 6 to give zero. Therefore, $(3 \cdot {}^{-}2)$ must be the integer

B
$$2 + {}^{-}2 = \text{▒}$$
$${}^{-}3 \cdot (2 + {}^{-}2) = \text{▒}$$
$$({}^{-}3 \cdot 2) + ({}^{-}3 \cdot {}^{-}2) = \text{▒}$$
$${}^{-}6 + {}^{-}3 \cdot {}^{-}2 = \text{▒}$$
$${}^{-}3 \cdot {}^{-}2 = \text{▒}$$

This number adds to $^{-}6$ to give zero. Therefore, $({}^{-}3 \cdot {}^{-}2)$ must be the integer

C
$$4 + {}^{-}4 = \text{▒}$$
$${}^{-}5 \cdot (4 + {}^{-}4) = \text{▒}$$
$$({}^{-}5 \cdot 4) + ({}^{-}5 \cdot {}^{-}4) = \text{▒}$$
$${}^{-}20 + {}^{-}5 \cdot {}^{-}4 = \text{▒}$$
$${}^{-}5 \cdot {}^{-}4 = \text{▒}$$

This number adds to $^{-}20$ to give zero. Therefore, $({}^{-}5 \cdot {}^{-}4)$ must be the integer

Use the correct word (**positive** or **negative**) to complete each statement.

A The product of a positive integer and a negative integer is a ? integer.

B The product of a negative integer and a negative integer is a ? integer.

think

A first number is multiplied by a second, and the product is 12 802. When the first number is multiplied by a number one larger than the second, the product is 13 148. Determine the first and the second number.

UTILIZATION

Exercise 2 is designed to help students think about some multiplication problems as repeated addition. Thus $3 \cdot {}^{-}4$ can be thought of as $^{-}4 + {}^{-}4 + {}^{-}4$. However, if both integers are negative a repeated addition interpretation is not applicable.

Parts B and C of Exercise 4 will help the student see why the product of two negative integers is a positive integer. Although Exercise 4 is more difficult than some of the other exercises, it is not starred because it presents vital material, to which most of your students should be exposed.

EXTENSION

Page 113 of the *Workbook* offers exercises that would be appropriate for an extra practice assignment.

Ask students to give the missing number in each step.

1.
$$3 + {}^{-}3 = \text{▒}\ (0)$$
$$5 \cdot (3 + {}^{-}3) = \text{▒}\ (0)$$
$$(5 \cdot 3) + (5 \cdot {}^{-}3) = \text{▒}\ (0)$$
$$15 + 5 \cdot {}^{-}3 = \text{▒}\ (0)$$
$$5 \cdot {}^{-}3 = \text{▒}\ ({}^{-}15)$$
$${}^{-}3 \cdot 5 = \text{▒}\ ({}^{-}15)$$

This is the number that adds to 15 to give 0.

2.
$$5 + {}^{-}5 = \text{▒}\ (0)$$
$${}^{-}3 \cdot (5 + {}^{-}5) = \text{▒}\ (0)$$
$$({}^{-}3 \cdot 5) + ({}^{-}3 \cdot {}^{-}5) = \text{▒}\ (0)$$
$${}^{-}15 + {}^{-}3 \cdot {}^{-}5 = \text{▒}\ (0)$$
$${}^{-}3 \cdot {}^{-}5 = \text{▒}\ (15)$$
$${}^{-}5 \cdot {}^{-}3 = \text{▒}\ (15)$$

This is the number that adds to $^{-}15$ to give 0.

Think Solution

By adding 1 to one of the factors, the product is increased by 346. This increase must be one of the factors. By division, the other factor is found to be 37.

Assignments

Minimum 1–3, written; 4, 5, oral.
Average 1–3, written; 4, 5 oral.
Maximum 1–5.

Objective

Given two integers, the student can find their product.

PREPARATION

Since this lesson continues the work with multiplication of integers, students may begin work on the Investigation immediately.

INVESTIGATION

You may either have the students try the Investigation on their own or use it as part of the Discussion stage.

The purpose of the Investigation is to present a physical interpretation of multiplication of integers. This interpretation requires the following agreements: The speed when walking to the *right* is denoted by a positive integer; to the left, by a negative integer. A distance to the right of zero is denoted by a positive integer; to the left, by a negative integer. Finally, the time that has passed ("hours ago") is denoted by a negative integer; future time ("hours from now"), by a positive integer.

DISCUSSION

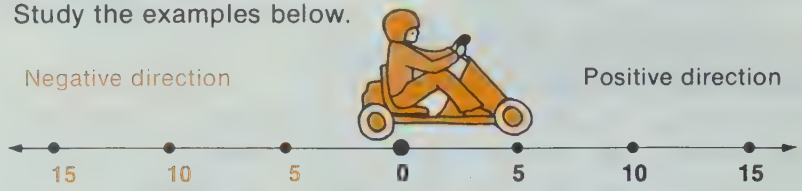
The situation described in Exercise 1 can be interpreted so as to yield four different equations.

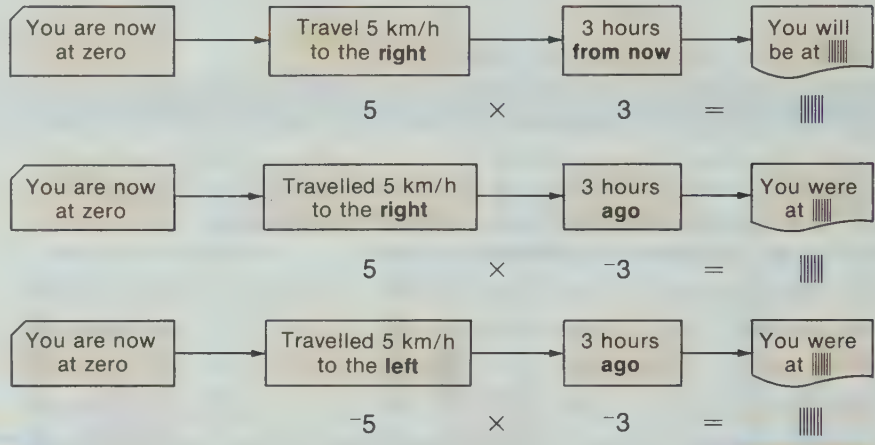
1. If you are now at 0 and travel 7 km/h to the right, then 2 hours from now you will be 14 km to the right. This gives the equation $7 \cdot 2 = 14$.
2. If you are now at 0 after having travelled 7 km/h to the right, then 2 hours ago you were 14 km to the left of where you are now. This is expressed by the equation $7 \cdot {}^{-}2 = {}^{-}14$.
3. If you are now at 0 after having travelled 7 km/h to the left, then 2 hours ago you were 14 km to the right of your present position. The equation for this would be ${}^{-}7 \cdot {}^{-}2 = 14$.
4. If you are now at 0 and travel 7 km/h to the left, then 2 hours from now you will be 14 km to the left. This gives the equation ${}^{-}7 \cdot 2 = {}^{-}14$.

Multiplying Integers

Investigating the Ideas

Sometimes it may be helpful to think about a physical example that illustrates multiplication of integers. Study the examples below.




You are now at zero
Travel 5 km/h to the right
3 hours from now
You will be at
5 × 3 =
You are now at zero
Travelled 5 km/h to the right
3 hours ago
You were at
5 × ⁻3 =
You are now at zero
Travelled 5 km/h to the left
3 hours ago
You were at
⁻5 × ⁻3 =

Can you give the missing number in each equation?

Discussing the Ideas

1. Explain how you can use the number line above to help you write an equation if the speed was 7 km/h to the right or to the left and time was either 2 hours from now or 2 hours ago.

2. What multiplication equation does this word problem suggest?
   A man is now at zero and riding a bicycle to the left at 12 km/h. Where will he be 2 hours from now?

3. Find the products. Think about the number line if necessary.
   A $6 \cdot {}^{-}4$ **B** $3 \cdot 4$ **C** ${}^{-}6 \cdot {}^{-}2$ **D** ${}^{-}7 \cdot 5$

Using the Ideas

1. Write a multiplication equation for each part.
Think about the number line.

- **A** You are now at zero, walking 4 km/h to the right. Where were you 3 hours ago?
- **B** You are now at zero walking 2 km/h to the left. Where were you 4 hours ago?
- **C** You are now at zero walking 3 km/h to the left. Where will you be 3 hours from now?
- **D** You are now at zero and running 6 km/h to the right. Where will you be 1 hour from now?

2. Find the product **P** in each equation.

A $4 \cdot {}^{-}5 = P$	**D** $1 \cdot {}^{-}5 = P$	**G** $^{-}2 \cdot {}^{-}5 = P$	**J** $^{-}5 \cdot {}^{-}5 = P$
B $3 \cdot {}^{-}5 = P$	**E** $0 \cdot {}^{-}5 = P$	**H** $^{-}3 \cdot {}^{-}5 = P$	**K** $^{-}6 \cdot {}^{-}5 = P$
C $2 \cdot {}^{-}5 = P$	**F** $^{-}1 \cdot {}^{-}5 = P$	**I** $^{-}4 \cdot {}^{-}5 = P$	**L** $^{-}7 \cdot {}^{-}5 = P$

3. Find the products.

A $2 \cdot {}^{-}4$	**G** $^{-}8 \cdot 0$	**M** $^{-}5 \cdot 0$	**S** $13 \cdot {}^{-}13$
B $^{-}4 \cdot 3$	**H** $12 \cdot {}^{-}3$	**N** $^{-}7 \cdot 6$	**T** $^{-}13 \cdot {}^{-}13$
C $^{-}4 \cdot {}^{-}3$	**I** $^{-}3 \cdot 12$	**O** $^{-}7 \cdot {}^{-}6$	**U** $10 \cdot {}^{-}27$
D $^{-}1 \cdot 1$	**J** $^{-}3 \cdot {}^{-}12$	**P** $^{-}12 \cdot 13$	**V** $^{-}100 \cdot 38$
E $^{-}1 \cdot {}^{-}1$	**K** $^{-}4 \cdot 8$	**Q** $^{-}12 \cdot {}^{-}13$	**W** $^{-}100 \cdot {}^{-}38$
F $^{-}2 \cdot {}^{-}4$	**L** $^{-}4 \cdot {}^{-}8$	**R** $14 \cdot {}^{-}12$	**X** $468 \cdot {}^{-}1000$

4. Find the products.

A $(2 \cdot 3) \cdot {}^{-}2$	**D** $({}^{-}5 \cdot {}^{-}2) \cdot 2$	**G** $^{-}2\,({}^{-}5 \cdot {}^{-}3)$	**J** $5 \cdot (3 \cdot 2)$
B $(4 \cdot {}^{-}4) \cdot 2$	**E** $(8 \cdot {}^{-}1) \cdot 1$	**H** $^{-}6 \cdot (2 \cdot {}^{-}3)$	**K** $^{-}1 \cdot ({}^{-}1 \cdot {}^{-}1)$
C $(1 \cdot {}^{-}1) \cdot {}^{-}3$	**F** $(3 \cdot {}^{-}3) \cdot {}^{-}3$	**I** $^{-}2 \cdot ({}^{-}3 \cdot {}^{-}4)$	**L** $({}^{-}1 \cdot 1) \cdot ({}^{-}1 \cdot 1)$

5. Solve the equations.

A $^{-}7 \cdot 8 = n$	**D** $8 \cdot r = 32$	**G** $^{-}4 \cdot 13 = m$	**J** $12 \cdot w = 108$
B $6 \cdot {}^{-}5 = s$	**E** $z \cdot {}^{-}5 = 15$	**H** $^{-}11 \cdot q = 110$	**K** $3 \cdot (x + 1) = {}^{-}12$
C $^{-}9 \cdot {}^{-}7 = t$	**F** $^{-}3 \cdot k = 24$	**I** $^{-}7 \cdot b = {}^{-}49$	**L** $^{-}5 \cdot (y - {}^{-}2) = 35$

More practice, page S-30, Set 53

UTILIZATION

The physical interpretation of multiplication is presented again in Exercise 1. There is no need to overemphasize the model if students are able to compute products without it. The rules for multiplying integers are quite simple, and the majority of students can reach these generalizations on their own.

In Exercise 5 students can solve the equations by thinking about missing factors. Division of integers will be discussed in the next lesson.

EXTENSION

To provide additional practice, make assignments from Supplementary Exercise Set 53 on page S-30 and *Duplicator Masters* page 75. Now might also be an appropriate time to recommend Research Project A on page D-27 for some students.

Remedial: Provide additional practice exercises in multiplication of integers.

Enrichment: Review the fact that $n^2 = n \cdot n$. Therefore if $n^2 = 4$, the solution can be either $n = 2$ or $n = {}^{-}2$, because $2^2 = 4$ and $({}^{-}2)^2 = ({}^{-}2) \cdot ({}^{-}2) = 4$. Ask students to solve some equations involving exponents, such as the ones below. (Answers are given in parentheses.)

1. $x^2 = 25$ ($x = 5$ or $^{-}5$)
2. $n^3 = {}^{-}8$ ($n = {}^{-}2$)
3. $r^2 = 100$ ($r = 10$ or $^{-}10$)
4. $s^2 - 2 = 34$ ($s = 6$ or $^{-}6$)
5. $1 + y^2 = 82$ ($y = 9$ or $^{-}9$)
6. $(k + 1)^2 = 9$ ($k = 2$ or $^{-}4$)
7. $z^3 - 1 = {}^{-}28$ ($z = {}^{-}3$)
8. $p^2 + p = 0$ ($p = 0$ or $^{-}1$)

Assignments

Minimum 1, 2, 3A–L.
Average 1, 2, 3M–X, 4. Maximum 1–5.

Objective

The student can find the quotient of two integers by using the inverse relationship between multiplication and division.

PREPARATION

Give some oral problems of multiplication of integers. Include questions in which students must name missing factors; for example: "What number times $^{-}3$ gives 12?" "If 5 is multiplied by an integer the product is $^{-}20$. What is that integer?"

INVESTIGATION

The flow charts in the Investigation illustrate the inverse relation between multiplication and division of integers. Since most students will be familiar with this relationship from their work with whole numbers and fractional numbers, the only new idea is that this relationship will enable them to find quotients of integers.

DISCUSSION

For Exercise 1, have several students place their equations on the chalkboard and explain how the quotients may be found.

Exercise 2 illustrates the method students should use in finding quotients of integers. Using this method, your students should find the ideas involved in the division of integers fairly simple. Students who understand the general rules for multiplying any two integers should be able to understand the idea of dividing any two integers.

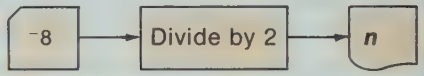
Multiplication and Division of Integers

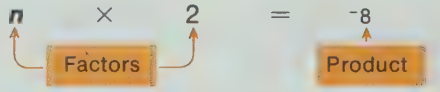
Investigating the Ideas

You can find the quotient of two integers by thinking about missing factors. If you can find the missing factor in a multiplication equation, then you will have found the quotient for a division equation. Study the flow charts and equations.

MULTIPLICATION FLOW CHART

DIVISION FLOW CHART





Can you find the number for n in either flow chart?

Write five more division equations and then solve them by finding missing factors.

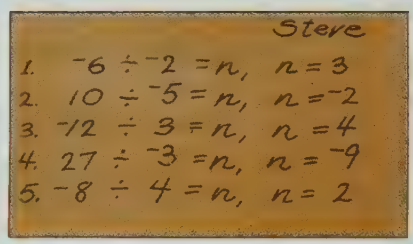
Discussing the Ideas

1. Explain how you found the quotients for the division equations that you wrote for the Investigation?

2. Steve checked his first division problem like this:

   P F F F F P
   $^{-}6 \div {}^{-}2 = 3$ because $3 \cdot {}^{-}2 = {}^{-}6$

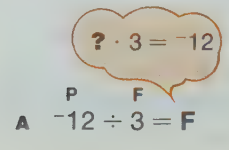
   Check the rest of his problems.
   Did Steve make any mistakes?
   Can you correct his mistakes?

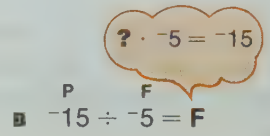


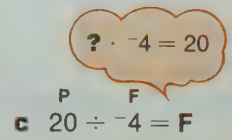
3. What is the missing factor in each equation?

   F F P F F P F F P F F P
   A $^{-}2 \cdot x = {}^{-}8$ B $n \cdot 7 = {}^{-}21$ C $r \cdot {}^{-}4 = 32$ D $3 \cdot t = 24$

4. Explain how to find the quotients.







Using the Ideas

1. Find the missing factors.

	F	F	P		F	F	P		F	F	P		F	F	P
A	$-2 \cdot a =$		-8	F	$r \cdot 4 =$		0	K	$3 \cdot n =$		-18	P	$n \cdot 20 =$		80
B	$8 \cdot b =$		-16	G	$u \cdot 5 =$		-5	L	$s \cdot 5 =$		-30	Q	$n \cdot -20 =$		80
C	$-3 \cdot x =$		3	H	$m \cdot 3 =$		-9	M	$2 \cdot y =$		-10	R	$n \cdot 20 =$		-80
D	$3 \cdot x =$		12	I	$-3 \cdot y =$		-3	N	$r \cdot -2 =$		-8	S	$n \cdot 11 =$		-55
E	$c \cdot -2 =$		6	J	$t \cdot -1 =$		8	O	$n \cdot -3 =$		12	T	$-12 \cdot y =$		36

2. Find the quotients. Exercise 1 may help you.

A	$-9 \div 3 = m$	F	$-8 \div -2 = a$	K	$36 \div -12 = y$	P	$-30 \div 5 = s$
B	$12 \div 3 = x$	G	$80 \div -20 = n$	L	$12 \div -3 = n$	Q	$-5 \div 5 = u$
C	$-18 \div 3 = n$	H	$-10 \div 2 = y$	M	$-80 \div 20 = n$	R	$0 \div -4 = r$
D	$-8 \div -2 = a$	I	$8 \div -1 = t$	N	$-55 \div 11 = n$	S	$-3 \div -3 = y$
E	$6 \div -2 = c$	J	$-18 \div 3 = n$	O	$3 \div -3 = x$	T	$-16 \div 8 = b$

3. Find the quotients. Think about missing factors.

A	$-35 \div 7 = n$	G	$12 \div 4 = n$	M	$12 \div -4 = n$
B	$-12 \div 4 = n$	H	$-12 \div -4 = n$	N	$8 \div -8 = n$
C	$-8 \div -8 = n$	I	$0 \div -6 = n$	O	$20 \div -5 = n$
D	$28 \div -7 = n$	J	$-28 \div 7 = n$	P	$-28 \div -7 = n$
E	$-16 \div -1 = n$	K	$-16 \div 1 = n$	Q	$-45 \div 15 = n$
F	$-45 \div -15 = n$	L	$-2448 \div -16 = n$	R	$(-8 \cdot 7) \div -8 = n$

4. Sue is thinking of a number. If you multiply Sue's number by itself and then add it to that product, you get 20. Give two possibilities for the number Sue might be thinking about.

think

1. Copy each exercise and insert any of the symbols $+$, $\times$, $-$, $\div$, or $( )$ in order to make the sentence true.

EXAMPLE: $2 = (4 + 4) \div 4$

A	$0 = 4\ 4\ 4$	C	$4 = 4\ 4\ 4$	E	$12 = 4\ 4\ 4$	G	$32 = 4\ 4\ 4$
B	$3 = 4\ 4\ 4$	D	$5 = 4\ 4\ 4$	F	$20 = 4\ 4\ 4$	H	$64 = 4\ 4\ 4$

2. Copy each exercise and insert any of the symbols $+$, $\times$, $-$, $\div$, $( )$, $!$, or $\sqrt{\ }$ in order to make the sentence true.

EXAMPLES: $\sqrt{4} = 2$ (Read $\sqrt{4}$ as "the square root of 4.")

$4! = 4 \times 3 \times 2 \times 1$ (Read $4!$ as "4 factorial.")

A	$1 = 4\ 4\ 4$	B	$6 = 4\ 4\ 4$	C	$8 = 4\ 4\ 4$	D	$25 = 4\ 4\ 4$

More practice, page S-31, Set 54

UTILIZATION

Exercises 1 and 2 are designed to help students think about the inverse relation between multiplication and division. If the student can find the missing factor in Exercise 1A, $-2 \cdot a = -8$, then he can find the quotient in Exercise 2F, $-8 \div -2 = a$.

An equation for Exercise 4 is $n^2 + n = 20$. Students are expected to solve the equation by trial and error methods, rather than algebraically.

EXTENSION

Appropriate further practice assignments may be selected from Supplementary Exercise Set 54 on page 31, *Workbook* page 114, and *Duplicator Masters* page 76.

Think Solution

1. Some additional possible answers:
   - A. $0 = (4 - 4) \times 4$
   - C. $4 = (4 - 4) + 4$
   - D. $5 = (4 \div 4) + 4$
   - E. $12 = (4 \times 4) - 4$

2. Some additional possible answers:
   - A. $1 = \sqrt{4} - (4! \div 4!)$
   - B. $6 = 4! \div (\sqrt{4} + \sqrt{4})$
     $6 = (4 + 4) - \sqrt{4}$
   - C. $8 = \sqrt{4} \times (\sqrt{4} + \sqrt{4})$
     $8 = (4! \div \sqrt{4}) - 4$
     $8 = (\sqrt{4} + \sqrt{4}) + 4$
   - D. $25 = (4 \div 4) + 4!$

You might also ask how many numbers from 1 to 20 can be obtained by combinations of three 4's. These are some possibilities:

$2 = (4 + 4) \div 4$	$13 = (4! + \sqrt{4}) \div \sqrt{4}$
$7 = (4! + 4) \div 4$	$14 = (4 \times 4) - \sqrt{4}$
$10 = (4! - 4) \div \sqrt{4}$	$16 = \sqrt{4} \times \sqrt{4} \times 4$
$11 = (4! - \sqrt{4}) \div \sqrt{4}$	$18 = (4 \times 4) + \sqrt{4}$
$12 = 4 + 4 + 4$	$20 = (4! - \sqrt{4}) - \sqrt{4}$

You may wish to have the very able students pursue the same idea using numbers other than 4.

Assignments

Minimum 1, 2. Average 1–3. Maximum 1–4.

Objective

Given two integers, the student can state which is the smaller or larger integer.

PREPARATION

A brief review of inequalities for whole numbers would be beneficial. Review the meaning of the symbols $>$ and $<$.

INVESTIGATION

The Investigation introduces the idea of inequality of integers in the context of a familiar game in which negative integers are used to record scores "in the hole" or "going set."

Suggest that the students record their answers to the Investigation questions in the form of a table like the one shown below.

Game	1	2	3	4
Best Score	Brian(7)	Ann(2)	Ann(6)	Brian(0)
Points needed for tie	10	7	15	1

5	6	7	8
Ann(6)	Brian($^-2$)	Ann(8)	Brian($^-5$)
14	5	2	5

For each game the student is selecting the larger of two integers and also deciding what *positive* integer must be added to the smaller integer to give the larger integer.

DISCUSSION

Exercise 1 presents a number line interpretation of inequality of integers. A more formal approach is given in Exercise 3. Formally stated, $a > b$ if $a = b + c$ where c is a positive integer.

Inequalities

Investigating the Ideas

Ann and Brian played eight games of cards and kept a record of their point scores for each game. The negative integers indicate scores "in the hole" or "going set."

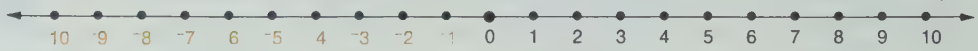
Games	1	2	3	4	5	6	7	8
Brian's scores	7	$^-5$	$^-9$	0	8	2	6	$^-5$
Ann's scores	$^-3$	2	6	$^-1$	6	7	8	$^-10$

Can you tell who had the best score for each game?

For each game, tell how many points the loser needed in order to tie the winner.

Discussing the Ideas

1. On the number line below, the integers are matched with points so that the point for the larger of two numbers is to the right of the point for the smaller number.



   - **A** Since 1 is to the right of $^-10$, which is the larger number?
   - **B** Since $^-4$ is to the left of 3, which is the larger number?
   - **C** Since $^-6$ is to the right of $^-10$, which is the larger number?

2. Use **greater than** or **less than** to complete each sentence.
   - **A** Any positive integer is ? zero.
   - **B** Any negative integer is ? zero.
   - **C** Any positive integer is ? any negative integer.
   - **D** Any negative integer is ? any positive integer.
   - **E** Any integer is ? any other integer that is to the left of it on the number line.
   - **F** Any integer is ? any other integer that is to the right of it on the number line.

3. We say that $^-4$ **is less than** $^-1$ and write this as $^-4 < {}^-1$ because we must add a **positive** number to $^-4$ in order to get $^-1$.
   $^-4 + 3 = {}^-1$; 3 is a positive number.
   We also say that $^-1$ **is greater than** $^-4$ and write $^-1 > {}^-4$
   What addition equation would show that the following inequalities are true?
   A $^-1 < 1$ **B** $^-5 < 0$ **C** $^-6 < 7$ **D** $^-7 < 4$

Using the Ideas

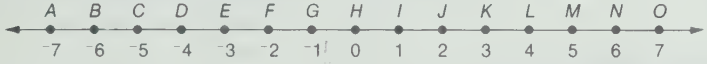
1. Give the correct symbol ($<$ or $>$) for each ▒

- **A** Since $3 = 2 + 1$, we know that 3 ▒ 2.
- **B** Since $^-3 = {}^-4 + 1$, we know that $^-3$ ▒ $^-4$.
- **C** Since $^-5 = {}^-6 + 1$, we know that $^-5$ ▒ $^-6$.

2. Give the correct symbol ($<$ or $>$) for each ▒

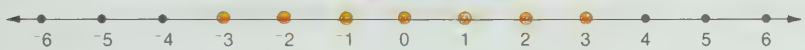
- **A** 7 ▒ $^-3$
- **B** $^-8$ ▒ $^-4$
- **C** 100 ▒ $^-1$
- **D** $^-12$ ▒ $^-8$
- **E** 0 ▒ 2
- **F** 0 ▒ $^-2$
- **G** $^-99$ ▒ $^-100$
- **H** $^-53$ ▒ $^-68$

3. Write the letters for the points in each set.



- **A** The set of points for numbers greater than $^-4$ and less than zero
- **B** The set of points for numbers less than or equal to $^-2$
- **C** The set of points for numbers greater than 3 or less than $^-5$

4. The example shows the graph of the set of integers greater than $^-4$ and less than 4. We say that we have graphed all possible integers for n in the inequality $^-4 < n < 4$.



Draw number lines and graph each set.

- **A** Integers greater than 0 and less than 5
- **B** Integers greater than $^-2$ and less than 3
- **C** Integers greater than $^-6$ and less than 6
- **D** Integers less than 0 and greater than $^-6$
- **E** Integers less than 1 and greater than $^-1$
- **F** Integers greater than $^-8$ and less than $^-2$
- **G** Integers less than $^-5$ and greater than $^-9$
- ★ **H** Integers for n in the inequality $^-2 < n < 5$
- ★ **I** Integers for n in the inequality $^-6 < n < {}^-1$

think

Arrange the digits 1 through 7 in the hexagons, so that the sum of the digits is the same number in any three hexagons in a row.

★ **5.** Let $A = \{^-2, ^-1, 0, 1, 2\}$, $B = \{^-5, ^-4, ^-3, ^-2, ^-1, 0\}$, and $C = \{0, 1, 2, 3, 4, 5\}$. Graph each of the following sets on a number line.

- **A** $A \cup B$
- **B** $A \cup C$
- **C** $A \cap B$
- **D** $A \cap C$
- **E** $B \cup C$
- **F** $B \cap (A \cup C)$
- **G** $(A \cup B) \cup C$
- **H** $(A \cap B) \cap C$

UTILIZATION

Review graphing on the whole number line. After such preparation, your students should be able to work through the graphing exercises.

Answers, Exercises 4 and 5

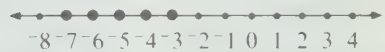
4 A.

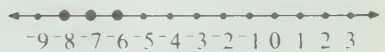
B.

C.

D.

E.

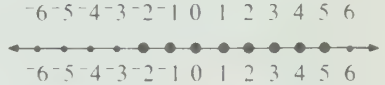
F.


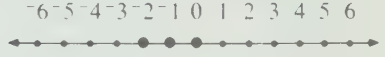
G.


H.

I.

5 A.

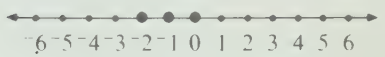
B.


C.


D.

E.

F.

G.


H.

Think Solution

Assignments

Minimum 1–3. Average 1–3, 4A–G. Maximum 1–5.

Objective

Given an ordered pair of integers, the student can graph the point in the co-ordinate plane for the pair.

PREPARATION

Materials: Graph paper or co-ordinate paper (*Duplicator Masters*) page 83 or 86.

Students' previous experience in graphing pairs of whole numbers (Unit C) provides an excellent background for the work in this lesson. The students will have already graphed points in the first quadrant, and this lesson will extend these ideas to all four quadrants. Review these ideas before students begin the Investigation.

INVESTIGATION

After the students have read the expository material in the Investigation, have them give the integer pairs for points E through H. It would be helpful to make an overlay of a co-ordinate grid for the overhead projector and give the students additional practice in associating integer pairs with points and conversely.

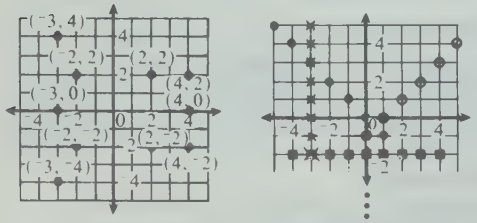
DISCUSSION

The words *axis*, *co-ordinate axes*, and *origin* are introduced in Exercise 1. Encourage students to use them correctly.

The discussion exercises provide experience in locating points in the co-ordinate plane and learning the co-ordinates of given points in the plane.

Answers, Exercise 1, page D-23

1A. Graph B: ■. Graph C: ×. Graph D: •. Graph E: o.

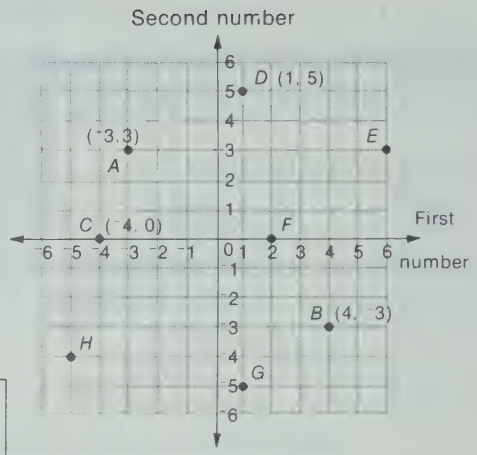


F. All integers on the horizontal axis.

Graphing Integer Pairs

Investigating the Ideas

You can use two intersecting number lines to show the graph of any pair of integers. The pair of integers that describe the location of a point are the **co-ordinates** of the point. The point of the pair is called the **graph of the integer pair**. The graphs of some integer pairs are shown on the co-ordinate grid.



Can you give the missing co-ordinates in the table below?

Point	A	B	C	D	E	F	G	H
Coordinates	(⁻3, 3)	(4, ⁻3)	(⁻4, 0)	(1, 5)	(▒, ▒)	(▒, ▒)	(▒, ▒)	(▒, ▒)

Discussing the Ideas

1. The two intersecting number lines are called the **co-ordinate axes** and each number line is an **axis**.
   The point of intersection of the co-ordinate axes is called the **origin**.
   What are the co-ordinates of the origin?

2. Give the co-ordinates of a point whose location can be described as
   - A 3 units to the right of the origin and 4 units down
   - B 5 units to the left of the origin and 6 units up
   - C 2 units to left of the origin
   - D 4 units to the left of the origin and 5 units down
   - E 3 units down from the origin
   - F 6 units to the right of the origin and 6 units up

3. Name some integer pairs so that the numbers in each pair are opposites of each other. The pair (⁻3, 3) is an example.
   What can you say about the graph of all such pairs of opposites?

4. List some integer pairs such that the second number is one more than the first number. Include some negative integers.

Using the Ideas

1. For each exercise, use graph paper and draw the co-ordinate axes. Mark the points for integers from $^{-}5$ through 5 on each axis. Then graph the following sets of integer pairs.
   - **A** $\{(4, 2), (4, ^{-}2), (4, 0), (^{-}3, 4), (^{-}3, ^{-}4), (^{-}3, 0), (^{-}2, 2), (2, ^{-}2), (^{-}2, ^{-}2), (2, 2)\}$
   - **B** Graph the set of integer pairs whose second number is $^{-}2$; that is, $(4, ^{-}2)$, $(3, ^{-}2)$, and so on.
   - **C** Graph the set of integer pairs whose first number is $^{-}3$.
   - **D** Graph the set of integer pairs in which the first and second number add to give zero.
   - **E** Graph the set of integer pairs in which the first number is one more than the second number.
   - **F** Graph the set of integer pairs in which the second number is 0.

★2. Draw the co-ordinate axes with points for integers from $^{-}10$ through 10 on each axis. Then graph the set of integer pairs in which
   - **A** the first number is equal to the second.
   - **B** the first number is greater than the second.
   - **C** the second number is greater than the first.
   - **D** the second number is two times the first number.
   - **E** the second number is the first number squared.
   - **F** the sum of the first and second number in each pair is 4.
   - **G** twice the second number is three times the first.

★3. Draw the graph of each of the two number pairs in each part. Then give the co-ordinates of the point that is halfway between the two points.

A $(4, ^{-}4)$ and $(4, 2)$	**C** $(1, 1)$ and $(5, 3)$	**E** $(^{-}4, ^{-}1)$ and $(2, ^{-}5)$
B $(^{-}5, 6)$ and $(^{-}5, 0)$	**D** $(^{-}5, 3)$ and $(5, ^{-}3)$	**F** $(^{-}1, ^{-}1)$ and $(5, ^{-}5)$

think

Ben argued that his \$5.00 a week allowance was not enough. His father refused to raise the amount.

Ben offered the following "bargain." He asked his father to give him 1¢ for the first week, 2¢ for the second week, 4¢ for the following week, 8¢ the following week, and so on.

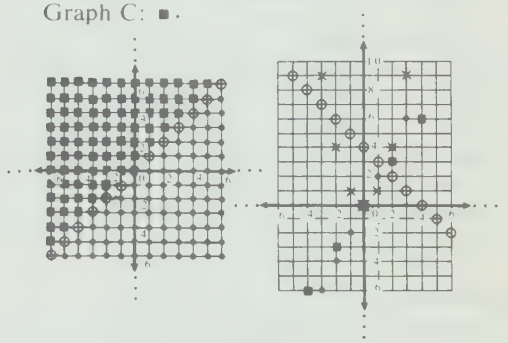
Should Ben's father accept the bargain? Explain your answer.

UTILIZATION

In Exercise 2, the student is given a relation between the numbers in each of the integer pairs and is required to sketch a graph of these pairs. Although the exercise is starred, most students will be able to do it, and it will provide a good background for the next lesson.

Answers, Exercise 2

2. Graph A: ○. Graph B: ●. Graph C: ■. Graph D: ● Graph E: × Graph F: ○ Graph G: ■



EXTENSION

Workbook page 115 provides appropriate additional practice exercises.

Think Solution

Ten weeks of allowance under the old plan would cost Ben's father \$50, whereas under the new plan, ten weeks cost \$10.23 ($1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512$ cents). At this point, the new plan would sound better to Ben's father. But, by the thirteenth week (when the week's allowance would reach \$40.96), Ben's father would probably begin to complain.

Assignments

Minimum 1. Average 1, 2A–F. Maximum 1–3.

Objective

Given a function rule and a set of integer input numbers, the student can find the corresponding output number and draw a graph of the input-output pairs.

PREPARATION

Materials: graph paper or co-ordinate grid paper. (*Duplicator Masters*, page 83 or 86).

Review the use of the function machine, using simple function rules with whole-number inputs.

INVESTIGATION

This lesson reinforces and extends the work of the previous lesson. Distribute graph paper or co-ordinate grid paper for the Investigation. When students have completed finding the integer pairs and their graph, have someone display their graph on the overhead projector or chalkboard.

DISCUSSION

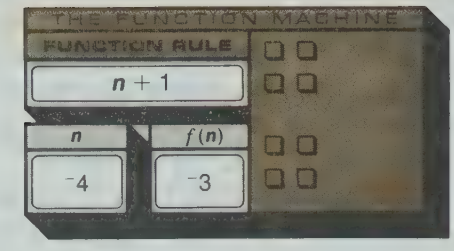
The points for the integer pairs of the function rule $f(n) = n + 1$ all fall on the same line; however, the graph is *not* all the points on the line since only integer co-ordinates are being used.

If the graph of the function rule in Exercise 2 is graphed on the same grid as in the Investigation, the student can readily see that the graph for $f(n) = n + 2$ is one unit higher than for $f(n) = n + 1$. You might ask students if they can guess without drawing graphs what the graph of $f(n) = n + 3$ or $f(n) = n - 1$ might be like.

Graphing Functions

Investigating the Ideas

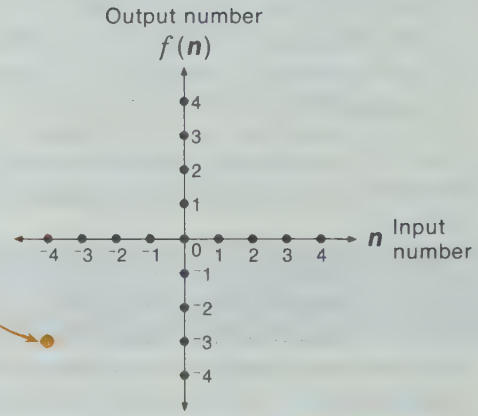
For each function rule and set of input numbers, you can find a set of number pairs. In each pair, the first number is the input number n and the second number is the output number $f(n)$. You can draw graphs for these sets of number pairs.



Function Rule: $n + 1$

n	$f(n)$	Integer pairs
$^-4$	$^-3$	$\rightarrow (^-4, ^-3)$
$^-3$		$\rightarrow (^-3,$)
$^-2$		$\rightarrow (^-2,$)
$^-1$		$\rightarrow (^-1,$)
0		$\rightarrow (0,$)
1		$\rightarrow (1,$)
2		$\rightarrow (2,$)

Graph of the integer pair



Can you find the missing numbers in the function table, the integer pairs, and then graph each pair on a grid?

Discussing the Ideas

1. Do the points for the integer pairs above form a pattern on your grid? Describe the pattern if there is one.

2. A Complete the function table for the rule $n + 2$. Then give the integer pairs.
   B Graph the integer pairs.
   C How does the graph for this function rule compare with the graph in the Investigation?

Function Rule: $n + 2$

n	$f(n)$	Integer pairs
$^-3$	$^-1$	$\rightarrow (^-3,$)
$^-2$		$\rightarrow ($,)
$^-1$		$\rightarrow ($,)
0		$\rightarrow ($,)
1		$\rightarrow ($,)
2		$\rightarrow ($,)

Using the Ideas

Copy and complete each function table. Then graph the integer pairs for each function.

1.

	Function Rule $n - 3$	
	n	$f(n)$
	-4	-7
A	-3	
B	-2	
C	-1	
D	0	
E	1	
F	2	
G	3	
H	4	

2.

	Function Rule $3 - n$	
	n	$f(n)$
	-4	7
A	-3	
B	-2	
C	-1	
D	0	
E	1	
F	2	
G	3	
H	4	

3.

	Function Rule $1 \cdot n$	
	n	$f(n)$
	-4	4
A	-3	
B	-2	
C	-1	
D	0	
E	1	
F	2	
G	3	
H	4	

★ **4.**

	Function Rule n^2	
	n	$f(n)$
	-4	16
	-3	9
A	-2	
B	-1	
C	0	
D	1	
	2	4
E	3	
F	4	

think

Find the number pairs for each exercise. Part A is done for you.

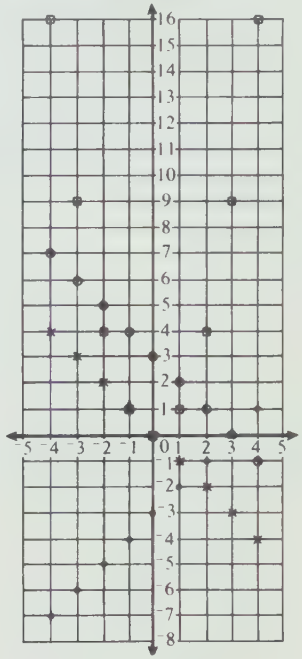
	Sums	✚		Products
A	0	5	5	-25
B	0			-49
C	-8			16
D	8			16
E	-4			0
F	2			-15
G	1			-20
H	-20			100

	Sums	✚		Quotients
A	-10	-5	-5	1
B	-9			8
C	-2			1
D	-4			-2
E	-20			1
F	8			-5
G	-40			7
H	-8			0

UTILIZATION

Considerable time can be saved by providing students with duplicated co-ordinate grids on which to sketch their graphs. After the students have completed the exercises, it would be helpful to have them display some of the graphs on the chalkboard or overhead projector.

Answers, Exercises 1–4



Graph for Exercise 1: ●
Graph for Exercise 2: ○
Graph for Exercise 3: ×
Graph for Exercise 4: □

EXTENSION

Appropriate additional practice exercises are provided on page 77 of the *Duplicator Masters*.

Enrichment: You may wish to challenge more able students with some more difficult functions, such as those below.

1. $f(n) = 2 \cdot n + 3$
2. $f(n) = 4n - 1$
3. $f(n) = n^2 + n$
4. $f(n) = n^2 + 1$
5. $f(n) = 12 \div n$
6. $f(n) = n^3$

Assignments

Minimum 1, 2. Average 1–3. Maximum 1–4.

Objective

The student can demonstrate the ability to work with the concepts presented in this module.

UTILIZATION

Since there are a large number of responses required in these review exercises, you should not feel that it is essential the students complete every part of each exercise. Many parts of the exercises could be answered orally rather than written, if you prefer.

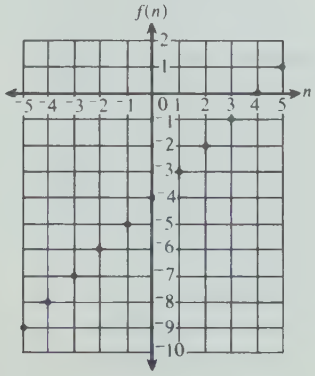
Workbook page 116 offers additional review exercises that would be appropriate for students who might benefit from further review of the ideas in this module.

Answers, Exercise 11

A.

n	−5	−4	−3	−2	−1	0	1	2	3	4	5
$f(n)$	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1

B.



REVIEWING THE IDEAS

1. Give the opposite of each integer.

A 5	C 27	E $^{-}365$	G $^{-}5806$
B $^{-}3$	D $^{-}16$	F 273	H 0

2. For each exercise, give the basic principle illustrated.

A $^{-}28 \cdot 1 = {}^{-}28$

B $^{-}7 \cdot 1 = 1 \cdot {}^{-}7$

C $^{-}12 + 12 = 0$

D $3 \cdot (8 + {}^{-}8) = (3 \cdot 8) + (3 \cdot 8)$

E $^{-}7 \cdot (5 \cdot {}^{-}2) = ({}^{-}7 \cdot 5) \cdot {}^{-}2$

F $^{-}68 + 0 = {}^{-}68$

G $^{-}8 + 6 = 6 + {}^{-}8$

A Opposites	E One
B Zero	F Commutative (×)
C Commutative (+)	G Associative (×)
D Associative (+)	H Distributive

3. Find the sums.

A $9 + {}^{-}9$	D $^{-}3 + {}^{-}8$	G $^{-}4 + 7$
B $10 + {}^{-}8$	E $^{-}8 + 5$	H $^{-}17 + 17$
C $^{-}7 + {}^{-}5$	F $3 + {}^{-}9$	I $^{-}27 + 0$

4. Find the missing addends.

A $0 + n = {}^{-}3$	D $2 + n = {}^{-}1$
B $n + 3 = {}^{-}2$	E $8 + n = 3$
C $^{-}6 + n = {}^{-}10$	F $^{-}6 + n = {}^{-}5$

5. Find the differences.

A $0 - 3$	D $^{-}4 - 3$	G $5 - {}^{-}1$
B $^{-}1 - 3$	E $0 - {}^{-}1$	H $^{-}5 - {}^{-}3$
C $^{-}2 - 3$	F $1 - {}^{-}1$	I $^{-}6 - {}^{-}10$

6. Find the products.

A $9 \cdot 8$	D $5 \cdot {}^{-}5$	G $^{-}86 \cdot {}^{-}10$
B $^{-}9 \cdot 4$	E $^{-}11 \cdot 11$	H $^{-}7 \cdot 20$
C $7 \cdot {}^{-}12$	F $^{-}8 \cdot {}^{-}7$	I $^{-}1 \cdot {}^{-}13$

7. Find the missing factors.

A $^{-}2 \cdot x = {}^{-}6$	D $^{-}1 \cdot p = 1$
B $7 \cdot y = {}^{-}35$	E $^{-}9 \cdot b = 45$
C $^{-}3 \cdot n = 27$	F $17 \cdot z = {}^{-}17$

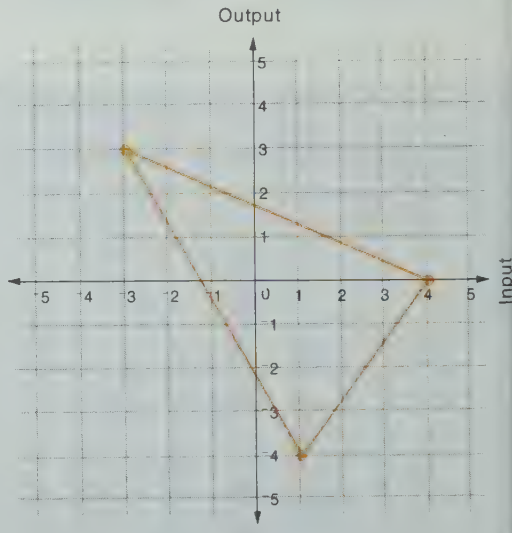
8. Find the quotients.

A $8 \div {}^{-}2$	G $^{-}72 \div {}^{-}9$	
B $^{-}16 \div {}^{-}4$	H $100 \div 20$	
C $^{-}20 \div 5$	I $^{-}48 \div {}^{-}6$	
D $42 \div {}^{-}7$	J $54 \div {}^{-}6$	
E $^{-}12 \div {}^{-}4$	K $^{-}32 \div 8$	
F $0 \div {}^{-}2$	L $15 \div {}^{-}15$	

9. Give the correct symbol ($<$ or $>$) each ◯.

A $^{-}1$ ◯ $^{-}2$	F $^{-}21$ ◯ $^{-}25$	
B 7 ◯ $^{-}7$	G $^{-}2$ ◯ 0	
C 10 ◯ $^{-}2$	H 100 ◯ $^{-}1000$	
D $^{-}2$ ◯ $^{-}3$	I 176 ◯ 716	
E 0 ◯ $^{-}5$	J $^{-}49$ ◯ $^{-}50$	

10. Give the co-ordinates of each vertex of the triangle.



11. A Make a function table for the rule $n + ({}^{-}4)$. Use the integers from $^{-}5$ through 5 as input numbers.

B Draw a graph for part A.

TEST YOURSELF

1. Give the opposite of each integer.
   A 6 B $^{-}9$ C 0

2. Complete the sentence:
   The sum of any integer and its opposite is ?.

3. Solve each equation. Think about the basic principles.
   A $8 + {}^{-}8 = n$
   B $^{-}23 + 18 = n + {}^{-}23$
   C $^{-}9 \cdot n = {}^{-}9$
   D $^{-}5 \cdot (^{-}4 + n) = 0$
   E $(^{-}1 + 8) + {}^{-}3 = 1 + (n + {}^{-}3)$

4. Find the sums.
   A $11 + {}^{-}6$ B $^{-}6 + {}^{-}8$ C $^{-}9 + 9$

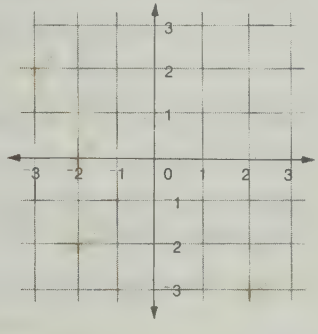
5. Find the differences.
   A $9 - {}^{-}5$ B $^{-}3 - 1$ C $^{-}7 - {}^{-}2$

6. Find the products.
   A $2 \cdot {}^{-}4$ B $^{-}6 \cdot {}^{-}3$ C $^{-}1 \cdot 1$

7. Find the quotients.
   A $^{-}9 \div 3$ B $^{-}24 \div {}^{-}3$ C $14 \div {}^{-}7$

8. Give the correct symbol ($<$ or $>$) for each ▨.
   A 3 ▨ $^{-}5$ B $^{-}9$ ▨ $^{-}7$

9. Give the co-ordinates of points *A*, *B*, *C*, and *D* on the graph.



10. Complete the function table.

	Function Rule: $n - 2$	
	n	$f(n)$
	3	1
A	2	▨
B	1	▨
C	0	▨
D	$^{-}1$	▨

RESEARCH PROJECTS

A There is no real number which when multiplied by itself will give a product of $^{-}1$. Because of this, mathematicians have invented the number i so that $i \cdot i = {}^{-}1$. The number i is called a **complex number**. Read about complex numbers. Find out how to add and multiply such numbers. (See *Realm of Algebra* by Issac Asimov; Boston: Houghton-Mifflin Company, 1961, pp. 151–161 available from Thomas Nelson & Sons (Canada) Ltd.)

B How long has man used negative numbers? How were they invented and what kind of symbols have been used to denote negative numbers? Read about negative numbers in an encyclopedia or in a mathematics history book. Report your findings to the class. (See *History of Mathematics* by D. E. Smith; New York: Dover Publications, 1953, Vol. II, pp. 257–260 available from General Publishing Co., Ltd.)

TEST YOURSELF

The self-evaluation test covers the basic operations with integers, inequalities, and graphs of integers in the co-ordinate plane. Students who score well on this test should be expected to do well on an achievement test covering the module. Encourage students to rework those parts that they missed and to seek help with those exercises they do not understand.

Note that the module objectives are correlated with individual test items by annotations on the student text page.

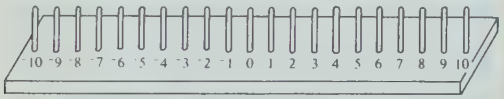
RESEARCH PROJECTS

Students who wish to explore Project A will find that any intermediate algebra book will contain material on complex numbers, but it is likely to be too difficult for students at this level. The best reference is probably that cited in the student text. Graphs of complex numbers could also be a part of this project.

Any history of mathematics book can serve as a source of information for students who undertake Project B. Especially recommended is Carl B. Boyer's *A History of Mathematics;* New York: John Wiley and Sons, Canada Ltd., 1968.

MATHEMATICAL RECREATION

A more durable model for this number line game can be made by placing 21 short pieces of dowel rod in a board as shown below. Rings or washers can be used to mark a player's position on the number line.



An interesting variation of the rules that will often shorten the game is the stipulation that, if a player's move takes him to a point occupied by the other player, he wins the game.

MATHEMATICAL RECREATIONS

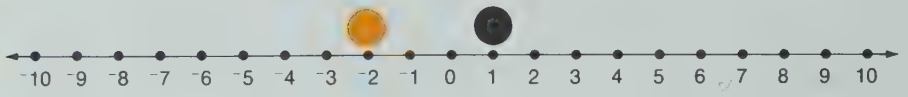
THE NUMBER LINE GAME

This is a game for two people.

Make a spinner like the one shown.

Draw a number line with points for the integers from $^-10$ through 10.

Each player also needs a counter or marker.



The players take turns spinning the spinner.

Starting at zero, each player moves his counter to the **right** the number of units on the spinner if the arrow lands on a positive integer. The player moves his counter to the left if the arrow lands on a negative number.

The player who reaches 10 or $^-10$ first, wins the game.

MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. A $^-6$; B 9; C 0 **2.** zero **3.** A 0; B 18; C 1; D 4; E 8; **4.** A 5; B $^-14$; C 0 **5.** A 14; B $^-4$; C $^-5$ **6.** A $^-8$; B 18; C $^-1$ **7.** A $^-3$; B 8; C $^-2$ **8.** A $>$; B $<$ **9.** $A:(^-3)$, $B:(2,^-3)$, $C:(^-2,^-2)$, $D:(^-2,0)$ **10.** A 0; B $^-1$; C $^-2$; D $^-3$

D-28

UNIT D: INTEGERS, PROBABILITY, AND STATISTICS

Module 2: Experiments in Probability

General Objectives

To provide an intuitive introduction to some probability concepts.
To introduce sample spaces and equally likely outcomes.
To enable the student to determine the probability of the outcomes of some probability experiments.
To provide probability experiments in which the student can actively compare experimental probability with theoretical probability.
To introduce the idea of sampling and drawing conclusions from samples.

	Performance Objectives	Pupil Text	Reteach-Reinforce	Related Activities
133	**YELLOW** Given a probability experiment and an appropriate figure (if needed), the student can list all the outcomes for the experiment.	D-32, 33	**WB** 117	**SWM 2** 156
134	**YELLOW** Given a probability experiment and an appropriate figure (if needed), the student can determine whether the outcomes of the experiment are equally likely.	D-30, 31		**SWM 2** 144, 145
135	**YELLOW** The student can find the theoretical probability for each outcome of a probability experiment.	D-34, 35	**WB** 118 **DM** 78	**SWM 2** 148, 155, 157
136	**YELLOW** Given a probability experiment, the student can compute the theoretical expectations for the experiment.	D-36, 37		**SWM 2** 155
137	***GREEN*** *The student can use samples of a large set to draw conclusions about the set.*	D-38, 39 D-40, 41	**WB** 119	**SWM 2** 160 **SWM 3** 144-147
	Reviewing the Ideas	D-42	**WB** 120	

MATHEMATICS

When a penny is tossed, the result is always either a head or a tail (assuming that the coin will never land standing on edge). Each result is called an *outcome* and the set {heads, tails} is called a *sample space* for the experiment. Every trial of the experiment results in one of the elements of the sample space for the experiment. If the penny is a *fair coin,* we can expect that heads will come up in $\frac{1}{2}$ of the tosses and tails in $\frac{1}{2}$ of the tosses. We say, therefore, that the outcomes are *equally likely* and that the probability of each outcome is $\frac{1}{2}$, because there is 1 chance in 2 that a head will result and 1 chance in 2 that a tail will result when a penny is tossed.

In general, if an experiment has a sample space of n elements and if each outcome in the sample space is equally likely, then the probability of each outcome is $\frac{1}{n}$. Since there are n elements, the sum of the n numbers of $\frac{1}{n}$ is 1.

Of course, not every experiment has equally likely outcomes. If, for example, 1 white bead and 3 red beads are placed in a box and one bead is drawn at random, there is 1 chance in 4 of drawing a white bead and 3 chances in 4 of drawing a red bead. Thus, the probability of drawing a white bead is $\frac{1}{4}$ and the probability of drawing a red bead is $\frac{3}{4}$. In any probability experiment, the probability of any outcome is always a number from 0 to 1 and the sum of all the probabilities in the sample space is 1.

TEACHING THE MODULE

Materials

Pennies or plastic disks, dice, wooden or plastic cubes, spinners (homemade), beads, marbles or beans of various colors, playing cards, paper cups, graph paper, rules, tracing paper.

Vocabulary

chances
equally likely outcomes
experiment
outcome
probability
sample
sample space

The primary purpose of this module is to introduce some of the simplest notions of probability through a wide variety of experiments in which the student takes an active part. The amount of mathematical notation, computation, and formulas is very limited, and only a minimum of new terminology is introduced. Thus, the focus is on ideas and experiments rather than computational skills.

In teaching this module, your main goal should be to introduce the students to probability by means of a wide variety of probability experiments. This module, perhaps more than any other, lends itself to active participation on the part of the student.

It is essential that sufficient material be available so that several pairs of students can work on the same experiment simultaneously. While the materials list for this module is extensive, most of the materials are readily available or can be quickly made. Some suggestions concerning materials that are needed for specific probability experiments are made in the commentary for individual lessons.

Most students will not need any extra motivation for the work in this module. Indeed, many will want to extend the work to greater depth. If the time is available, be sure to capitalize on any such interest.

Perhaps the most important thing to emphasize in this module is the distinction between theoretical probability and probabilities derived from actual experimentation. Students will need to understand that, although the probability of getting a head when a coin is tossed is $\frac{1}{2}$, it is not likely that one will get exactly 50 heads in 100 tosses of a coin. Misconceptions about probability are quite common, and you will need to be alert to dispel them. For example, if a tossed coin results in 3 heads in a row, students may feel that the probability of getting a head on the next toss is no longer $\frac{1}{2}$, that somehow it is more likely that the next toss will come up tails.

Lesson Schedule

Although this module is relatively short, if time permits, you will probably want to spend more than one day for each lesson, especially when experiments are being performed by the students. You probably should allow a minimum of 6 days, but if you have the students carry out a large number of experiments, an additional 3 or 4 days of classwork may be needed.

Evaluation

Keep in mind that this module is designed to create interest and to stimulate students to carry out simple experiments and record their findings. Evaluate students largely on their daily participation rather than on their ability to remember how to solve probability problems.

RESOURCES FOR ACTIVE LEARNING

Activities

Developmental Math Cards, L3, Addison-Wesley

Experiences in Mathematical Ideas, Vol. 2, "Dealing with Uncertainty," pp. 174–207, NCTM

Experiments in Probability and Statistics, Midwest Publications

Nuffield Project: *Probability and Statistics; Problems*—Red Set, Nos. 15, 15A, 23, 23A, Wiley

Probability: The Science of Chance, Rand McNally

Towards Probability, Cuisenaire Co.

Manipulative Devices

Binostat (General Learning Corp.; Sigma)

Hexstat (LaPine; Gamco)

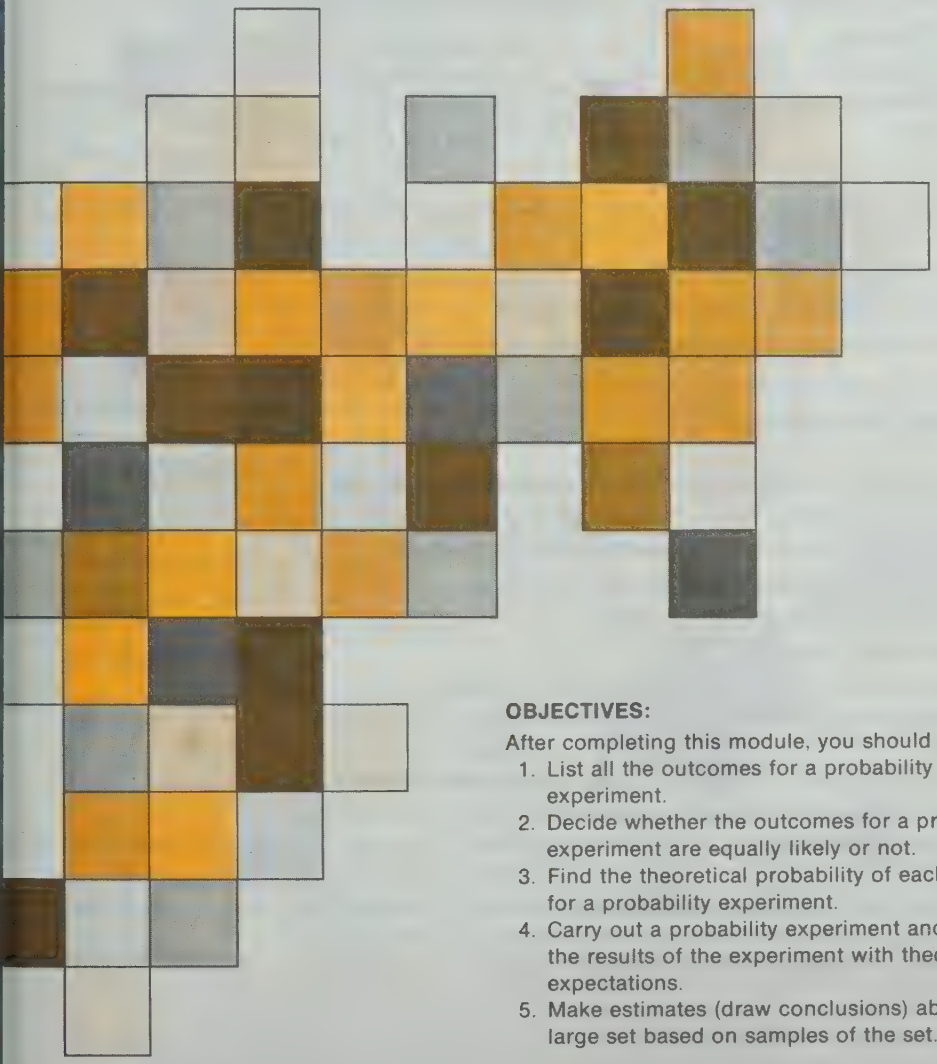
Probability Kit (Creative Publications; LaPine)

Spinner Kit (Houghton Mifflin)

Spinners and Dice (Creative Publications)

UNIT D: *Integers, Probability, and Statistics*

MODULE 2: *Experiments in Probability*



OBJECTIVES:

After completing this module, you should be able to:

1. List all the outcomes for a probability experiment.
2. Decide whether the outcomes for a probability experiment are equally likely or not.
3. Find the theoretical probability of each outcome for a probability experiment.
4. Carry out a probability experiment and compare the results of the experiment with theoretical expectations.
5. Make estimates (draw conclusions) about a large set based on samples of the set.

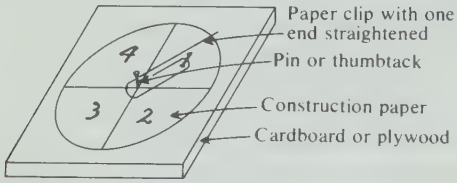
Objective

The student can carry out probability experiments with several outcomes and decide if the outcomes are equally likely.

PREPARATION

Materials: Pennies or plastic disks, dice or wooden cubes, spinners, marbles or beads of two different colors, thumbtacks.

Collect sufficient materials for students to carry out the Investigation. Small plastic disks or checkers could be substituted for pennies. If dice are not available, small wooden cubes can be marked to simulate dice. A satisfactory classroom spinner can be constructed as shown below.



INVESTIGATION

If students work in pairs or in small groups on the Investigation, less material will be needed and more interest may be generated. Some students may carry out the experiment and others may keep a tally of the outcomes.

DISCUSSION

In discussing *equally likely outcomes*, point out that, although the two outcomes in tossing a coin, theoretically, should be considered as equally likely, an actual experiment may not yield those results. Similarly, although each outcome in tossing a die is considered equally likely, in actual practice when a die is tossed only a few times, one face may come up more often than some of the other faces. In a large number of trials, however, the experimental results are likely to be very close to the theoretical results.

In discussing spinners like those shown in Exercise 4, bring out the fact that each region on the spinner must be the same size if the outcomes are to be equally likely.

Equally Likely Outcomes

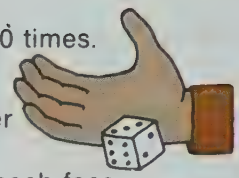
Investigating the Ideas

Try one of these experiments.

A Toss a penny 100 times. Keep a tally of heads and tails.

H	𝍸
T	𝍸 ||

B Toss a die 100 times. Keep a tally of the number of dots that come up on each face.



1	2	3	4	5	6
||	|	|||	|		||

Does your tally show that each result occurred about the same number of times?

Discussing the Ideas

1. When each of the possible results or **outcomes** for an experiment has the same chance of occurring, the outcomes are said to be **equally likely outcomes**.
   - **A** Do you think that the outcomes for the coin experiment are equally likely? Give your reasons.
   - **B** Do you think that the outcomes for tossing a die are equally likely? Give your reasons.

2. Jack thought that since he did not get *exactly* 50 tails and 50 heads in 100 tosses, this proved that outcomes were not equally likely. What do you think?

3. Combine the results of your experiment with those of your classmates who carried out the same experiment. Discuss what the combined results show.

4. If the arrow on each spinner is spun and the region in which it stops is tallied, which spinner do you think would have equally likely outcomes?

Spinner A

Spinner B

Using the Ideas

For each experiment described below, give the following:

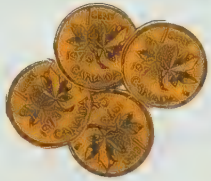
- **A** List all of the possible outcomes.
- **B** Tell whether the outcomes seem to be equally likely or not.
- **C** If one outcome is more likely than another, tell which one it is.

EXPERIMENT 1: Spin the arrow on the spinner. Record the number of the region in which the arrow stops.

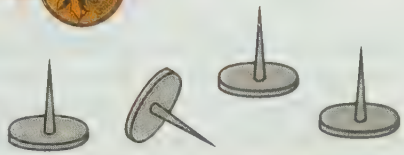
EXPERIMENT 2: Five faces of a cube have a letter **A** on them. The sixth face has the letter **B** on it. The cube is tossed and the letter on the top face is recorded.

EXPERIMENT 3: Spin the arrow on the spinner. Record the number of the region in which the arrow stops.

EXPERIMENT 4: Six red marbles and four blue marbles are placed in a box. Without looking, one marble is drawn from the box, its color recorded, and then replaced into the box.



EXPERIMENT 5: Four pennies are tossed at the same time. The number of heads and tails that come up are recorded.



★EXPERIMENT 6: Toss four thumb tacks. Record the number of tacks that land with their points up and on their side.

think

Use four 7's and any of the mathematical operation symbols of +, −, ×, ÷, and parentheses to write a numeral for 56.

UTILIZATION

An effective way to utilize these exercises would be to discuss each experiment with the students, then have the students carry out the experiment and compare their results with the ideas presented in the discussion. If the experiments are actually carried out, students should work in pairs or small groups. Each experiment should be performed for at least 100 trials in order to yield satisfactory data.

Answers, Experiments 1–6

1. Outcomes: {1, 2, 3, 4} Equally likely outcomes.
2. Outcomes: {A, B} A is more likely than B.
3. Outcomes: {1, 2, 3} Equally likely outcomes.
4. Outcomes: {Red marble, blue marble} Red marble is more likely than blue.
5. Outcomes: {4 heads; 3 heads, 1 tail; 2 heads, 2 tails; 1 head, 3 tails; 4 tails} Most likely outcome is 2 heads and 2 tails.
6. Outcomes: {4 up; 3 up, 1 on a side; 2 up, 2 on a side; 1 up, 3 on a side; 4 on a side} The results will depend on the kind of thumbtack used.

EXTENSION

Enrichment: Choose 100 or more telephone numbers at random from a directory and record the last digit of each number. Try to determine by experiment if each of the ten digits (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) is equally likely to occur.

Think Solution

$(7 \div 7 + 7) \times 7 = 56$

Some students might be encouraged to make up other puzzle problems to be solved by using four 7's and the basic operation symbols. They could then challenge classmates to solve them. Sample problems and solutions:

0 $= (7 - 7) \times 77$ **1** $= (7 + 7) \div (7 + 7)$
14 $= 7 + 7 + 7 - 7$ **28** $= 7 + 7 + 7 + 7$
49 $= (7 \times 7) \div (7 \div 7)$ **98** $= 7 \times 7 + 7 \times 7$
★**678, 223, 072, 849** $= 7^7 \times 7^7$

Assignments

Minimum 1–3. Average 1–4. Maximum 1–6.

Objective

The student can list the outcomes in the sample space for a given probability experiment.

PREPARATION

Materials: Cubes or dice, coins, spinners.

It is quite desirable to have sufficient material so that everyone can participate in the experiments. You may arrange several work stations in your room so that the students can work in small groups on the Investigation or other experiments.

INVESTIGATION

The purpose of the Investigation is to have the students determine all of the different outcomes from tossing two cubes. The set of all the different outcomes is the sample space for the experiment. Each trial of the experiment results in one outcome in the sample space.

It may not be evident to many students at the outset how many different outcomes are possible from the cube-tossing experiment, but as they work on the experiments they should discover that each face of one cube can be paired with each face of the other cube; thus, there are 6×6, or 36, different outcomes that are possible in the experiment.

DISCUSSION

Exercises 1 and 2 should bring out the idea that all 36 outcomes for the Investigation experiment can be listed without actually carrying out the experiment.

A *sample space*, introduced in Exercise 3, is simply the set of all possible outcomes for an experiment. While the elements in the sample space for the die tossing experiment in Exercise 4 consist of the numbers 1 through 6, the elements of the sample space for Exercise 5 are ordered pairs of letters. Point out that we must agree upon which letter of each pair refers to the penny or the dime. In this case, since the penny is listed first in the table, we could agree that the first letter in each ordered pair refers to the penny.

Sample Spaces

Investigating the Ideas

Mark the faces of a small cube with the letters **A**, **B**, **C**, **D**, **E**, and **F**.

Mark the faces of a second cube with the numerals 1, 2, 3, 4, 5, and 6.

Every time you toss the two cubes, a letter and a numeral on the top faces is an outcome.

For example, the outcome shown of an **E** and a 4 turning up could be written as (E, 4)

How many different outcomes can you get?

Discussing the Ideas

1. **A** How many different outcomes are there that have **E** as part of the outcome?
   B How many different letters can come up on the one cube?
   C How many different numerals can come up on the other cube?

2. Can you explain how to find all the possible outcomes without actually tossing the cubes?

3. The set of all possible outcomes for an experiment is called a **sample space** for the experiment.
   How many outcomes are in the sample space for the Investigation?

4. A sample space for the experiment of tossing a single coin can be designated by the set $\{H, T\}$ where the H stands for heads and the T stands for tails. What is a sample space for an experiment of tossing a die?

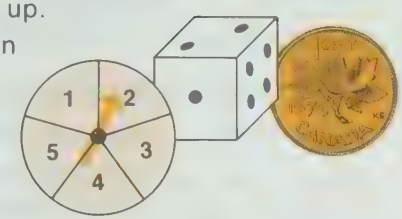
5. A penny and a dime are tossed. A table similar to the one shown can help to show the different outcomes. The sample space for the experiment can be written as $\{(H, H), (H, T), (T, H), (T, T)\}$.
   A How many outcomes are in the sample space?
   B What does the symbol (H, T) mean?
   C What does (T, H) mean?

PENNY	DIME
H	*H*
H	*T*
T	*H*
T	*T*

Using the Ideas

1. List all the possible outcomes for each experiment.

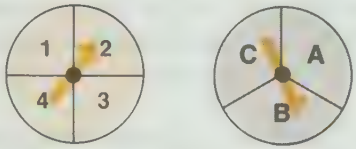
- **A** Toss a penny. Record which side comes up.
- **B** Toss a die. Record the number of dots on the top face.
- **C** Spin the arrow on a dial. Record the number of the region in which the arrow stops.



2. If a penny, a nickel, and a dime are tossed at the same time, you might want to make a table similar to the one at the right in order to find the sample space. Complete the table to show all the possible outcomes.

Penny	Nickel	Dime
H	*H*	*T*
⋮	⋮	⋮

3. One of the outcomes for spinning the arrows on the two dials like those at the right is (2, **B**). List all the other outcomes that could occur.



First dial Second dial

4. An experiment consists of tossing a penny and a die at the same time. Complete the sample space.

Sample space: $\{(H, 1), (T, 1), (H, 2), \ldots\}$

★ **5.** An experiment consists of shuffling a deck of 52 playing cards and then turning up cards until an ace is turned up. The outcome is the number of cards that must be turned up until the first ace is reached (including the ace).

- **A** The smallest outcome is obviously 1 and occurs when an ace is the first card turned up. What is the largest possible outcome?
- **B** On the average, how many cards do you think must be turned up before you reach the first ace? Try the experiment.

UTILIZATION

In Exercises 1 through 4, it should not be required that all answers be expressed in set notation. A table that shows all possible outcomes is a sufficient demonstration of students' understanding. For example the table suggested in Exercise 2 might be completed as follows.

Penny	Nickel	Dime
H	H	H
H	H	T
H	T	H
H	T	T
T	H	H
T	H	T
T	T	H
T	T	T

The experiment in Exercise 5 should prove interesting to some students. The experiment should be repeated for 20 or more trials, and the average number of cards turned up, to and including the first ace, should be computed. The theoretical result for this experiment is that there are 10.6 cards turned up, to and including the first ace.

EXTENSION

Appropriate additional practice exercises are provided on page 117 of the *Workbook*.

Enrichment: Use any of the first four exercises of this set as a basis for some experiments. Have students tally the outcomes to see if the outcomes seem to be equally likely.

Assignments

Minimum 1, 2. Average 1–4. Maximum 1–5.

Objective

Given a simple probability experiment, the student can find the probability of each outcome.

PREPARATION

Materials: Small cubes (1 per student or group of students).

Review the idea of outcomes, equally likely outcomes, and sample spaces of probability experiments.

INVESTIGATION

Before beginning the experiment in the Investigation, each student should estimate and record the number of times he expects each face to come up in 60 tosses. Have students work in pairs or in small groups in order to carry out the Investigation. Since each face of the cube is equally likely to come up when tossed, each face should come up about $\frac{1}{6}$ of the tosses. Since 3 of the faces are labeled "A," those faces should come up about $\frac{3}{6}$ of the time. The faces labeled "B" should come up about $\frac{2}{6}$ of the time, and face "C" should come up $\frac{1}{6}$ of the time.

DISCUSSION

Exercise 1 introduces the language of *chances* of an outcome. This is related to the idea of the *probability* of an outcome in Exercise 2.

Stress the fact that if a sample space contains n outcomes, each of which is equally likely, then the probability of each outcome is $\frac{1}{n}$. Exercise 4 can be used to illustrate this fact. Also bring out the fact that the probability of any outcome is always a number from 0 to 1 and that the sum of all the probabilities of the outcomes in a sample space must be 1.

Finding the Probability of an Outcome

Investigating the Ideas

Label 3 faces of a cube with the letter **A**;

Label 2 faces of the cube with the letter **B**.

Label 1 face of the cube with the letter **C**.

Guess how many times each one of the three letters would come up in 60 tosses of the cube.

Can you carry out the experiment of 60 tosses and compare your results with your guess?

Discussing the Ideas

1. Karen explained her guess of "30 **A**'s" for the experiment this way: "Since there are 3 *chances* in 6 of an **A** coming up, then about $\frac{3}{6}$ or $\frac{1}{2}$ of the 60 tosses should be **A**'s." How could she estimate the number of times the other letters should appear?

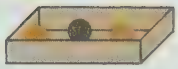
2. Since there are *3 chances in 6* that an **A** will appear on top, we say that the **probability** of getting an **A** is $\frac{3}{6}$ or $\frac{1}{2}$.
   - **A** There are *2 chances in 6* of getting a **B**, what is the **probability** of getting a **B**?
   - **B** What are the chances of getting a **C**?
   - **C** What is the probability of getting a **C**?

3. **A** What are the chances of getting a head when tossing a penny?
   - **B** What is the probability of getting a head?
   - **C** What are the chances of getting a tail?
   - **D** What is the probability of getting a tail?

4. **Equally likely outcomes have equal probabilities.**
   - **A** Since each number is equally likely for this spinner, what is the probability for each number?
   - **B** What is the sum of the probabilities for all three outcomes?

Using the Ideas

1. Two colored marbles and one black marble are placed in a box. Suppose that a blindfolded person is to draw a marble from the box.

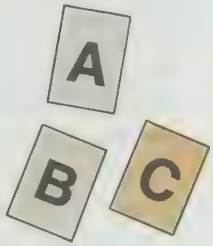


   A There are ||| *chances* in 3 of drawing a colored marble.
   B What is the **probability** of drawing a colored marble.
   C There are ||| chances in 3 of drawing a black marble.
   D What is the probability of drawing a black marble?

2. A box contains 9 blocks all the same size. Four blocks are red, three are black, and two are white. Suppose that the blocks are mixed together and a blindfolded person selects one of the blocks from the box. What is the probability of selecting

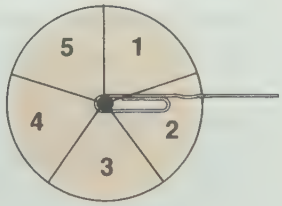
   A a red block? B a black block? C a white block?

3. Three cards with the letters **A**, **B**, or **C** printed on one side are turned face down on a table and then their positions are thoroughly mixed. Supply the correct information to complete each sentence.



   A The probability of turning up the card with letter **A** is ?.
   B The probability of turning up a card with either **A** or **B** on it is ?.
   C If an outcome is certain to happen, then the probability of that outcome is 1. The probability of turning up a card with either **A**, **B**, or **C** on it is ?.
   D If an outcome is impossible or can never happen, then the probability of that event is zero. If 1 of the 3 cards is to be turned up, then the probability that it has the letter **Z** on it is ?.

4. A spinner is constructed as shown at the right. When the arrow (paper clip) is spun, it is equally likely to stop in any of the five regions.



   A What is the probability that the arrow will stop on an odd-numbered region?
   B What is the probability that the arrow will stop on an even-numbered region?
   C What is the probability that the arrow will stop on a region whose number is less than 5?

UTILIZATION

The exercises can be discussed orally or assigned as written work. The exercises would provide active involvement on the part of the students if the students actually carry out the suggested experiments.

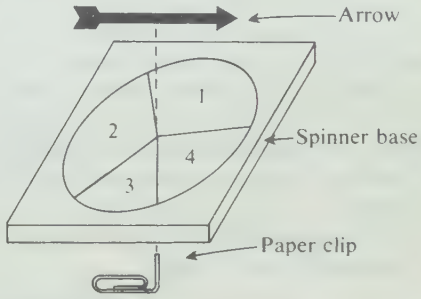
If the experimental approach is taken, it is important to compare the theoretical probability with the empirical probability that the students find. The students should observe that, in general, if a large number of trials are made for the experiment, the empirical probability is usually quite close to the theoretical probability.

EXTENSION

To provide extra practice, make assignments from *Workbook* page 18 and *Duplicator Masters* page 78.

Enrichment: Design a spinner experiment which has several outcomes which are not equally likely. Have students perform the experiment many times and estimate, from the experiment, the probability to assign to each outcome.

A spinner which is easily made from cardboard is shown below. One end of a paper clip is pushed through the spinner base. This acts as an axis on which the cardboard arrow turns. Tape the paper clip to the bottom side of the base.



Assignments

Minimum 1, 2. Average 1–3. Maximum 1–4.

Objective

The student can carry out a probability experiment and compare the results of the experiment with expected results.

PREPARATION

Materials: Pennies, marbles of 2 colors, playing cards, homemade spinners, dice or cubes, paper cups, centimetre graph paper (Duplicator Masters, page 83).

Set up work stations around the classroom containing the materials needed to carry out each experiment suggested by the exercises. The students can work in pairs or small groups. This will reduce the amount of material needed for the experiments.

UTILIZATION

The exercises will provide a wide variety of probability experiments for students to try. You may want to assign only a few problems to each student. If each pair or group of students is to complete all or most of the experiments, you should probably spend more than one class period on this lesson.

One of the main objectives of this lesson should be to compare the experimental or empirical probabilities with the theoretical probability. For example, in Exercise 1 the theoretical probability of getting 2 heads when two pennies are tossed is $\frac{1}{4}$ or 0.25. However, in carrying out the experiment students may find that they got 2 heads 21 times out of 100. Thus, the empirical probability of 2 heads is $\frac{21}{100}$ or 0.21, which is quite near the theoretical probability. If all student results for a given experiment are combined, experimental results will usually be quite near the theoretical probabilities.

Theoretical probabilities for the outcomes for each problem are as follows:

1. $P(2H) = \frac{1}{4}$; $P(1H, 1T) = \frac{1}{2}$; $P(2T) = \frac{1}{4}$
2. $P(\text{black marble}) = \frac{3}{10}$; $P(\text{white marble}) = \frac{7}{10}$
3. $P(\text{Ace}) = \frac{4}{52} = \frac{1}{13}$
4. $P(A) = \frac{1}{2}$; $P(B) = \frac{1}{4}$; $P(C) = \frac{1}{4}$

PROBABILITY EXPERIMENTS

In Exercises 1 through 9, try to carry out as many of the probability experiments as possible. Then compare your **experimental results** with the results that you would *theoretically expect* based on your knowledge of probability.

1. Toss two pennies 100 times and make a tally of the results.

Outcomes	Tally
2*H*	/
1*H* and 1*T*	//// /
2*T*	//

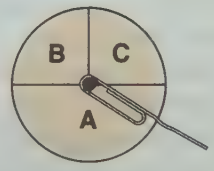
- **A** What is the ratio of the number of times two heads appear to the number of tosses?
- **B** What is the ratio of the number of times a head and a tail appear to the number of tosses?
- **C** In 100 tosses, what is the number of times that you would **expect** two tails to appear?

2. Put three black marbles and seven white marbles (all the same size) into a small sack. Without looking, draw a marble from the sack, record its color, and then replace it in the sack. Draw and replace the marbles 50 times.

- **A** The probability of drawing a black marble from the bag is $\frac{3}{10}$. Did you draw a black marble about $\frac{3}{10}$ of the time?
- **B** What is the probability of drawing a white marble from the bag? Does your experimentation agree with this?

3. There are four aces in an ordinary deck of 52 playing cards. Thus, if one card is drawn from the deck, the probability that it is an ace is $\frac{4}{52}$ or $\frac{1}{13}$. Shuffle a deck of playing cards and draw a card from the deck. Repeat the experiment 50 times. Keep a record of the number of aces drawn. Did you draw an ace about $\frac{1}{13}$ of the time?

4. Make a spinner like the one shown. The probability of the arrow stopping in region A is $\frac{1}{2}$. The probability of the arrow stopping in either region B or C is $\frac{1}{4}$. Keep a tally of the outcomes for 100 spins. Find each of these ratios:



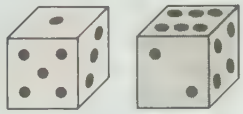
- **A** $\frac{\text{Number of A outcomes}}{100}$
- **B** $\frac{\text{Number of B outcomes}}{100}$
- **C** $\frac{\text{Number of C outcomes}}{100}$

5. Toss three pennies at a time 48 times. Keep a tally of your results.

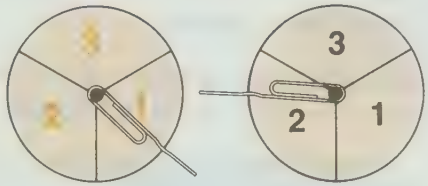
Outcomes	Probability	Tally
3*H*	$\frac{1}{8}$	
2*H*, 1*T*	$\frac{3}{8}$	
1*H*, 2*T*	$\frac{3}{8}$	
3*T*	$\frac{1}{8}$	

- **A** Write a fraction that compares the number of times each outcome occurred to the number of tosses. Compare your result with the probability shown in the table.
- **B** Write a fraction that compares the actual number of times that two heads and a tail appear to the number of tosses.

6. The probability of getting a sum of 7 when two dice are tossed is $\frac{1}{6}$. Toss a pair of dice 60 times and give a fraction that compares the number of times you got a sum of 7 with the number of tosses.



7. Make two spinners like those shown. Spin both arrows at the same time for 60 times. Keep a tally of the sums that you get.



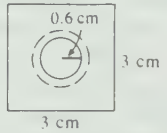
- **A** Which sum do you think is most likely to occur?
- **B** What is the experimental probability of getting that sum?

★ **8.** Drop a paper drinking cup on the floor 100 times. Keep a tally of the number of times it lands right side up, upside down, and on its side.

- **A** Determine the experimental probability of the cup landing in each position.
- **B** Compare your results with those of your classmates who performed the experiment.

★ **9.** Make a grid composed of 3-centimetre squares. Drop a penny on the grid from a height of about 30 centimetres. Determine the experimental probability of the coin landing **completely inside** one of the squares on the grid based on 100 drops of the coin.

5. Probabilities are given in the table.
6. $P(7) = \frac{1}{6}$
7. Most likely sum is 4. $P(4) = \frac{4}{9}$
8. The probabilities for the outcomes of this experiment will vary because of the shape of the cup, the distance the cup falls, or even how the cup is held as it is dropped. For most paper cups with bases smaller than their tops, it appears to be far more likely that they will land on the side than in any other position. For more consistent results, you might suggest that the experiment be performed by dropping the cup to the floor from the height of a student desk and releasing the cup with the open side facing the floor and without imparting any spin to it.
9. In carrying out this experiment, if the penny bounces off the grid, do not count the trial. If a soft pad is placed under the grid, the penny will not be as likely to bounce off it. In order for the penny not to touch any of the lines of the 3 cm grid, the centre of the penny must come to rest inside a circle at the centre of the square whose radius is about 0.6 cm (the shaded region of the figure below).



The radius of the penny is about 0.9 cm and the distance from the centre of the square to an edge is 1.5 cm. The ratio of the area of the shaded circular region to the area of the 3 cm square is the probability that the penny will land inside the lines of the grid. This probability is

$$\frac{(0.6^2)\ \pi}{3^2} = \frac{0.36\ \pi}{9} \approx 0.13.$$

Assignments
Minimum 1–4.
Average Odd-numbered exercises.
Maximum 1, 3, 5, 6, 7, 8, 9.

Objective

Given a sample of a large set, the student can make an estimate of the whole set.

PREPARATION

Materials: Tracing paper.

Review estimation by asking students to estimate, for example, the number of tiles on the floor, the number of acoustical tiles in the ceiling, or the number of square centimetres needed to cover their desk top. After discussing how such estimates are made, direct the students to the Investigation for another estimation problem.

INVESTIGATION

Students should record their estimates of the number of stars on a piece of paper. Some students will immediately want to check their estimates by direct counting, but the discussion will be better motivated if the exact number of stars remains unknown at this time.

DISCUSSION

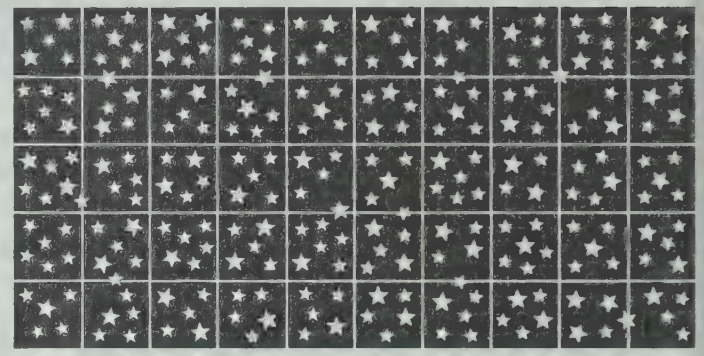
Emphasize that the stars inside one of the small squares constitute a *sample* of the data as a whole and that we can use the number of stars in one square centimetre to estimate the total number of stars. Since the estimate will depend upon which square is used as a sample, it should be noted that the number of stars in the squares vary from as few as 4 to as many as 7, depending on how the borderline stars are counted. It appears that most squares have 5 to 6 stars, so there should be between 250 and 300 stars.

If students wish to know the exact number of stars, supply tracing paper so that they can cover the picture with it and mark and count the stars. (There are 260 stars in the picture.)

Sampling

Investigating the Ideas

Suppose an astronomer made this photograph of a small portion of the sky and placed a 5 by 10 centimetre grid over the photograph.



Can you estimate the number of stars shown in the photograph?

Discussing the Ideas

1. What was your estimate of the number of stars in the photograph? Explain how you made your estimate.

2. **A** Look at one of the centimetre squares on the photograph. How many stars are in one square?
   B The stars that are in one of the centimetre squares is a **sample** of all the stars in the photograph. How could you use the number of stars in only one of the squares to help you make an estimate of the total number of stars?

3. **A** Find a centimetre square on the photograph that has as few stars as you can. How many are there?
   B Find a centimetre square that you think has the most stars on it. How many are there?
   C How can these samples help you make an estimate?

4. **A** If someone estimated that there were 100 stars, how would you convince him that his estimate was too small?
   B If someone estimated 1000 stars, how would you convince him that his estimate was too large?

5. Choose as a sample the stars that are in one of the columns of centimetre squares. Estimate the total number of stars. How does this estimate compare with your previous estimates?

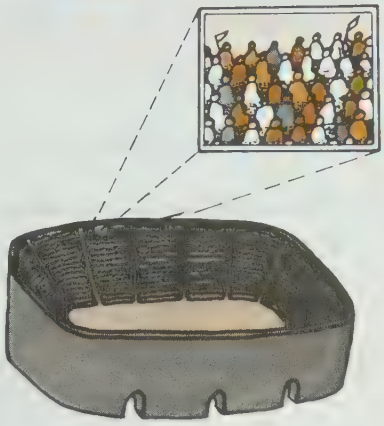
Using the Ideas

1. In order to estimate the number of beans in a plastic bag, Larry counted out a pile of 100 beans. Then he made other piles, without counting, that were about the same size as the pile of 100 beans. Altogether he had 16 piles. What was his estimate of the number of beans?

2. Marie used another method of estimating the number of beans. She counted the number of beans that would fill one mug and found that there were 254 beans. Then she emptied the beans from the plastic bag by mugfuls and found that there were $5\frac{1}{2}$ mugs of beans in the bag. What was Marie's estimate for the number of beans?

3. Jack found that in a normal walking gait, the length of his step was about 60 centimetres. Estimate the number of steps that Jack would have to take to walk a distance of one kilometre.

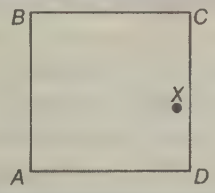
4. A photograph of a group of 64 spectators at a football game showed 20 women and 44 men. Estimate the number of women at the game if the attendance was 58 560 people.



5. Three dice were tossed and the sum of the top faces was recorded. In 216 tosses of the three dice, a sum of 14 was obtained 15 times. Based on this sample, about how many times would a sum of 14 occur in 1000 trials?

think

Trace square *ABCD* and point *X* on another piece of paper and cut out the square region. Make one straight cut through point *X* so that the square region is divided into two pieces that are the same size and shape.



UTILIZATION

Assign the exercises as needed, allowing sufficient time for discussing them afterwards. Have the students identify the sample size of each problem and describe how information about the sample was used in making their estimates.

EXTENSION

Exercises 1 and 2 would make excellent problems for student involvement. Play a "Guess the number of beans in the jar" game. Students can use samples of any size to make their estimate, but should avoid direct counting. You can award a "Champion Guesser" certificate to the student who makes the closest estimate. Of course, someone must make an exact count of the beans in order to determine who is the winner.

Exercises 3 and 5 also lend themselves to active student involvement.

Think Solution

Any segment containing the centre of a square and having end points on the boundary of the square divides the square into two congruent parts. Thus, the segment containing x and the centre of the square will divide it as required.

Assignments

Minimum 1–3. Average 1–4. Maximum 1–5.

Objective

Given a sample of a set, the student can make some inferences about the set.

PREPARATION

Since the Investigation will take somewhat more than the usual time to complete, begin with it immediately.

INVESTIGATION

If feasible, divide the class into four groups and have each group carry out a different one of the surveys suggested in the Investigation. Allow the students to conduct their surveys and collect the information in whatever manner seems easiest to them. Students who conduct survey A, for example, might display the results of their survey on a chart like the one below.

Program	1st Choice	2nd Choice	3rd Choice	4th Choice
A	2	6	2	0
B	1	1	3	5
C	5	2	1	2
D	2	1	4	3

From this information, one could conclude that program C was the first choice of $\frac{1}{2}$ of the students, that program A was the second choice of $\frac{6}{10}$ of the students, and so on.

DISCUSSION

You will want to allow enough time to discuss the results of the surveys in detail. Bring out such points as the fact that the results of the survey depend on the choice of the sample. For example, a survey of television programs should include both boys and girls if it is intended to reflect the preferences of students in general. Similarly, a survey of favorite subjects should include both boys and girls.

In Exercise 4A, students may make such observations as that most students ($\frac{7}{10}$) have either 1 or 2 brothers or sisters and that the number of students having no brothers or sisters is the same as the number having 3 brothers and sisters and the number having 4 brothers and sisters.

Drawing Conclusions from Samples

Investigating the Ideas

Can you choose one of the surveys below and carry it out?

A List four popular television programs. Find out which is the most popular program by surveying ten students.	**B** List the school subjects that your classmates are taking. Find the favorite subject of ten of your classmates.
C Find out what part of the students in your room wear glasses or contact lenses.	**D** Find the number of brothers and sisters for ten of your classmates.

Discussing the Ideas

1. When you conduct a survey or questionnaire of only ten students, you are using a **sample** of all the students. Do you think that the results you get depend upon which students you choose for your sample?

2. If you want to determine the most popular television program, should your sample include both boys and girls? Why?

3. Suppose you found out that about $\frac{1}{10}$ of the students in your room wear glasses or contact lenses. How could you use this information to estimate the number of students in your **school** that wear glasses or contact lenses?

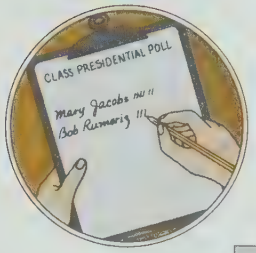
4. Larry carried out investigation D.
   - **A** How would you interpret the results of his sample?
   - **B** What was the total number of brothers and sisters of the ten students sampled?
   - **C** What was the average number of brothers and sisters for the ten students?

Number of brothers or sisters	Student tally
0	|
1	|||
2	||||
3	|
4	|

Using the Ideas

1. **A** Based upon this poll, who should win the class presidential election?

B If 100 students vote, estimate the number of votes Mary and Bob will each receive.



2. In a sample of 25 beads from a box of 1000 red and black beads, 15 were red and 10 were black.

A Estimate the number of red beads in the box.

B Estimate the number of black beads in the box.

red	𝍸 𝍸 𝍸
black	𝍸 𝍸

3. From previous samples taken, a company expects to find that 2 out of every 1000 parts manufactured will not meet manufacturing standards.

A What is the probability that a given part will not meet manufacturing standards?

B If the company manufactures 500 000 parts per year, about how many of these parts will not meet the standards?

4. A sample of voters polled in a province showed these percentages for each of three candidates. If there are 2 124 000 voters in the province, estimate the number of votes each of the candidates might expect to receive.

Candidate	Percentage
A	46%
B	34%
C	20%

5. Using a sample of 200 people in a certain city, it was found that 72% of the people had a television set in their homes. If there are 40 000 households in the city, about how many television sets are there in the city?

6. A light bulb manufacturer expects, from experience with many shipments (samples) of light bulbs, that about $1\frac{1}{2}\%$ of the bulbs shipped will arrive at their destination either defective or broken. About how many defective or broken bulbs would there be in a shipment of 10 000 bulbs?

UTILIZATION

Assign the exercises as independent work for the students. Afterwards, most students will benefit from a discussion of the reasoning behind each problem. In each problem, the basic assumption is that the sample described has the same general properties or proportions as the entire set (population) in question. Thus, in the sample in Exercise 1, 7 out of 10 students state a preference for Mary, so one would conclude that, based on this sample, about 70 out of 100 students would be expected to vote for Mary. Be sure to note that the actual outcome may be quite different because of the smallness of the sample and other factors such as whether the sample is truly representative of the student voters as a whole.

EXTENSION

Workbook page 119 may be assigned for appropriate further practice with ideas presented in this lesson.

Enrichment: Exercise 2 could be used as an experiment involving samples. Mix together 2 different colors of beads or beans. (For example, count out 100 white beans and 400 brown beans and mix them together.) Students then choose a small sample of the beans and count how many of each color are in their sample. On this basis they can predict the proportion of each color of bean; they can also predict the number of each color, if they are given the total number. Use some small measure such as a coffee measure to obtain samples that contain about the same number of beans.

Assignments

Minimum 1–3. Average 1–5. Maximum 1–6.

Objective

The student can demonstrate the ability to work with the concepts presented in this module.

UTILIZATION

The review exercises can be assigned as independent work or can be used as a basis of a discussion reviewing the main ideas of the module.

Additional appropriate review exercises are provided on page 120 of the *Workbook*.

REVIEWING THE IDEAS

1. Which spinner is made to have equally likely outcomes?

A

B

2. Use the letters *H* (heads) and *T* (tails) to construct a sample space for an experiment of tossing a penny and a nickel.

3. A If you toss a penny 50 times, about how many heads would you expect to turn up?

B What is the probability of getting a head?

4. What is the probability of getting a head and a tail if two coins are tossed?

5. What is the probability of getting a number less than 5 if a die is tossed?

6. A dial is divided into five equal sized regions and the arrow is equally likely to stop on any of the regions. (Assume that the arrow does not stop on the lines between the regions.)

A What is the probability that the arrow will stop on 2?

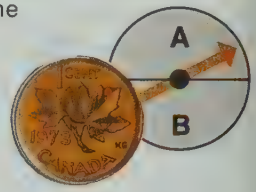
B What is the probability that the arrow will stop on an even-numbered region?

C What is the probability that the arrow will stop on an odd-numbered region?

7. An experiment consists of tossing a penny and spinning the arrow on a spinner, and then recording the outcomes.

A List all possible outcomes in the sample space for the experiment.

B What is the probability of getting the outcomes (tails, B) for the experiment?



8. Refer to the spinner at the right.

A What is the probability of each outcome?

B If the arrow is spun 60 times, about how many times would you expect it to stop on 3?

9. Forty students were asked to name their favorite sport.

Football	𝍸 𝍸 𝍸 ||
Basketball	𝍸 𝍸
Baseball	𝍸
Hockey	𝍸
Tennis	||
Golf	|

A On the basis of this sample, about how many students out of 200 would choose football?

B About how many out of 200 would choose basketball?

C About how many out of 200 would choose hockey?

10. In a population of 10 000 000 registered voters, a poll shows that the leading candidate in an election can expect to get 45% of the votes. About how many votes is this?

TEST YOURSELF

1. If a spinner like the one shown is used in a probability experiment, what is the sample space for the experiment?

2. Are the outcomes in the experiment for Exercise 1 equally likely outcomes?

3. Give the probability of each outcome in the sample space for the experiment in Exercise 1.

4. Which outcome is more likely to occur when two pennies are tossed?
   - A Two heads
   - B A head and a tail

5. What is the probability of getting a 7 when you toss one die?

6. Donna and Karen tossed a penny 220 times. Heads came up 117 times. How many more or less times did heads come up than they would expect from probability theory?

7. Three black marbles and two white marbles are placed in a sack. The marbles are to be drawn from the sack without looking.
   - A If a single marble is drawn, what is the probability that it will be black?
   - B If a single marble is drawn, what is the probability that it will be white?

8. The sum of the probabilities of all the outcomes in a sample space is ?.

9. A sample of cars stopping at a toll booth of a bridge showed that 1 out of every 3 cars had only one person in the car. If 24 000 cars cross the bridge each day, about how many cars have only one person?

10. Using a sample of 300 families in a certain city, it was found that 69 of them go camping during the summer. If there are 21 000 families in the city, how many go camping?

RESEARCH PROJECTS

A Find the probability of getting certain combinations of cards in bridge or poker. (See *Mathematics* by David Bergamini and the Editors of Life; New York: Time, Inc., 1963, pp. 126–147 available from GLC Educational Materials and Services, Ltd.)

B Determine the possible outcomes of the sample space for the experiment of throwing three dice. Which sum or sums are most likely to occur when three dice are thrown? Can you find the probability of each outcome?

C What is the probability that two persons will have the same birthday (the same day of the month, not necessarily the same year)? What is the fewest number of people needed in order that the probability of two people having the same birthday is $\frac{1}{2}$? (See *Probability: A First Course* by Frederick Mosteller, Robert E. Rourke, and George B. Thomas: Don Mills, Ontario Addison-Wesley (Canada) Ltd., 1970.)

TEST YOURSELF

Students who have a clear understanding of the material covered in this module should be able to score well on the self-evaluation test. Encourage students who have missed some questions to recheck their work and the related objectives (see student page annotations) and then review the appropriate portions of the module.

RESEARCH PROJECTS

Probability is a topic that lends itself to many research projects. The three suggested in the student text are only samples of projects that may prove of interest.

Although many of the probabilities or odds in cards and other games of chance are difficult to compute since they involve some sophisticated mathematical ideas, students may find it intriguing to discover through Project A just what some of those probabilities are.

Project B could be carried out by a group of students. There are 216 possible outcomes for tossing 3 dice. The sums most likely to occur are 10 and 11, each of which has a probability of $\frac{1}{8}$.

Project C will yield findings that will probably be quite surprising to any students who undertake it. If about 23 people are selected at random, the probability that 2 of the people have the same birthday is about $\frac{1}{2}$. For 30 people the probability of the same birthday is about $\frac{7}{10}$. For 60 or more people, the probability is very close to, but less than, 1.

MATHEMATICAL RECREATION

This mathematical recreation is somewhat similar to Exercise 9 on page D-37. The grid should be made as large as possible. If a soft mat is placed under the grid, the penny will not be as likely to bounce off when it is dropped on the grid.

To show how π is derived from this experiment, the following argument can be made.

Let r be the radius of the penny; then the area of the penny is πr^2. This is the area of the inscribed circle (labelled C in the figure). The side of the red square (C) is $2r$, so its area is $4r^2$. The probability that the coin hits on an intersection is

$$\frac{\pi r^2}{4r^2} \text{ or } \frac{\pi}{4}.$$

The experimental probability, $\frac{\text{hits}}{\text{tosses}}$, should be very near to $\frac{\pi}{4}$ for a large number of tosses.

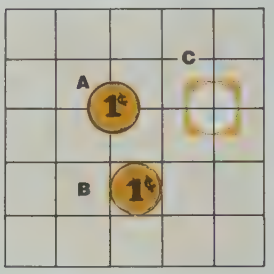
Hence, $\frac{\text{hits}}{\text{tosses}} \approx \frac{\pi}{4}$ and $\frac{4 \times \text{hits}}{\text{tosses}} \approx \pi$.

MATHEMATICAL RECREATIONS

Finding a Value for π by Probability

The number π is the ratio of the circumference of any circle to its diameter and it about 3.14159 . . . . The method below uses a probability experiment to find an approximate value for π.

A Draw a grid of squares on a large sheet of paper, making the squares 1.8 centimetres on each side (the same as the diameter of a penny). If you drop a penny on the grid, it will either cover an intersection point on the grid as in Example A, or it will miss as in Example B, but the outcomes are not equally likely.



B Drop a penny on the grid 100 times and keep a record of hits and misses on the intersection points.

Hits	𝍸 𝍸 ||
Misses	𝍸 |

C Multiply the number of hits in 100 tosses by 4 and then divide this answer by the number of tosses (100). Your answer should be a rough approximation for π ($\pi = 3.14159$ . . .).

Figure C may suggest why you get an approximation for π as a result of this experiment. If the centre of the coin falls inside the white circle, the coin will touch the intersection point. If the centre falls inside the colored portion of the square, it will not touch the intersection point. Find the ratio of the area of the circle to the area of the square.

MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. {A, B, C} **2.** No **3.** A: $\frac{1}{4}$; B: $\frac{1}{4}$; C: $\frac{1}{2}$ **4.** B **5.** $\frac{1}{6}$
6. Heads came up 7 more times **7.** A $\frac{3}{5}$; B $\frac{2}{5}$ **8.** 1
9. 8 000 **10.** 4830

UNIT D: INTEGERS, PROBABILITY, AND STATISTICS

Module 3: Statistics

General Objectives

To introduce some fundamental ideas of statistics and statistical data.
To provide experience in interpretation of statistical data presented by bar graphs, line-segment graphs, circle graphs, and picture graphs.
To review the idea of arithmetic mean and to provide applications of the concept.
To introduce the idea of the median of a set of numbers.
To provide experience with frequency distributions and the mode (or modes) of a set of numbers.

	Performance Objectives	Pupil Text	Reteach-Reinforce	Related Activities
138	**RED** The student can arrange and interpret numerical data in a table.	D-46, 47	**WB** 121	**SWM 2** 237
139	**RED** Given a bar graph, the student can interpret the information shown by the graph.	D-48, 49	**WB** 122 **DM** 79	**SWM 1** 55, 239
140	**RED** Given a line-segment graph, the student can interpret the information shown by the graph.	D-50, 51	**WB** 123 **DM** 80	**SWM 2** 241, 258, 259
141	**RED** The student can interpret the information given in a picture graph.	D-52	**WB** 124	**SWM 2** 244, 245
142	**RED** Given a circle graph, the student can interpret data illustrated by the graph.	D-53	**WB** 124	
143	**RED** Given a set of numbers, the student can compute the arithmetic mean.	D-54, 55	**WB** 125	**ASC** W-13
144	**YELLOW** The student can find the median of a set of numbers.	D-56, 57	**WB** 126	
145	**YELLOW** The student can find the mode(s) of a set of numbers.	D-58, 59	**WB** 127 **DM** 81	**SWM 2** 242, 243
	Reviewing the Ideas	D-60	**WB** 128	

MATHEMATICS

Descriptive statistics is concerned with ways of organizing and presenting numerical data in a way that will facilitate interpretation of the data. In this module, statistical data is presented in the form of tables, bar graphs, line-segment graphs, picture graphs (also called pictographs), and circle graphs.

Closely associated with sets of numerical data are certain statistics that are called *measures of central tendency*. In this module, the three measures of central tendency that are explored are *mean*, *median*, and *mode*.

The *arithmetic mean* (often called the *average*) of a set of numbers is the sum of all the numbers in the set divided by the number of numbers in the set. This is equivalent to replacing each number in the set by the arithmetic mean without changing the sum of the numbers in the set.

The *median* of a set of numbers is the middle number of the set when the numbers are arranged in order of size. If there is no single middle number, the median is the arithmetic mean of the two middle numbers.

The *mode* of a set of numbers is the most frequently occurring number (or numbers) in the set. A set of numbers can have more than one mode. If, however, no number in a set occurs more than once, the set has no mode.

TEACHING THE MODULE

Materials

Reference books, almanacs, encyclopedias, graph paper, newspapers, pairs of dice.

Vocabulary

arithmetic mean
average
bar graph
circle graph
data
frequency
frequency distribution graph
line-segment graph
mean
median
mode(s)
picture graph (pictograph)
statistics

The basic objective of this module is to introduce the student to some basic ideas of descriptive statistics. The first lesson deals with collecting and interpreting data. This is followed by several lessons dealing with graphical data. Bar graphs, line-segment graphs, picture graphs, and circle graphs are introduced. This is followed by some lessons on measures of central tendency—mean, median, and mode(s).

Most students will have had some experience with graphical data, but not in so concentrated a form as in this module. A majority of students will probably have computed arithmetic means, but the work with median, modes, and frequency distribution graphs will probably be new to them.

The main effort with graphical work should be directed toward analyzing and interpreting graphical data. Less emphasis should be placed on the construction of graphs.

Lesson Schedule

The amount of time you spend on this module will depend, largely, on the amount of emphasis you give to the material. A minimum coverage of the material should require about 6 days. However, if you elect to carry out many of the suggestions for extension or have students engage in other special projects, you may need from 8 to 10 days to complete the module.

Evaluation

Your evaluation should focus on understanding and interpretation of graphical data rather than on students' ability to construct various graphs based on statistical data. While there is some value in knowing how to construct graphs, it is far more important that students be able to interpret data presented by a graph.

The self-evaluation test at the end of the module can serve as a model for a module achievement test, or you might administer the achievement test which is available from the publisher as part of the evaluation program for this series.

RESOURCES FOR ACTIVE LEARNING

Activities

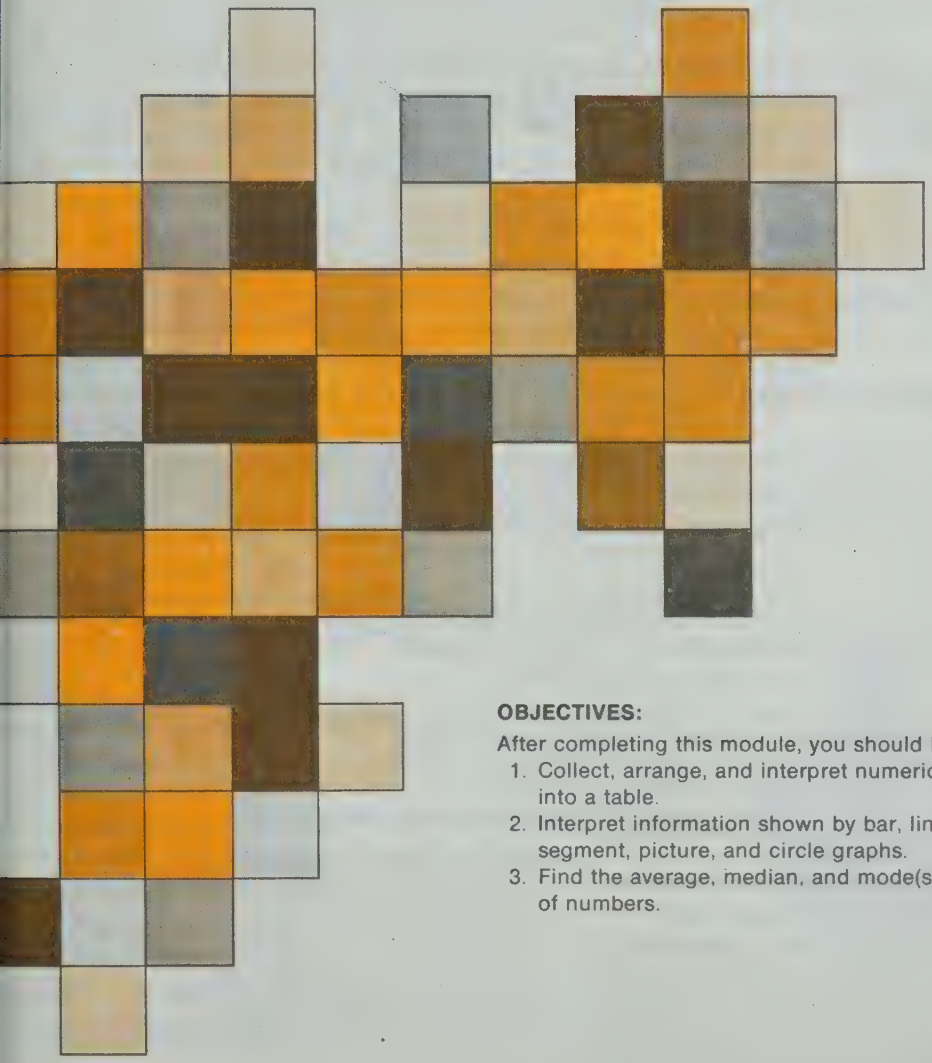
Developmental Math Cards, L7, Addison-Wesley
ESS: Pendulums, Webster, McGraw-Hill
Experiences in Mathematical Ideas, Vol. 2, "Graphs" and "Organizing Data," pp. 96–173, NCTM
Experiments in Mathematics, Stage 3, pp. 16–19, Houghton Mifflin
Inquiry in Mathematics via the Geoboard, "Linear graphs," Card sets 25 and 26, Walker
Nuffield Project: *Probability and Statistics; Speed and Gradient 1,* Wiley
Statistics, Cuisenaire Co.
Statistics by Example, Addison-Wesley

Manipulative Devices

Hexstat (Gamco; LaPine)
Probability and Statistics Lab (Gamco; LaPine)

UNIT D: *Integers, Probability, and Statistics*

MODULE 3: *Statistics*



OBJECTIVES:

After completing this module, you should be able to:

1. Collect, arrange, and interpret numerical data into a table.
2. Interpret information shown by bar, line-segment, picture, and circle graphs.
3. Find the average, median, and mode(s) of a set of numbers.

Objective

The student can collect, arrange, and interpret data about a given topic.

PREPARATION

Materials: Almanacs, encyclopedias, or other reference books.

Display some examples of statistical data taken from newspapers, magazines, or books. Show various tables and graphs and point out that we often need to be able to read and interpret such data.

INVESTIGATION

Have the students study the Investigation and select which data they wish to try to find. You may prefer to divide the class into groups and assign a particular problem to each group. You might suggest that some students find information for both problems. If possible, have several different sources of information available for the students to use. Some variation in answers will probably result. Use your school library if reference books are not available in your classroom.

DISCUSSION

In the Discussion, bring out the fact that in collecting numerical data about an object or topic, somewhat different data may be found by two people using different sources of information. Such discrepancies may result from use of different rounding procedures in the presentation of numerical data, or especially in the case of population figures may reflect differences in the date the data were gathered.

In arranging numerical data, students should use some systematic scheme such as increasing or decreasing order.

Investigating the Ideas

You may often find it necessary to use many different sources in order to gather information about a problem. Consider the following questions:

1. What is the population of each of the five largest cities in your province?

2. What are the record high and low temperatures for your city or town and for four other cities?

Can you find some information about the questions above?	Record your information.	

Discussing the Ideas

1. The science of analyzing numerical information is called **statistics**. Could two persons find different information when answering the questions above? Explain.

2. Facts about an object or a topic are called **data**. Judy copied the following data from a reference book in order to answer question 2 above.

   Calgary 36°C, $^{-}$45°C London 41°C, $^{-}$33°C Montreal 36°C, $^{-}$34°C
   St. John's 34°C, $^{-}$29°C Victoria 35°C, $^{-}$19°C

   What are some ways in which Judy could have arranged the data in order to make it easier to read and interpret?

3. Frank arranged his material for question 1 in a table. Would you have arranged it differently? In what way?

CITY	POPULATION
"A"	452 524
"B"	750 879
"C"	275 425
"D"	383 818
"E"	540 025

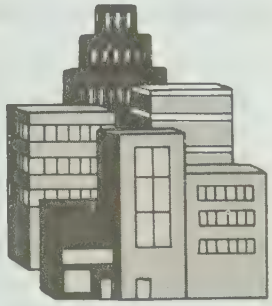
4. After finding information for the Investigation questions above, arrange the data in a manner so that it would be easy for others to understand. What are some things that you should consider when arranging your data?

Using the Ideas

1. The melting point temperatures of some metals in °C are as follows: gold 1063°C, iron 1535°C, aluminum 649°C, copper 1083°C, lead 328°C, nickel 1453°C, tungsten 3410°C, silver 960°C. Arrange the data in a table.
   - A Which metal has the highest melting point?
   - B Which metal has the lowest melting point?
   - C Which metal has a melting point temperature about twice as high as lead?
   - D Which two metals have the closest melting-point temperatures?

2. Annual sales in dollars for some large international companies in 1972 were Exxon Corporation 22 billion, Royal Dutch Shell 14 billion, Unilever 8.9 billion, Volkswagenwerk 5.4 billion, General Motors 28.3 billion, Hitachi 4.4 billion and Nestle 4.1 billion. Arrange the data in a table.
   - A Which company had the largest amount of sales?
   - B Which company had about half of the amount of sales as Unilever?
   - C Which company had about four times as many sales as Volkswagenwerk?
   - D Which company had sales that were about seven billion less than the sales of the Exxon Corporation?

3. Find the population of the five largest cities in the world. Arrange this information in a table.



4. Collect and arrange information about these topics.
   - A The names and heights of the six tallest buildings in the world.
   - B The names and areas of the five largest fresh water lakes.
   - C The names and heights of the five highest mountains in the world.

5. Find the names of all the major league baseball players who have hit more than 50 home runs in one season, the number of home runs that they hit, and year in which they were hit.

think

Is this a **magic square**?

4356	3267	8712
9801	5445	1089
2178	7623	6534

UTILIZATION

Exercises 1 and 2 present numerical data which the students must arrange and interpret.

Exercises 3, 4, and 5 require that data be found in reference sources. Students should be able to use the school library, or reference books should be provided, if these problems are assigned. Some variation in answers may be expected on some problems depending on the sources and the dates on which the information was gathered and published. Have students display the data collected for these exercises and discuss the possible reasons for any variations in data.

EXTENSION

Additional practice exercises are provided on page 121 of the *Workbook*.

Think Solution

The square is a magic square with a sum of 16335. This particular square has some other interesting properties. Considering only the digits in the ones' place of each numeral, a magic square with a sum of 15 is formed. Considering only the digits in the tens' place of each numeral a magic square whose sum is 12 is formed. The digits in the hundreds' place and the digits in the thousands' place also form magic squares.

Assignments

Minimum 1, 2. Average 1–3. Maximum 1–5.

Objectives

The student can interpret the data presented by a bar graph.

Given appropriate data, the student can depict the data by means of a bar graph.

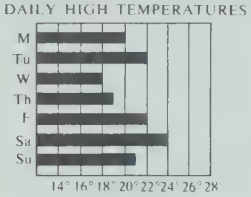
PREPARATION

Materials: Examples of bar graphs from magazines, newspapers, or other sources; graph paper (*Duplicator Masters*, page 83).

To introduce the idea of interpreting data from a bar graph, you or your students might bring to class and discuss examples of bar graphs from newspapers or magazines.

INVESTIGATION

Have the students study the information in the table and the bar graph. If you prefer, you may discuss Exercises 1 and 2 of the Discussion section before having the students construct their graphs of the daily high temperatures.



DISCUSSION

In your discussion of the graphs, bring out the fact that bar graphs depict data in a way that makes it easy for us to compare quantities visually. In discussing Exercise 2, point out that data is often rounded in order to show it on a graph.

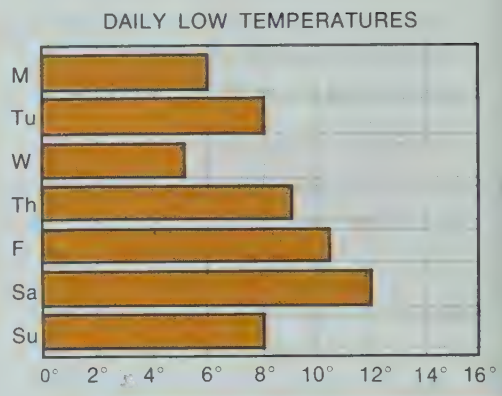
Exercises 3 and 4 illustrate how graphs of statistical data can be distorted. In the graph in Exercise 3, the bar for Fred was placed beside the bar for Bill, thus making Bill's allowance look very small. Note also that the scale starts with $1.00. The graph in Exercise 4 is drawn so that it appears that each boy gets very nearly the same allowance, although Bill's allowance is still the least. In both graphs, the scales were adjusted so as to emphasize certain aspects of the data.

Bar Graphs

Investigating the Ideas

Using the information in the table, a student made a bar graph to show the daily low temperatures for each day in a certain week.

HIGH AND LOW TEMPERATURES (°C)		
Day	Low	High
Monday	6°	20°
Tuesday	8°	22°
Wednesday	5°	18°
Thursday	9°	19°
Friday	10°	22°
Saturday	12°	24°
Sunday	8°	21°



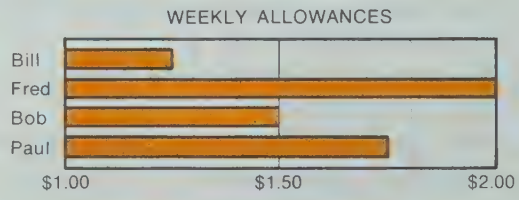
Can you make a bar graph that will show the daily high temperatures for the week?

Discussing the Ideas

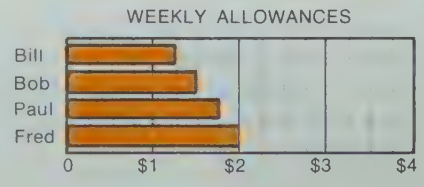
1. If you wanted to compare daily low temperatures, which would you find easier to use, the table of data or the bar graph? Explain your answer.

2. Does the bar graph show the data as accurately as the table?

3. Bill drew this bar graph to show how his weekly allowance compared with three of his friends. Estimate each boy's allowance.

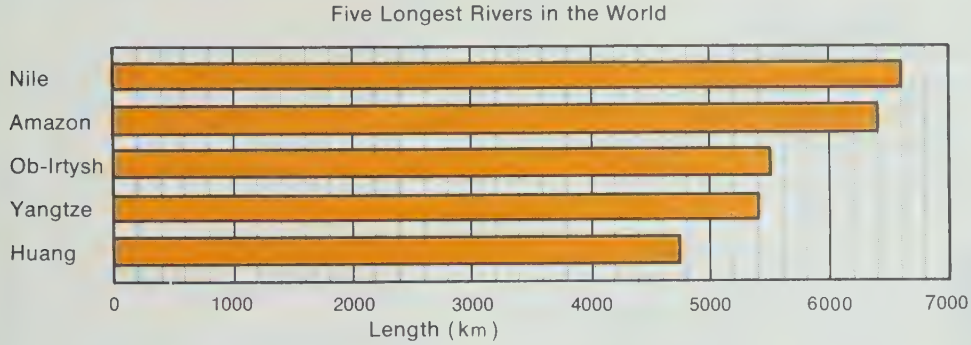


4. Bill's father said that Bill's graph was misleading. He drew a graph like this one. Which graph do you think gives the information best? Why?



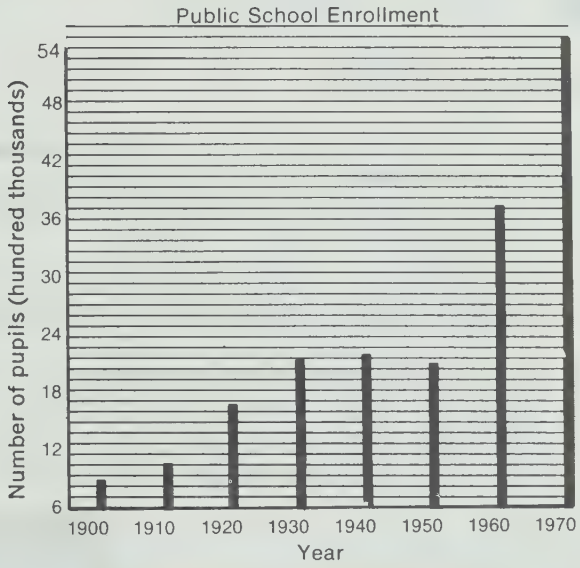
Using the Ideas

1. Estimate the length of each river to the nearest hundred kilometres.



2. The bar graph at the right shows public school enrollment in Canada.

- **A** Give the approximate school enrollment for each year shown on the graph.
- **B** How many more pupils were enrolled in 1970 than in 1900?
- **C** In which 10-year period did school enrollment grow most rapidly? Give the increase.
- **D** During which years did it decrease?



3. Construct a bar graph giving the approximate population of Canada by decades from 1900 to 1970.

Year	Approximate population (in 100 000)
1900	54
1910	72
1920	87
1930	104
1940	115
1950	141
1960	183
1970	217

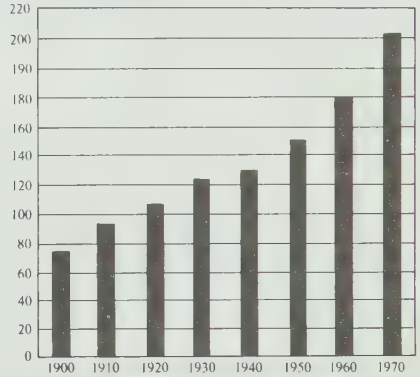
4. Using the graph that you constructed for Exercise 3, make an estimate of the population of Canada for 1980 and 2000.

UTILIZATION

Exercises 1 and 2 require reading and interpreting the data presented in the graphs. Exercise 3 requires construction of a bar graph, which may have either horizontal or vertical bars.

In Exercise 4 the students are asked to estimate the U.S. population for 1980 and 2000. This process of projecting data beyond that which is given is known as *extrapolation*. You may wish to spend some time discussing the usefulness of extrapolation as well as its inherent dangers.

Answer, Exercise 3



EXTENSION

Workbook page 122 and *Duplicator Masters* page 79 contain exercises suitable for additional practice assignments.

Remedial: Collect several different bar graphs from magazines and newspapers and have students estimate the numerical data represented by each bar.

Enrichment: Have students choose a set of numerical data from some reference source and depict it by means of a bar graph.

Assignments

Minimum 1–2. Average 1–3. Maximum 1–4.

Objectives

Given a line-segment graph, the student can interpret the data shown by the graph.

Given appropriate data, the student can construct a line-segment graph depicting the data.

PREPARATION

Materials: Hourly temperatures for one day, graph paper (*Duplicator Masters*, page 83).

To save time and provide additional motivation, you might ask students to obtain the hourly temperature data before this lesson is presented. This might be done in conjunction with the previous lesson on bar graphs of temperatures. Many daily newspapers carry hourly temperatures, and this data could be used by students if you prefer.

INVESTIGATION

If students do not have their own sets of daily temperatures, write a set of temperatures for them to use. Supply graph paper as needed. You may find it beneficial to discuss Exercises 1, 2, and 3 before having the students draw their graphs for the Investigation.

DISCUSSION

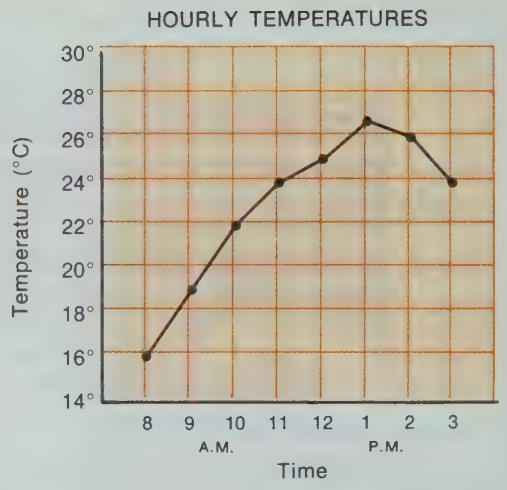
Discuss Exercises 1 and 2 in relation to the graph shown in the Investigation and the graph that the students have drawn. When discussing Exercise 1B, you should point out that the temperature at 10:30 a.m. appears to be 23°C. Drawing a conclusion such as this from a graph is called *interpolation*. We assume that the temperature between the measured intervals rises or falls along a straight line.

Note that negative numbers occur in the data presented in Exercise 3. Discuss how the scale for a graph depicting such data would be made. Encourage students actually to construct a line-segment graph of this data.

Line-segment Graphs

Investigating the Ideas

Pam kept a record of hourly temperatures for a certain school day. Then she made a graph like the one at the right in order to show her data.



Can you find the hourly temperatures for one day and draw a graph like this one to show your data?

Discussing the Ideas

1. Graphs like the one Pam made are **line-segment graphs**. Points are marked corresponding to the time and temperatures, and then successive points are joined with line segments.
   - A Read the temperature for each hour shown on the graph.
   - B Can you estimate the temperature at 10:30 using the graph?

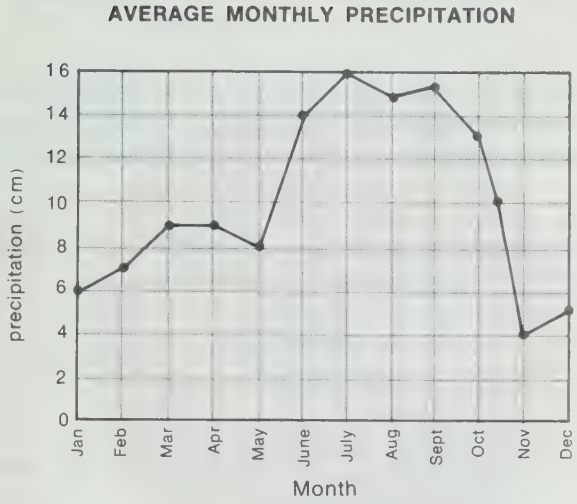
2. A What is the difference between the highest and the lowest temperatures shown on the graph?
   - B How can this difference help you to decide what scale that you should use on the temperature axis?

3. How would you construct a line-segment graph for the temperatures shown in the table?

Hour	Temperature (°C)
8:00	−10°
9:00	−4°
10:00	−1°
11:00	0°
12:00	2°
1:00	4°
2:00	1°
3:00	−2°

Using the Ideas

Refer to the precipitation graph for Exercises 1, 2, and 3.

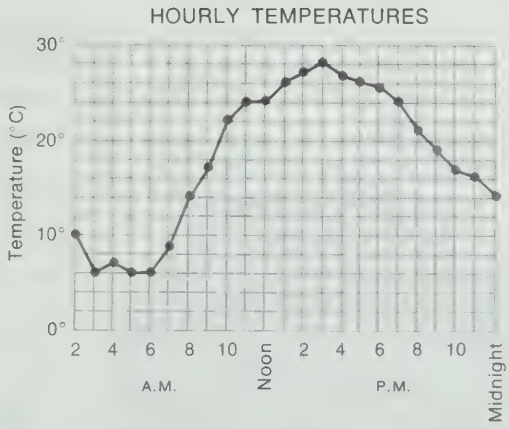


1. Give the approximate average rainfall for each month to the nearest centimetre.

2. In which months is the average rainfall
   - A less than 6 centimetres ?
   - B more than 12 centimetres ?

3. Find the approximate annual rainfall for the area graphed.

4. Refer to the hourly temperature graph at the right.
   - A What is the highest temperature on the graph? When did it occur?
   - B What is the lowest temperature? When did it occur?
   - C How much did the temperature rise between 6:00 A.M. and noon?



5. The average daily high temperatures for the month of July for a certain locality is shown in the table. Construct a line-segment graph to show the data.

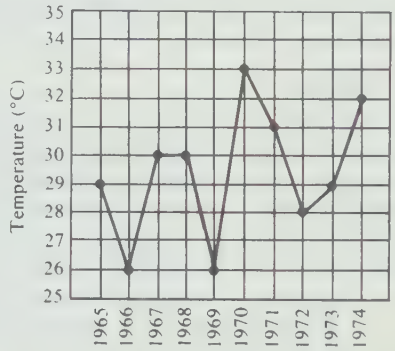
Year	Average daily high (°C)	Year	Average daily high (°C)
1965	29°	1970	33°
1966	26°	1971	31°
1967	30°	1972	28°
1968	30°	1973	29°
1969	26°	1974	32°

UTILIZATION

Exercises 1 through 4 require only reading and interpretation of the graphs. Therefore, you may prefer to handle these exercises orally and assign Exercise 5 as independent work. Students will need graph paper for Exercise 5.

Answer, Exercise 5

5.



EXTENSION

To provide further practice with the ideas presented in this lesson, make assignments from *Workbook* page 123 and *Duplicator Masters* page 80.

Enrichment: Have students keep a record of daily high temperatures or daily low temperatures for a week or more. Then they can construct a line-segment graph to depict the data. This could be done as a class project in which students construct a large graph on posterboard. A new segment could be added to the graph each day.

Assignments

Minimum 1–4, oral.
Average 1–4, oral; 5.
Maximum 1–4, oral; 5.

Objective

Given a picture graph, the student can estimate the numerical data depicted by the graph.

PREPARATION

If possible, collect some picture graphs from newspapers or magazines. A brief comparison of these graphs to the kinds of graphs previously studied would be helpful preparation for this lesson.

UTILIZATION

Picture graphs are also called *pictographs* or *pictograms*. Bar graphs can usually portray the same data as a picture graph, but picture graphs are often used when attractivenss of the display is important and when only approximations of the data need be presented. Since picture graphs appear quite frequently in newspapers and other printed material, it is important that students be able to analyze and interpret them correctly. It is less important that they be able to construct picture graphs, especially since such graphs usually require some artistic skill.

In discussing the exercises, call attention to the fact that the partial figures are used to show parts of the basic unit of the graph. Considerable variation should be expected in estimates of numbers represented by the partial figures. For example, in Exercise 1 estimates of the number of English-speaking people may vary from about 320 million to about 350 million. Obviously illogical estimates, of course, should be corrected.

EXTENSION

Enrichment: Have students construct a picture graph of the population growth of Canada since 1900. Refer them to Exercise 3 on page D-49 for the needed data.

Assignments

Minimum 1, 2. Average 1–3. Maximum 1–3.

PICTURE GRAPHS

Picture graphs are often found in newspapers, magazines, and reference books. They can help you to interpret statistical data quickly, but only show *approximate* data.

1. The pictograph at the right shows the approximate number of people that speak each of the languages listed.

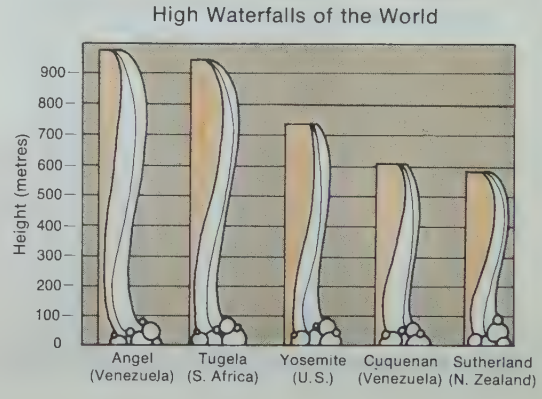
- **A** How many people does each symbol in the graph represent?
- **B** Estimate the number of people (to the nearest ten million) who speak each language.

2. The pictograph at the right shows automobile sales during 5-year periods from 1955 through 1970.

- **A** In which year was there the greatest number of sales?
- **B** In which year was there the least number of sales?
- **C** Estimate the number of sales (to the nearest million) for each year.

3. The graph at the right combines some features of a bar graph with some of a picture graph.

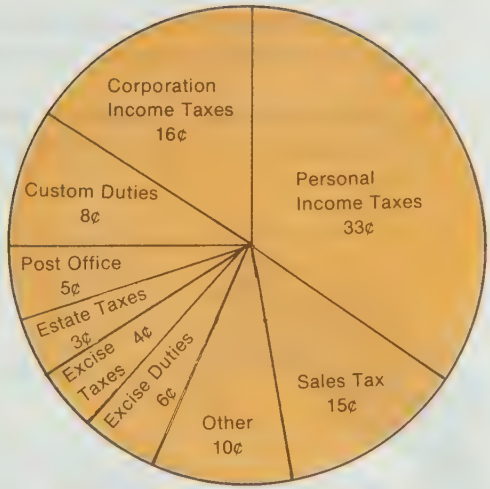
- **A** Estimate the height of each waterfall to the nearest hundred metres.
- **B** How much higher is Angel Falls than Yosemite Falls?
- **C** How much higher is Tugela Falls than Cuquenán Falls?



Circle Graphs

Circle graphs are often used to show how a given quantity is divided into fractional parts.

The graph at the right shows the sources and part of each dollar of government income. For example, 16% of the income (16¢ out of each dollar) comes from corporation income taxes.

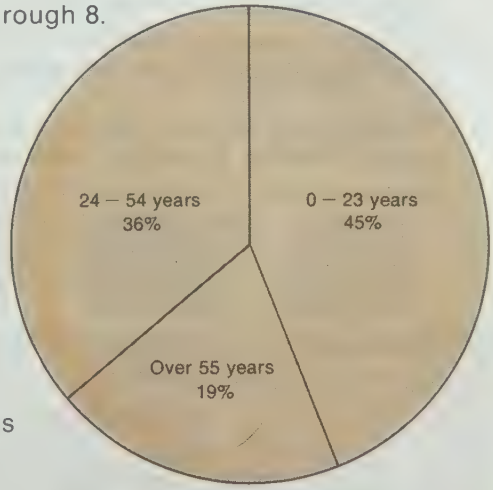


Government Income 1971–1972

1. From where does the government receive the largest part of its income?
2. What item accounts for one-third of government income?
3. The estimated total government income for 1971–1972 was $14 226 600 000. If 33¢ of each dollar came from individual income taxes, what is the total amount of income taxes?
4. What is the amount of income received from sales taxes? from other excise taxes?

Use the population graph for Exercises 5 through 8.

5. In what age range is the largest portion of the population?
6. The population of Canada in 1971 was approximately 21 569 000 people.
   - A About how many people were over 55 years old?
   - B About how many people were under 24 years old?
7. About 9% of the population is between the ages of 15 and 19. How many persons is that?
8. About $6\frac{1}{2}$% of the population is over 70 years old. About how many persons are over 70 years old?



Population Distribution by Age (1971)

Objective

The student can interpret the data presented in a circle graph.

PREPARATION

A brief review of finding a percent of a number will be helpful for students as preparation for the work on these exercises.

UTILIZATION

Circle graphs show how a total amount is divided into parts. Point out that the sum of all the parts must be 100% of the total amount.

In this lesson no instruction is given on the construction of circle graphs. The emphasis is on understanding and correctly interpreting the data presented by the graph rather than on construction of the graphs.

Assign the exercises as independent work. After the students have completed the work, have some of them place their solutions on the chalkboard and discuss their method of solution.

EXTENSION

For suitable further practice exercises, refer to page 124 of the *Workbook*.

Enrichment: Students might be challenged to make a circle graph that shows how they spend their time during each day (24 hours). Included in their graph should be hours spent in school, recreation, sleeping, eating, and other activities. You may need to give students some guidance on how to divide the circular regions into sectors to show the various amounts of time devoted to each activity.

Assignments

Minimum 1–4. Average 1–7. Maximum 1–8.

Objective

Given a list of numbers, the student can compute the arithmetic mean of the numbers.

PREPARATION

Materials: Newspapers or magazines.

Each student will need a short paragraph from a newspaper or magazine to carry out the Investigation. Have these materials available for the students to use.

INVESTIGATION

The students should study the example in the Investigation, then try to find the average length of the words in the paragraph they select. It might be of interest for the students to identify the section of the paper from which they choose their paragraph. This could engender a discussion of whether the average length of the words from the sports section, for example, is greater or less than that of the words from the front page or the editorial page, and so on.

You may find it necessary to agree that any numeral in a paragraph should be counted as a word and that each digit in the numeral should be counted as a letter.

DISCUSSION

Since most students will already be familiar with the idea of the average or arithmetic mean of a set of numbers, the material will be a review for most. When discussing the flow chart method for finding the mean in Exercise 2, emphasize that this method is equivalent to replacing each number in the set by a single number so that the sum remains the same. Thus,

$35 + 47 + 83 + 69 + 71 = 305$ and
$61 + 61 + 61 + 61 + 61 = 305$.

Therefore 61 is the arithmetic mean of the numbers in the set.

In discussing Exercise 3, students should observe that the arithmetic mean of the set might be any number between 43 and 129.

Finding Averages

Investigating the Ideas

There are 54 words in this particular newspaper clipping made up of 277 letters.

The average number of letters in each word is about 5.1 (to the nearest tenth).

$$
\begin{array}{r}
5.12 \\
54\overline{)277.00} \\
270 \\
\hline
70 \\
54 \\
\hline
160 \\
108 \\
\hline
52
\end{array}
$$

The prime rate is the minimum interest a bank charges its top corporate customers. Floating primes are adjusted more or less automatically in line with other money market rates, while fixed rates are adjusted administratively.

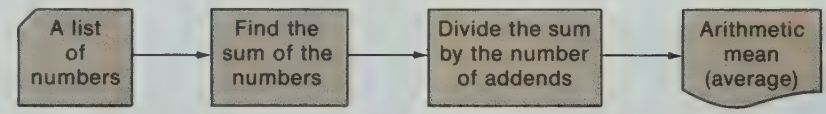
Short-term money market rates have been firming in recent weeks, paving the way for these prime rate boosts.

Can you select a newspaper clipping containing less than 100 words and find the average length of each word?

Discussing the Ideas

1. A What was the average number of letters per word of your newspaper clipping?
   B Compare the average that you found with that of another classmate. Are the averages about the same?

2. The **average** of a set of numbers is often called the **arithmetic mean** of the set. The flow chart below may help you review how to compute arithmetic means.



   Use the flow chart to help you find the arithmetic mean of the numbers in this set:

   {35, 47, 83, 69, 71}

3. In a long list of numbers, the smallest number is 43 and the largest number is 129. Without any other information, what could you say about the size of the arithmetic mean of this set of numbers?

Using the Ideas

1. Find the arithmetic mean of each set of numbers to the nearest tenth.
   - A {235, 279, 303, 369, 346}
   - B {1, 4, 6, 10, 15, 20, 26, 39, 46}
   - C {16.4, 23.9, 30.1}
   - D {1.2, 0.8, 2.4, 3.7, 1.6, 3.8}

2. What is the arithmetic mean of $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{5}{6}$?

3. Bob found that the ages of six of his teachers at school were 26, 29, 59, 38, 48, and 34. What is the arithmetic mean of their ages?

4. Larry scored 83, 78, 92, and 87 on four mathematics tests. What was his average score?

5. A person has an average salary of $193 per week. What is his yearly salary?

6. Beth has 21¢, Mary has 34¢, Diane has 15¢, and Lisa has 44¢. If they put their money together and divide it equally among themselves, how much would each girl have?

7. Four starting players on a professional football team weigh 92, 100, 89, and 95 kilograms. What is the average weight of the men?

8. The arithmetic mean of a list of 32 numbers is 66.4. What is the sum of all the numbers in this list?

★ 9. Suppose the average income for four people is $100 000 per year. Three of the men earn more than $10 000 per year. What is the maximum amount of income the fourth man can earn?

Think

The answers to a short True–False quiz are shown for three students. If one of the students got all three of the problems correct, and no student missed all three, find each student's score.

Arnold	Barbara	Chris
1. True	1. True	1. false
2. False	2. True	2. True
3. True	3. False	3. false

More practice, page S-31, Set 55

UTILIZATION

Assign the exercises as independent work for the students. Allow time for discussion when the students have completed the exercises.

Exercise 9 illustrates the effect that one relatively large number can have on the arithmetic mean of a set of numbers in which the remaining numbers are relatively small. If three of the men earn no more than $10 000 per year, then the fourth man must earn a minimum of $370 000 per year in order that the four men average $100 000 per year. Thus, an "average" of $100 000 per year is not truly descriptive of the earnings of the four men.

EXTENSION

To provide further practice with the ideas presented in this lesson, make selective assignments from Supplementary Exercise Set 55 on page S-31, *Workbook* page 125, and *Arithmetic Skill Card* W-13.

Enrichment: Divide the students into 3 or 4 groups. Have each group work together and find the average height or weight of the students in that group. Then have them find which student in the group is closest to "average" on the basis of the data found.

Think Solution

If Arnold had answered all three problems correctly, then Chris would have missed all three and vice versa. Hence, Barbara must have answered all three problems correctly. Therefore, Arnold has one right answer and Chris has two right answers.

Assignments

Minimum 1–4. Average 1–7. Maximum 1–9.

Objective

Given a list of numbers, the student can find the median of the numbers.

PREPARATION

Discuss with the students the fact that *averages* or *arithmetic means* are called *measures of central tendency* in statistics. The mean of a set of numbers is one such measure. Then point out that another measure of central tendency is introduced in this lesson.

INVESTIGATION

Allow the students to use any technique they wish in trying to find which student had the middle score. Some students may list the scores in order of size and then find the middle score. Others may estimate the middle score and then check their estimate. For example if their estimate was 83, then by counting the scores they would find 15 scores above 83. Since there are 27 scores in all they would know that the middle score must be somewhat higher. Further estimation and counting would reveal 85 to be the middle score.

DISCUSSION

The median of a set of numbers is the middle number of the set when the numbers are arranged in order of size, or the arithmetic mean of the two numbers nearest the middle in cases where there is an even number of items in the list.

The flow chart in Exercise 2 gives the steps in finding the median of a list of numbers. When discussing Exercise 3, consider examples such as the following:

$$A = \{1, 2, 3, 4, 5\} \text{ and}$$
$$B = \{1, 1, 2, 2, 99\}$$

In set A, the mean and the median are the same number, 3. In set B, the mean is 21 but the median is 2.

Finding Medians

Investigating the Ideas

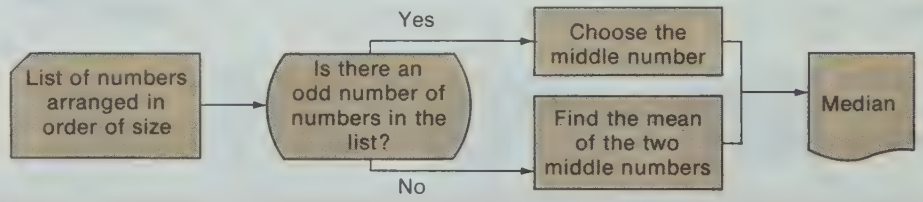
Beth A.	82	Elaine H.	80	Paul P.	77
Art B.	72	Ruth H.	80	Pat R.	86
Jim B.	69	Robert J.	83	Jerry S.	92
Ginny D.	92	Sarah J.	85	Ralph S.	84
Marie D.	96	Max K.	95	Lisa T.	78
Allen E.	88	William L.	93	Diane V.	98
Barbara F.	90	Margaret M.	81	Virginia W.	75
Dave F.	89	Susan N.	76	Jackie W.	87
Ellen G.	79	Eric O.	94	Frank Y.	90

Can you find the student that made this statement?
"Just as many students made lower scores than mine as made higher scores than mine."

Discussing the Ideas

1. The *middle* score in the list above is called the **median** score. How can you arrange the data so that the middle score is easily found?

2. When there is an even number of numbers in a list, there is no single middle number. The flow chart below shows how the median is found in any case.


List of numbers arranged in order of size
Is there an odd number of numbers in the list?
Yes
No
Choose the middle number
Find the mean of the two middle numbers
Median

Use the flow chart to determine the median of each set of numbers.

- **A** {14, 15, 17, 21, 29}
- **B** {22, 24, 25, 27, 31, 33, 34, 34, 37, 41}
- **C** {83, 86, 92, 90, 93, 93, 93, 89, 90, 97, 94, 86}

3. Do you think that the **mean** and the **median** of a set of numbers are usually about the same? Give some examples.

Using the Ideas

1. Find the median for each set of numbers.
   - **A** {27, 38, 46, 49, 52}
   - **B** {12, 12, 18, 26}
   - **C** {4, 7, 7, 8, 9, 13, 13, 14, 15}
   - **D** {274, 329, 351, 376, 382, 390}
   - **E** {19, 12, 36, 27, 41, 13, 33}
   - **F** {9, 2, 5, 3, 8, 7, 4, 6, 1, 10}

2. On a certain test, six students made these scores:
   88, 82, 94, 91, 81, 92
   What was the median score?

3. Barry had the median score of 81 in his class of 33 students. No one else in the class had a score of 81. How many students in the class had scores that were higher than Barry's score?

4. The low temperatures for each day during a week were 8°, 4°, 16°, 3°, 4°, and 15°. Find the arithmetic mean and the median of this set of temperatures.

5. The heights of a group of students are given in the table at the right. What is the median height?

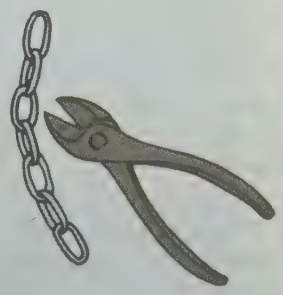
Heights of Students in Metres				
1.48	1.63	1.45	1.80	1.59
1.55	1.50	1.57	1.51	1.52
1.58	1.53	1.55	1.48	1.76
1.50	1.63	1.70	1.52	1.62
1.60	1.65	1.69	1.75	1.56

★ 6. What is the median of this set if the arithmetic mean is 57?

{23, 37, ▮▮▮, 76, 100}
↑ median

think

A traveller wishes to stay at an inn for several days. He has no money, but he has a valuable gold chain. The innkeeper agrees to let him stay on the condition that each day he pay one link of his chain. The traveller must pay every day; that is, at the end of one day, the innkeeper must have one link, at the end of two days, two links, and so forth. The traveller must cut his chain in order to make these payments. If the chain has seven links and the traveller stays for seven days, show how by cutting only one link in all he can pay for his lodging every day.



More practice, page S-32, Set 56

UTILIZATION

Assign the exercises as independent work. Remind the students to use the flow chart in Discussion Exercise 2 as an aid in finding medians. When the list of numbers is extensive, as in Exercise 5, students will find it necessary to rearrange the numbers in order of size to find the median.

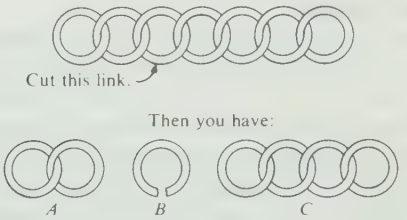
EXTENSION

Appropriate further practice assignments may be made from Supplementary Exercise Set 56 on page S-32 and *Workbook* page 126.

Enrichment: Exercise 5 could be used as an enrichment problem. Students can estimate and then compute the arithmetic mean of the set of numbers.

An interesting class activity would be to determine the median height of the students in the class.

Think Solution



1st day: Give one link (B).
2nd day: Give two links (A); get one back (B).
3rd day: Give one link (B).
4th day: Give four links (C); get three links back (A, B).
5th day: Give one link (B).
6th day: Give two links (A); get one back (B).
7th day: Give one link (B).

Assignments

Minimum 1–3. Average 1–5. Maximum 1–6.

Objective

The student can determine the mode of a given set of data.

PREPARATION

Material: Pairs of dice, graph paper (*Duplicator Masters*, page 83).

Distribute the materials students will need for the Investigation. Students can make their own dice from wooden or cardboard cubes, if not enough commercial dice are available.

INVESTIGATION

The Investigation could be carried out by students individually, but if they work in pairs they can more easily record the outcomes. Their graphs should be patterned after the one shown in the text. The student graphs may vary considerably with each other and with the one shown in the text, depending on the outcomes of their tosses.

DISCUSSION

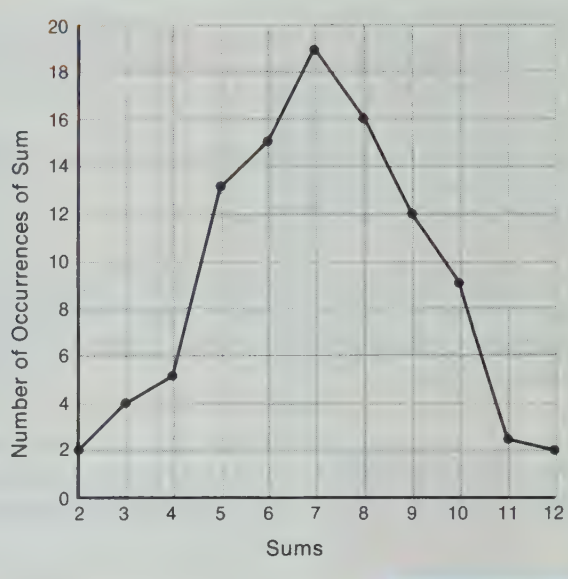
Mode is another measure of central tendency of a set of data and, in probability experiments, provides a more useful description of the data than either the mean or median. In the dice tossing experiment, for example, we are concerned with the number of times each sum occurs and which sum occurs most frequently rather than with the average or median of all the sums tossed.

The Discussion can be extended by collecting the data on the frequencies found by each of the students and combining them. Then a frequency distribution graph for all of the data could be made by the students.

Frequency Distribution – Modes

Investigating the Ideas

Keith and Brian worked together on a probability experiment. They tossed two dice 100 times and kept a tally of the different sums on the tosses. Then they made a graph to show the number of times each sum occurred.



Can you perform the same experiment with a classmate and draw a graph to show your results?

Discussing the Ideas

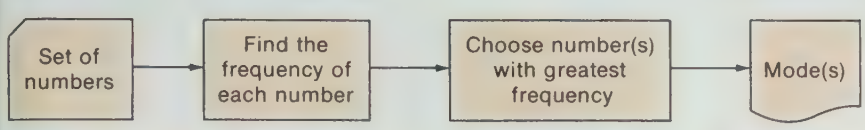
1. A Is the shape of your graph somewhat like the one shown in the Investigation?
   B Which sum occurred most often in your experiment?
   C Is it the same number as the one Keith and Brian found?

2. The number of times a number occurs in a set of data is called the **frequency** of the number. The graph that you made is called a **frequency distribution graph**. What is the frequency of each sum in the graph in the Investigation?

3. In a set of numerical data, the number or numbers with the largest frequency is called **mode** or **modes** of the data. The mode for the data in the Investigation is 7 because a sum of 7 occurred most frequently. What was the mode(s) for your experiment?

Using the Ideas

1. Find the mode(s) of each set of numbers. The flow chart may help you.



A

TEST SCORES		
62	79	85
68	80	87
72	80	88
73	81	88
75	83	88
77	84	88
78	85	92
78	85	94

B

STUDENT HEIGHTS (metres)				
1.58	1.60	1.62	1.64	1.66
1.59	1.60	1.62	1.64	1.68
1.59	1.61	1.63	1.64	1.70
1.60	1.61	1.63	1.65	1.71
1.60	1.62	1.63	1.65	1.72

C

NUMBER OF BROTHERS AND SISTERS				
0	1	1	2	3
0	1	1	2	4
0	1	2	2	4
0	1	2	2	5
1	1	2	3	7

2. Jane and Diane tossed four coins at a time and recorded the number of heads and tails for each toss. Draw a frequency distribution chart for their experiment.

OUTCOME	FREQUENCY
4 heads	5
3 heads, 1 tail	24
2 heads, 2 tails	37
1 head, 3 tails	27
4 tails	7

3. The tally at the right shows the frequency of the last digits of one column of numbers from a telephone directory. What is the mode?

LAST DIGIT	TALLY
0	𝍸 ||
1	𝍸 ||
2	𝍸 |
3	𝍸 𝍸 |
4	𝍸 𝍸 𝍸 ||||
5	𝍸 𝍸 ||||
6	𝍸 𝍸 𝍸 |
7	𝍸 𝍸 |||
8	𝍸 𝍸 |||
9	𝍸 𝍸

4. Using a telephone directory, choose 100 telephone numbers and tally the last digit of the numbers.
   - **A** What is the mode(s) of the last digits?
   - **D** Make a frequency distribution graph of the last digits.

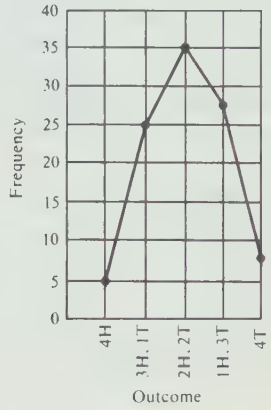
More practice, page S-32, Set 57

UTILIZATION

The flow chart at the top of the page shows the steps in finding the mode or modes of a set of numbers. Students will need graph paper for Exercises 2, 3, and 4.

In Exercise 4 students should try to get a random sample of the numbers by opening the directory at random and choosing one or more of the columns of listings as their sample.

Answer, Exercise 2



EXTENSION

Further practice assignments may be made from Supplementary Exercise Set 57 on page S-32, *Workbook* page 127, and *Duplicator Masters* page 81. This would also be an appropriate time to recommend that some students explore Research Project A on page D-61.

Enrichment: Some students might try Exercise 2 as an experiment and make a frequency distribution graph of the results. The most likely outcome is 2 heads and 2 tails, which has a probability of $\frac{3}{8}$. The probability of 4 heads is $\frac{1}{16}$; the probability of 3 heads and 1 tail is $\frac{1}{4}$; the probability of 1 head and 3 tails is $\frac{1}{4}$; and the probability of 4 tails is $\frac{1}{16}$.

Assignments

Minimum 1, 2. Average 1–3. Maximum 1–4.

Objective

The student can demonstrate the ability to work with the concepts presented in this module.

UTILIZATION

The review exercises can be assigned as individual work or can provide the basis for small- or large-group discussion.

All of the work with graphs in these exercises involves interpretation of the depicted data; the exercises do not call for any constructions, but students could be asked to construct a bar graph showing the data in Exercise 1 if you wish. Also, a frequency distribution graph of the data in the table accompanying Exercise 8 might be assigned as an optional activity for some students.

Workbook page 128 provides additional review exercises for this module.

REVIEWING THE IDEAS

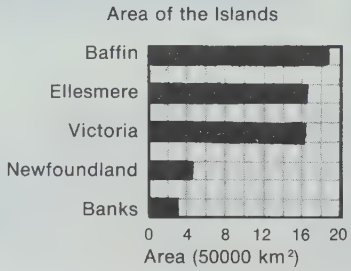
1. Use the table to answer the questions.

MOUNTAIN	HEIGHT (m)
Everest	8848
McKinley	6194
Popocatepetl	5452
Whitney	4418
Pikes Peak	4301

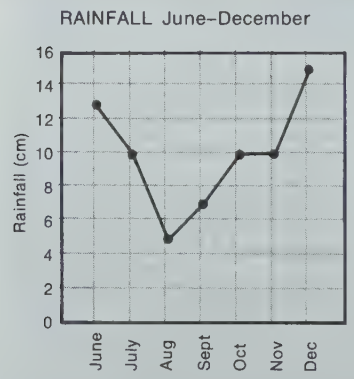
A How much higher is Everest than McKinley?

B Which mountain is closest to half the height of Everest?

2. Estimate the area of each island to the nearest 5000 square kilometres.

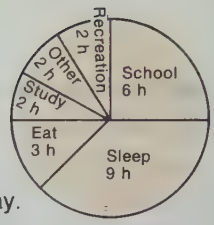


3. Use the line-segment graph to find the amount of rainfall for each month.



4. What was the average rainfall for the June-December period in Exercise 3?

5. Karen made this circle graph to show how she spent her time during a 24-hour day.



A What fractional part of the day was spent at school?

B What part of the day was spent sleeping?

C What part of the day was spent eating and sleeping?

6. Carmen checked the price of a certain article in four supermarkets. The chart shows how the prices compared. To the nearest cent, what was the average cost of the article?

Price (¢)
65
69
73
79

7. Find the median of each list of numbers.

A 54, 14, 29, 38, 68

B 74, 79, 88, 100, 68, 86

8. What is the mode of the test scores in this table?

Score	Frequency	Score	Frequency
75	1	86	7
76	0	87	7
77	1	88	5
78	3	89	6
79	4	90	4
80	3	91	3
81	5	92	2
82	6	93	1
83	7	94	2
84	9	95	1
85	8	96	1

TEST YOURSELF

Use the data in the table below for Exercises 1 through 5.

LAKE	AREA (km^2)
Superior	82 410
Huron	59 600
Michigan	58 020
Erie	25 740
Ontario	19 530

1. Which lake has the largest area?

2. How much larger is Lake Michigan than Lake Erie?

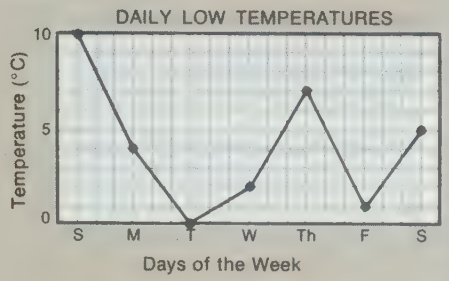
3. What is the total area of the five lakes?

4. What is the arithmetic mean of the areas?

5. What is the median area?

6. TRUE or FALSE:
   The arithmetic mean of any set of numbers is the same as the median of the set of numbers.

7. The line-segment graph shows the daily low temperatures for a week.



   A. What was the lowest temperature?
   B. On which day did the highest of the daily lows occur?

8. The mean of a list of 20 numbers is 16.4. What is the sum of all the numbers in the list?

9. The ages of ten students are 12, 12, 13, 13, 13, 14, 14, 14, 14, 15. What is the mode of their ages?

10. On a circle graph, the complete circular region represents one dollar. What part of the region would be used to represent 20 cents?

RESEARCH PROJECTS

A The letter "e" is said to be the most frequently used letter in the English language. Choose a paragraph from a newspaper or magazine and make a frequency distribution chart or a graph of each of the letters appearing in the paragraph. What do you think is the order of the letters in the alphabet if arranged according to their frequency of use?

B Find a book that has the decimal representation of π to several hundred decimal places. (The book *The Lore of Large Numbers* by Philip J. Davis; New Mathematical Library, Yale, 1961, contains 4000 decimal places for π.) Make a tally of the digits appearing in the decimal. Then make a frequency distribution graph to show your results. Does one digit occur more often than any others or are there several modes?

TEST YOURSELF

The self-evaluation test should enable students to assess their understanding of the main topics presented in this module and prepare them for a module achievement test.

Correlations between the module objectives and individual test items are annotated on the student book page at the left.

RESEARCH PROJECTS

Project A may prove interesting to many students. The frequency of occurrence of the letters will vary depending on the sample taken, but based on a large sample, a likely order of frequency of occurrence is the following:

E T O A N I R S H D L C U M F P
Y B G W V K X J Q Z

Research Project B deals with the problem of the *normality* of π. A number is said to be *normal* if in its decimal representation each digit occurs $\frac{1}{10}$ of the time. Studies seem to indicate that π may be normal, but the problem has not yet been solved theoretically. A frequency distribution based on the first 100 decimal places of π is given below.

Digit	0	1	2	3	4	5	6	7	8	9
Frequency	8	8	12	11	10	8	9	8	12	14

The activity suggested by this project would be quite time consuming if carried out by individual students, so you might want to encourage several students to work together on it. The decimal representations of π might be broken up into smaller groups of, say, 50 and individual students might be responsible for determining the frequency distribution for each such group. Alternatively, each of the ten digits could be tallied by a different student. In either case, individual findings could then be combined to make the graph.

Objective

The student can demonstrate the ability to work with the concepts presented in this learning unit.

UTILIZATION

Most students will benefit from a cumulative review of the three modules of this learning unit, especially if an achievement test over the unit is to be administered.

Since there are a large number of problems, you may find it necessary to be selective in making an assignment. If so, you should be sure to include some exercises from each of the modules that has been covered by your class (Module 1: Exercises 1–9; Module 2: Exercises 10–14; Module 3: Exercises 15–21).

Encourage students to work on those kinds of problems they feel they need to practice. If feasible, provide answers to the problems, so that students can check their own work.

Having small groups of students work together and discuss the exercises might be an effective way to manage the review.

CUMULATIVE REVIEW

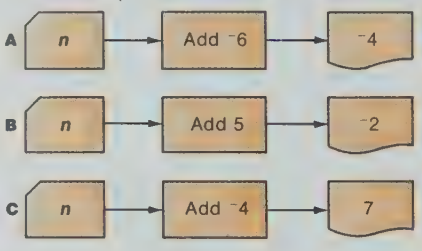
1. Give the opposite of each integer.

A 6 C $^{-}29$ E 99 G 1
B $^{-}4$ D 46 F $^{-}99$ H 0

2. Find the sums.

A $8 + {}^{-}3$
B $^{-}2 + 9$
C $^{-}4 + {}^{-}5$
D $9 + 8$
E $6 + {}^{-}1$
F $^{-}7 + 2$
G $^{-}7 + 9$
H $^{-}6 + 0$
I $^{-}9 + {}^{-}2$
J $^{-}10 + {}^{-}1$

3. Give the input number for each flow chart.



4. Find the differences.

A $0 - 8$
B $3 - 7$
C $^{-}1 - 2$
D $8 - {}^{-}1$
E $^{-}5 - 6$
F $9 - {}^{-}3$
G $^{-}4 - {}^{-}5$
H $^{-}3 - {}^{-}3$

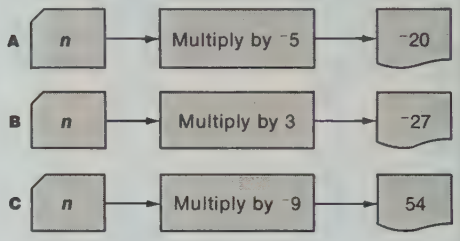
5. Complete each sentence with the words **positive** or **negative**.

A The product of two positive integers is a ___?___ integer.
B The product of two negative integers is a ___?___ integer.
C The product of a positive integer and a negative integer is a ___?___ integer.

6. Find the products.

A $8 \cdot {}^{-}3$
B $^{-}5 \cdot 2$
C $^{-}2 \cdot {}^{-}7$
D $^{-}3 \cdot 16$
E $^{-}8 \cdot {}^{-}9$
F $^{-}1 \cdot {}^{-}1$

7. What is the input number for each flow chart?



8. Find the quotients.

A $9 \div {}^{-}3$
B $^{-}18 \div {}^{-}6$
C $^{-}20 \div 4$
D $2 \div {}^{-}2$
E $^{-}12 \div 3$
F $24 \div {}^{-}8$

9. Give the correct symbol ($<$ or $>$) for each ◯.

A $^{-}9$ ◯ 10
B $^{-}6$ ◯ $^{-}8$
C 8 ◯ $^{-}8$
D $^{-}100$ ◯ $^{-}1$

10. A probability experiment consists of spinning two spinners like those below and recording the number pairs on the spinners.

First spinner

Second spinner

A List the number pairs in the sample space.
B What is the probability of getting the pair (3, 2)?

11. Two red marbles, 3 yellow marbles, and 4 white marbles are placed in a box and mixed. Without looking, a student draws a marble from the box. What is the probability that the marble drawn will be

A red? B yellow? C white?

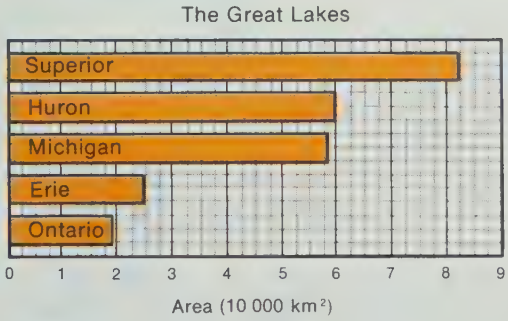
12. If one of the whole numbers from one through ten is selected at random, what is the probability that it will be larger than seven?

13. Mary Ann is going to toss a single die 1000 times as a probability experiment. Which number is the best estimate for the number of times the face with three dots should come up in the 1000 tosses?

[A] 120 [B] 170 [C] 230 [D] 300

14. A political candidate expects to get 34% of the votes of the registered voters in a certain city. If there are 108 000 registered voters in the city, how many votes can the candidate expect? (Round your answer to the nearest thousand.)

15. Use the bar graph to estimate the area of each lake to the nearest thousand square kilometres.



16. On a circle graph, the complete circular region represents one dollar. What fractional part of the region would represent $12\frac{1}{2}$ cents?

17. The price per share of a certain kind of stock for one week is given in the table below.

M	T	W	T	F
$35\frac{1}{2}$	34	$33\frac{1}{2}$	$34\frac{3}{4}$	36

Construct a line-segment graph to illustrate this information.

18. A Estimate the average (arithmetic mean) of the numbers in this set.
{28, 43, 71, 36, 58, 49, 67}

B Compute the average of the set of numbers. Round your answer to the nearest whole number.

19. John made the following scores on seven mathematics tests:
86, 94, 91, 85, 80, 96, 90

A What was his median score?

B What was his average score to the nearest whole number?

C Which is higher, his median score or his average score?

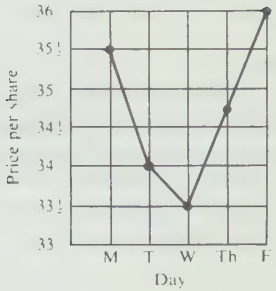
20. Karen had a bowling average of 124 pins per game. Then she bowled three more games and scored 119, 136, and 108.

A Did she bowl as much as her average in the last three games?

B How much more or less than her average did she bowl in the last three games?

21. On a mathematics test more students made a score of 82 than any other score. What is the number 82 called for this set of test scores?

Answer, Exercise 17



MATHEMATICAL RECREATION

Checkers or any other counters may be used for this game. The game can also be played at the chalkboard by marking 13 x's and having each player, in turn, erase 1, 2, or 3 of the x's.

Solutions

1. Five counters are left for your move. After your move, your opponent can win by taking enough counters to leave only one; that is, he will always be able to take a number of counters which when added to your choice gives 4, so that $5 - 4 = 1$.

Counters to remove	Your choice	Opponent's choice	Counters remaining
5	3	1	1
5	2	2	1
5	1	3	1

2. If there are seven counters to remove, you can easily win by removing 2 counters. Then your opponent is faced with the same situation you had in Example 1.

A variation of this game would be one in which the first player names a number from 1 to 3, the second adds a number from 1 to 3, and so forth. If your students are interested, this game can be played in class very effectively.

MATHEMATICAL RECREATIONS

THE UNLUCKY 13 GAME

There is an interesting counting game that can be challenging to everyone. This game is played with 13 counters laid in a row.

Two persons play against each other. Each one must, at his turn, take one, two, or three counters. The player who is forced to take the last counter loses the game.

Try to answer the two questions below to help you to understand the game and to develop some strategies.

1. If there are five counters left and it is your turn to play, will you win or will you lose?
2. If there are seven counters left and it is your turn to play, how many should you take in order to win the game?

Play the game with someone. Can you develop some winning strategies?

MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. Superior **2.** 32 280 km^2 **3.** 245 300 km^2
4. 49 060 km^2 **5.** 58 020 km^2 **6.** False
7. A 0°; B Sunday **8.** 328 **9.** 14 **10.** $\frac{1}{5}$ or 20%

Supplementary Exercises

SET 1

1. Let $X = \{a, b, c, d, e\}$, $Y = \{a, c, t\}$, and $Z = \{b, a, d\}$. List the elements in each of the following sets.

   A $X \cup Y$ **C** $X \cup Z$ **E** $Y \cup Z$ **G** $X \cup \phi$

   B $X \cap Y$ **D** $X \cap Z$ **F** $Y \cap Z$ **H** $X \cap \phi$

2. Let $R = \{0, 2, 4, 6, 8\}$, $S = \{1, 3, 5, 7, 9\}$, and $T = \{0, 3, 6, 9\}$. List the elements in each of the following sets. Consider the sets in the parentheses first.

   A $R \cup S$ **E** $S \cup T$ **I** $S \cup \phi$ **M** $(R \cap S) \cup T$

   B $R \cap S$ **F** $S \cap T$ **J** $S \cap \phi$ **N** $R \cap (S \cup T)$

   C $R \cup T$ **G** $R \cup R$ **K** $(R \cup S) \cup T$ **O** $(R \cup S) \cap (R \cup T)$

   D $R \cap T$ **H** $R \cap R$ **L** $(R \cup S) \cap T$ **P** $(R \cup S) \cap (R \cap T)$

SET 2

1. Simplify each of the following by using 10 with an exponent.

   A $10 \cdot 10 \cdot 10 \cdot 10$ **C** $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$

   B $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$ **D** $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$

2. Give the usual numeral for each power of ten.

   A 10^3 **C** 10^1 **E** 10^7 **G** 10^6 **I** 10^{15} **K** 10^{12}

   B 10^5 **D** 10^9 **F** 10^{10} **H** 10^0 **J** 10^{23} **L** 10^8

3. Find the products.

   A $10^2 \cdot 10^2$ **C** $10^5 \cdot 10^1$ **E** $10^4 \cdot 10^4$ **G** $10^6 \cdot 10^0$

   B $10^3 \cdot 10^5$ **D** $10^3 \cdot 10^3$ **F** $10^3 \cdot 10^5$ **H** $10^0 \cdot 10^0$

4. Find the quotients.

   A $10^3 \div 10^1$ **C** $10^5 \div 10^3$ **E** $10^6 \div 10^1$ **G** $10^3 \div 10^0$

   B $10^4 \div 10^2$ **D** $10^3 \div 10^3$ **F** $10^5 \div 10^4$ **H** $10^7 \div 10^5$

SET 3

1. Write the usual numeral for each.

A $3 \cdot 10^2$ **C** $6 \cdot 10^4$ **E** $4 \cdot 10^7$ **G** $5 \cdot 10^0$ **I** $8 \cdot 10^1$

B $8 \cdot 10^5$ **D** $9 \cdot 10^6$ **F** $2 \cdot 10^{10}$ **H** $6 \cdot 10^9$ **J** $4 \cdot 10^{13}$

2. Express each numeral in scientific notation.

A 600 **C** 4000 **E** 500 000 **G** 30 000 000

B 50 **D** 80 000 **F** 9 000 000 **H** 50 000 000 000 000

3. Use scientific notation to represent each number name.

A forty **E** five million **I** two hundred thousand

B three hundred **F** eighty million **J** four hundred million

C twenty thousand **G** six billion **K** nine hundred billion

D nine thousand **H** thirty billion **L** five trillion

SET 4

1. Give the correct symbol ($<$, $=$, or $>$) for each ◍.

A 35 ◍ 39 **E** 7146 ◍ 7064 **I** 58 755 ◍ 59 855

B 79 ◍ 97 **F** 3909 ◍ 3990 **J** 638 710 ◍ 638 075

C 349 ◍ 294 **G** 8838 ◍ 8783 **K** 5 400 500 ◍ 5 450 000

D 678 ◍ 687 **H** 1010 ◍ 1100 **L** 100 100 100 ◍ 101 000 000

2. Write the numeral for each number indicated.

A one less than one hundred **G** 1000 less than 100 000

B 10 less than one hundred **H** 901 less than 100 000

C 100 less than one thousand **I** 1111 less than 10 000

D one less than one thousand **J** 100 000 less than one million

E 1000 less than 10 000 **K** 10 001 less than one million

F 99 less than 10 000 **L** 100 001 less than one million

SET 5

1. Find the sums. Use base-eight numerals.

A $4_{(8)} + 2_{(8)}$ **B** $14_{(8)} + 2_{(8)}$ **C** $27_{(8)} + 1_{(8)}$ **D** $36_{(8)} + 2_{(8)}$ **E** $14_{(8)} + 6_{(8)}$ **F** $32_{(8)} + 15_{(8)}$

G $54_{(8)} + 14_{(8)}$ **H** $27_{(8)} + 43_{(8)}$ **I** $15_{(8)} + 25_{(8)}$ **J** $73_{(8)} + 12_{(8)}$ **K** $55_{(8)} + 23_{(8)}$ **L** $67_{(8)} + 14_{(8)}$

M $132_{(8)} + 205_{(8)}$ **N** $426_{(8)} + 133_{(8)}$ **O** $265_{(8)} + 241_{(8)}$ **P** $377_{(8)} + 301_{(8)}$ **Q** $555_{(8)} + 123_{(8)}$ **R** $547_{(8)} + 251_{(8)}$

2. Find the differences. Use base-eight numerals.

A $7_{(8)} - 3_{(8)}$ **B** $10_{(8)} - 1_{(8)}$ **C** $15_{(8)} - 4_{(8)}$ **D** $14_{(8)} - 5_{(8)}$ **E** $26_{(8)} - 12_{(8)}$ **F** $47_{(8)} - 35_{(8)}$

G $63_{(8)} - 5_{(8)}$ **H** $50_{(8)} + 7_{(8)}$ **I** $32_{(8)} - 15_{(8)}$ **J** $70_{(8)} - 54_{(8)}$ **K** $126_{(8)} - 35_{(8)}$ **L** $101_{(8)} - 46_{(8)}$

SET 6

1. Find the products. Use base-eight numerals.

A $4_{(8)} \times 3_{(8)}$ **B** $10_{(8)} \times 5_{(8)}$ **C** $23_{(8)} \times 2_{(8)}$ **D** $14_{(8)} \times 5_{(8)}$ **E** $34_{(8)} \times 2_{(8)}$ **F** $13_{(8)} \times 5_{(8)}$

G $40_{(8)} \times 2_{(8)}$ **H** $15_{(8)} \times 6_{(8)}$ **I** $31_{(8)} \times 4_{(8)}$ **J** $54_{(8)} \times 5_{(8)}$ **K** $103_{(8)} \times 4_{(8)}$ **L** $215_{(8)} \times 3_{(8)}$

2. Find the quotients. Use base-eight numerals.

A $6_{(8)} \div 2_{(8)}$ **C** $14_{(8)} \div 4_{(8)}$ **E** $43_{(8)} \div 5_{(8)}$ **G** $104_{(8)} \div 4_{(8)}$

B $10_{(8)} \div 2_{(8)}$ **D** $30_{(8)} \div 4_{(8)}$ **F** $61_{(8)} \div 7_{(8)}$ **H** $137_{(8)} \div 5_{(8)}$

SET 7

1. Tell what number each expression represents.

A $(3 + 4) \cdot 5$

B $3 + 4 \cdot 5$

C $30 \div 5 + 1$

D $30 \div (5 + 1)$

E $56 - (8 \div 2)$

F $(56 - 8) \div 2$

G $18 + 5 \cdot 6$

H $(18 + 5) \cdot 6$

I $2 \cdot 3 \cdot 3 + 4$

J $(2 \cdot 3) \cdot (3 + 4)$

K $2 \cdot (3 \cdot 3 + 4)$

L $2 \cdot 3 \cdot 3 + 15 \div 5$

M $24 \div 3 + 5 \cdot 2$

N $(24 \div 3 + 5) \cdot 2$

O $24 \div (3 + 5) \cdot 2$

2. Solve the equations.

A $3 \cdot n + 1 = 10$

B $15 - 2 \cdot n = 3$

C $(n \div 5) + 1 = 2$

D $6 \cdot n - 25 = 29$

E $8 \cdot (n - 1) = 40$

F $(32 \div n) \cdot 3 = 24$

G $10 \cdot n + 3 \cdot n = 26$

H $5 \cdot n - 6 \cdot 2 = 8$

I $2 \cdot 5 \cdot 3 + n = 37$

J $12 \cdot n - 12 = 0$

K $24 \div (2 + n) = 3$

L $(25 \div n) + 5 = 10$

M $(n \div 2) - 1 = 49$

N $2 \cdot (n + 3) - 6 = 0$

SET 8

1. Find the sums.

A
$$\begin{array}{r} 25 \\ +\,69 \\ \hline \end{array}$$

B
$$\begin{array}{r} 87 \\ +\,43 \\ \hline \end{array}$$

C
$$\begin{array}{r} 456 \\ +\,\ 99 \\ \hline \end{array}$$

D
$$\begin{array}{r} 732 \\ +\,861 \\ \hline \end{array}$$

E
$$\begin{array}{r} 2568 \\ +\ 432 \\ \hline \end{array}$$

F
$$\begin{array}{r} 18\,063 \\ +\,15\,119 \\ \hline \end{array}$$

G
$$\begin{array}{r} 26 \\ 19 \\ +\,33 \\ \hline \end{array}$$

H
$$\begin{array}{r} 149 \\ 73 \\ +\,452 \\ \hline \end{array}$$

I
$$\begin{array}{r} 2809 \\ 503 \\ +\ \ 45 \\ \hline \end{array}$$

J
$$\begin{array}{r} 6829 \\ 5931 \\ 7522 \\ +\,1099 \\ \hline \end{array}$$

K
$$\begin{array}{r} 28\,176 \\ 44\,360 \\ 50\,905 \\ +\,11\,111 \\ \hline \end{array}$$

L
$$\begin{array}{r} 10\,068 \\ 1\,149 \\ 658 \\ +\ \ \ 5 \\ \hline \end{array}$$

2. Find the sums.

A $58 + 79 + 103$

B $125 + 47 + 1500$

C $135 + 8 + 46$

D $2700 + 34 + 200$

E $149 + 383 + 76 + 168$

F $18\,457 + 10\,790 + 999$

G $4315 + 653 + 27$

H $965 + 966 + 967$

I $47\,973 + 140\,750$

J $488 + 97 + 23\,705$

K $6513 + 8805 + 7009$

L $7900 + 79\,000 + 790\,000$

SET 9

1. Find the differences.

A 53 − 39	**B** 87 − 74	**C** 100 − 76	**D** 235 − 47	**E** 563 − 334	**F** 905 − 833
G 4986 − 1595	**H** 5740 − 2981	**I** 8800 − 5411	**J** 7651 − 5555	**K** 4001 − 3802	**L** 9000 − 547
M 29 795 − 13 686	**N** 61 837 − 8 535	**O** 76 000 − 39 436	**P** 85 000 − 83 952	**Q** 348 966 − 149 865	**R** 743 009 − 356 075

2. Find the differences.

A 71 − 45	**D** 2000 − 1371	**G** 27 126 − 26 139
B 348 − 199	**E** 9001 − 49	**H** 408 970 − 98 790
C 1423 − 746	**F** 10 000 − 950	**I** 900 009 − 101 010

SET 10

1. Find the products.

A 7 · 10	**G** 30 · 10	**M** 20 · 60	**S** 200 · 60
B 100 · 9	**H** 300 · 10	**N** 300 · 50	**T** 500 · 20
C 6 · 1000	**I** 300 · 100	**O** 90 · 40	**U** 300 · 200
D 5 · 10 000	**J** 30 · 1000	**P** 400 · 20	**V** 800 · 400
E 100 000 · 3	**K** 300 · 1000	**Q** 50 · 70	**W** 3000 · 40
F 4 · 1 000 000	**L** 30 · 10 000	**R** 80 · 50	**X** 600 · 5000

2. Find the quotients.

A 240 ÷ 6	**G** 1500 ÷ 300	**M** 60 000 ÷ 3000	**S** 120 000 ÷ 4000
B 800 ÷ 4	**H** 4200 ÷ 600	**N** 40 000 ÷ 8000	**T** 150 000 ÷ 500
C 350 ÷ 5	**I** 4000 ÷ 50	**O** 21 000 ÷ 700	**U** 270 000 ÷ 900
D 600 ÷ 10	**J** 2800 ÷ 40	**P** 32 000 ÷ 400	**V** 480 000 ÷ 80
E 900 ÷ 30	**K** 6300 ÷ 90	**Q** 56 000 ÷ 70	**W** 100 000 ÷ 50
F 1200 ÷ 600	**L** 7200 ÷ 800	**R** 20 000 ÷ 50	**X** 8 100 000 ÷ 90

SET 11

1. Find the products.

A 38×3	**D** 158×6	**G** 853×9	**J** 9638×7
B 53×7	**E** 356×5	**H** 508×4	**K** $23\,604 \times 6$
C 78×9	**F** 740×8	**I** 945×7	**L** $430\,895 \times 3$

2. Find the products.

A 23×17	**D** 68×52	**G** 249×21	**J** 6589×35
B 49×23	**E** 97×86	**H** 809×73	**K** 8359×97
C 63×36	**F** 76×93	**I** 777×46	**L** $42\,895 \times 64$

3. Find the products.

A 217×175	**D** 340×216	**G** 508×749	**J** 1697×806
B 345×444	**E** 459×107	**H** 671×488	**K** 2304×350
C 199×675	**F** 901×901	**I** 606×999	**L** 5893×649

SET 12

1. Find the quotients.

A $3\overline{)84}$	**D** $5\overline{)270}$	**G** $8\overline{)536}$	**J** $3\overline{)885}$	**M** $7\overline{)2527}$
B $7\overline{)224}$	**E** $6\overline{)288}$	**H** $4\overline{)548}$	**K** $6\overline{)2970}$	**N** $9\overline{)6201}$
C $4\overline{)664}$	**F** $3\overline{)561}$	**I** $9\overline{)504}$	**L** $5\overline{)2670}$	**O** $8\overline{)12\,432}$

2. Find the quotients and remainders.

A $5\overline{)83}$	**D** $6\overline{)238}$	**G** $9\overline{)210}$	**J** $4\overline{)8631}$	**M** $6\overline{)5005}$
B $7\overline{)99}$	**E** $8\overline{)310}$	**H** $8\overline{)531}$	**K** $7\overline{)2444}$	**N** $8\overline{)12\,153}$
C $4\overline{)123}$	**F** $3\overline{)199}$	**I** $6\overline{)527}$	**L** $5\overline{)4301}$	**O** $9\overline{)25\,984}$

SET 13

1. Find the quotients and remainders.

A $21\overline{)473}$ **D** $69\overline{)795}$ **G** $81\overline{)6583}$ **J** $48\overline{)19\,870}$

B $17\overline{)538}$ **E** $43\overline{)1095}$ **H** $75\overline{)5738}$ **K** $89\overline{)46\,795}$

C $33\overline{)865}$ **F** $58\overline{)3710}$ **I** $92\overline{)6047}$ **L** $65\overline{)100\,008}$

2. Find the quotients and remainders.

A $191\overline{)578}$ **D** $403\overline{)2590}$ **G** $921\overline{)34\,795}$ **J** $873\overline{)413\,888}$

B $316\overline{)683}$ **E** $278\overline{)1193}$ **H** $748\overline{)15\,000}$ **K** $609\overline{)793\,502}$

C $599\overline{)1246}$ **F** $611\overline{)5034}$ **I** $542\overline{)79\,851}$ **L** $777\overline{)333\,333}$

SET 14

1. Write an equation for each flow chart.

A $n \rightarrow -16 \rightarrow 57$

B $6 \rightarrow \cdot n \rightarrow 96$

C $n \rightarrow \div 8 \rightarrow 13$

D $n \rightarrow 2 \rightarrow +3 \rightarrow 25$

E $n \rightarrow \div 5 \rightarrow +5 \rightarrow 15$

F $7 \rightarrow +n \rightarrow \cdot 3 \rightarrow 36$

2. Write an equation for each sentence.

A When a certain number is subtracted from 45, the difference is 31.

B The product of a certain number and 9 is 108.

C When a number is divided by 8, the quotient is 25.

D When 3 is added to 5 times a number, the result is 18.

E When a certain number is divided by 6 and then 9 is subtracted from the quotient, the result is 0.

F When the sum of a number and 8 is multiplied by 3, the result is 60.

SET 15

Write an equation or draw a diagram for each problem.
Then solve the problem.

1. The sum of a certain number and 5 is 27.

2. When 7 is subtracted from the product of a certain number and 3, the result is 35.

3. Tom is 165 centimetres tall. Tom's brother is 18 centimetres shorter than he is. How tall is Tom's brother?

4. A man travelled 371 kilometres in 7 hours. What was his average speed?

5. When 8 is added to a number and then the sum is divided by 5, the result is 10.

6. The perimeter of a triangle is 69 centimetres. If all the sides are the same length, what is the length of each side of the triangle?

7. A commuter train is travelling at 96 kilometres per hour. How far will it travel in 3 hours?

8. Susan is three years more than twice her brother's age. Her brother is 8 years old. How old is Susan?

9. A piece of ribbon was 199 centimetres long. If there is 7 centimetres left over after cutting it into 3 pieces of the same length, how long is each piece?

10. The perimeter of a rectangle is 30 metres. The length is 3 metres longer than the width. Find the length and width of the rectangle.

11. The population of Rockville is 18 less than twice the size of Emmeryton. If Rockville has 2754 people, how many people are in Emmeryton?

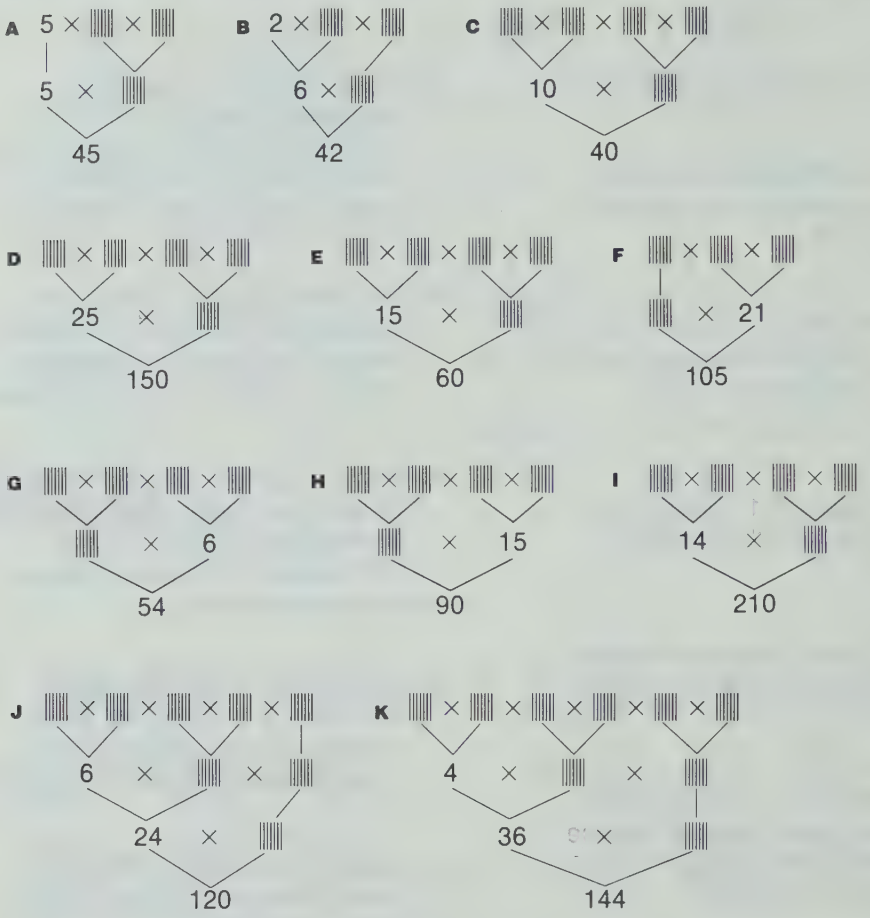
12. If the length of a rectangle is 5 times as long as the width and the width is 6 metres, what is the perimeter of the rectangle?

13. Jeremy paid a total of $4.52 for 3 ball point pens. If 26¢ of the total amount was sales tax, how much was each pen?

14. Karen bought several barrettes that cost 79 cents each and a comb for 63 cents. She spent $3.79 in all, how many barrettes did she buy?

SET 16

1. Copy and complete the factor trees.



2. Give the prime factorization of each number.

A 8	**E** 27	**I** 35	**M** 51	**Q** 54	**U** 150
B 15	**F** 30	**J** 39	**N** 80	**R** 72	**V** 210
C 18	**G** 31	**K** 40	**O** 90	**S** 96	**W** 500
D 24	**H** 32	**L** 41	**P** 103	**T** 100	**X** 512

SET 17

1. **A** List the factors of 15. **B** List the factors of 9.
   C List the common factors of 15 and 9.
   D What is the greatest common factor (GCF) of 15 and 9?

2. **A** List the factors of 24. **B** List the factors of 40.
   C List the common factors of 24 and 40.
   D What is the greatest common factor (GCF) of 24 and 40?

3. Find the GCF of each pair of numbers.

A 6 and 10	**F** 25 and 20	**K** 10 and 45	**P** 51 and 39
B 15 and 7	**G** 18 and 27	**L** 36 and 45	**Q** 200 and 45
C 20 and 12	**H** 48 and 36	**M** 25 and 40	**R** 80 and 200
D 15 and 18	**I** 31 and 51	**N** 75 and 100	**S** 630 and 240
E 21 and 14	**J** 40 and 15	**O** 24 and 108	**T** 121 and 70

SET 18

1. **A** List the first 8 multiples of 4. **B** List the first 8 multiples of 12.
   C List the common multiples of 4 and 12.
   D What is the least common multiple (LCM) of 4 and 12?

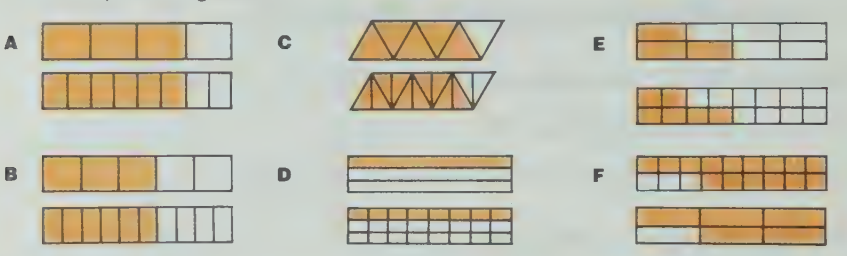
2. **A** List the first 8 multiples of 9. **B** List the first 8 multiples of 15.
   C List the common multiples of 9 and 15.
   D What is the least common multiple (LCM) of 9 and 15?

3. Find the LCM of each group of numbers.

A 3 and 7	**F** 15 and 20	**K** 2, 4, and 5	**P** 3, 8, and 12
B 2 and 11	**G** 10 and 25	**L** 2, 3, and 9	**Q** 3, 12, and 16
C 5 and 15	**H** 6 and 33	**M** 2, 5, and 10	**R** 5, 8, and 16
D 9 and 12	**I** 30 and 45	**N** 3, 5, and 20	**S** 3, 10, and 15
E 8 and 10	**J** 25 and 40	**O** 2, 7, and 8	**T** 25, 30, and 75

SET 19

1. Name a pair of equivalent fractions suggested by the shading in each pair of girues.



2. Give the next three fractions for each set of equivalent fractions.

A $\left\{\frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \ldots\right\}$

B $\left\{\frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \ldots\right\}$

C $\left\{\frac{3}{5}, \frac{6}{10}, \frac{9}{15}, \ldots\right\}$

D $\left\{\frac{3}{7}, \frac{6}{14}, \frac{9}{21}, \ldots\right\}$

E $\left\{\frac{1}{8}, \frac{2}{16}, \frac{3}{24}, \ldots\right\}$

F $\left\{\frac{3}{1}, \frac{6}{2}, \frac{9}{3}, \ldots\right\}$

G $\left\{\frac{3}{10}, \frac{6}{20}, \frac{9}{30}, \ldots\right\}$

H $\left\{\frac{6}{5}, \frac{12}{10}, \frac{18}{15}, \ldots\right\}$

I $\left\{\frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \ldots\right\}$

J $\left\{\frac{5}{8}, \frac{10}{16}, \frac{15}{24}, \ldots\right\}$

K $\left\{\frac{7}{9}, \frac{14}{18}, \ldots\right\}$

L $\left\{\frac{11}{4}, \frac{22}{8}, \ldots\right\}$

M $\left\{\frac{6}{1}, \frac{12}{2}, \ldots\right\}$

N $\left\{\frac{0}{1}, \frac{0}{2}, \ldots\right\}$

O $\left\{\frac{1}{25}, \frac{2}{50}, \ldots\right\}$

SET 20

1. Give the lowest-terms fraction for each set.

A $\left\{\frac{6}{12}, \frac{10}{20}, \frac{15}{30}\right\}$

B $\left\{\frac{5}{15}, \frac{6}{18}, \frac{10}{30}\right\}$

C $\left\{\frac{6}{10}, \frac{12}{20}, \frac{300}{500}\right\}$

D $\left\{\frac{70}{100}, \frac{100}{1000}, \frac{10\,000}{100\,000}\right\}$

E $\left\{\frac{3}{12}, \frac{5}{20}, \frac{15}{60}\right\}$

F $\left\{\frac{9}{30}, \frac{15}{50}, \frac{27}{90}\right\}$

G $\left\{\frac{100}{60}, \frac{20}{15}, \frac{55}{33}\right\}$

H $\left\{\frac{27}{36}, \frac{36}{48}, \frac{48}{64}\right\}$

I $\left\{\frac{18}{66}, \frac{24}{88}, \frac{33}{121}\right\}$

2. Give the lowest-terms fraction for each given fraction.

A	$\frac{6}{10}$	**D**	$\frac{9}{12}$	**G**	$\frac{8}{20}$	**J**	$\frac{14}{27}$	**M**	$\frac{12}{20}$	**P**	$\frac{15}{36}$	**S**	$\frac{28}{42}$	**V**	$\frac{24}{64}$
B	$\frac{8}{12}$	**E**	$\frac{6}{15}$	**H**	$\frac{9}{21}$	**K**	$\frac{16}{28}$	**N**	$\frac{20}{32}$	**Q**	$\frac{27}{36}$	**T**	$\frac{36}{45}$	**W**	$\frac{27}{37}$
C	$\frac{6}{9}$	**F**	$\frac{13}{18}$	**I**	$\frac{15}{24}$	**L**	$\frac{6}{16}$	**O**	$\frac{21}{35}$	**R**	$\frac{16}{40}$	**U**	$\frac{18}{60}$	**X**	$\frac{12}{200}$

SET 21

1. Give the correct symbol ($=$ or $\neq$) for each ▒.

A $\frac{2}{3}$ ▒ $\frac{6}{9}$ **E** $\frac{1}{8}$ ▒ $\frac{5}{40}$ **I** $\frac{7}{7}$ ▒ $\frac{7}{1}$ **M** $\frac{1}{6}$ ▒ $\frac{6}{100}$

B $\frac{4}{6}$ ▒ $\frac{10}{15}$ **F** $\frac{3}{1}$ ▒ $\frac{28}{9}$ **J** $\frac{4}{20}$ ▒ $\frac{2}{8}$ **N** $\frac{0}{5}$ ▒ $\frac{0}{10}$

C $\frac{9}{20}$ ▒ $\frac{4}{5}$ **G** $\frac{4}{5}$ ▒ $\frac{36}{45}$ **K** $\frac{18}{6}$ ▒ $\frac{6}{3}$ **O** $\frac{5}{25}$ ▒ $\frac{12}{60}$

D $\frac{3}{6}$ ▒ $\frac{9}{12}$ **H** $\frac{9}{12}$ ▒ $\frac{6}{8}$ **L** $\frac{30}{50}$ ▒ $\frac{9}{15}$ **P** $\frac{49}{64}$ ▒ $\frac{7}{8}$

2. Solve the equations.

A $\frac{2}{3} = \frac{n}{9}$ **E** $\frac{n}{40} = \frac{4}{5}$ **I** $\frac{5}{8} = \frac{n}{40}$ **M** $\frac{35}{100} = \frac{7}{n}$

B $\frac{5}{6} = \frac{10}{n}$ **F** $\frac{6}{6} = \frac{n}{24}$ **J** $\frac{3}{1} = \frac{9}{n}$ **N** $\frac{25}{n} = \frac{1}{3}$

C $\frac{n}{3} = \frac{8}{12}$ **G** $\frac{n}{18} = \frac{10}{15}$ **K** $\frac{n}{20} = \frac{16}{80}$ **O** $\frac{9}{12} = \frac{45}{n}$

D $\frac{6}{7} = \frac{n}{70}$ **H** $\frac{5}{9} = \frac{5}{n}$ **L** $\frac{n}{15} = \frac{24}{90}$ **P** $\frac{n}{9} = \frac{0}{6}$

SET 22

1. Give the correct symbol ($<$ or $>$) for each ▒.

A $\frac{1}{4}$ ▒ $\frac{1}{5}$ **F** $\frac{2}{3}$ ▒ $\frac{27}{30}$ **K** $\frac{1}{8}$ ▒ $\frac{1}{9}$ **P** $\frac{3}{8}$ ▒ $\frac{5}{16}$

B $\frac{4}{3}$ ▒ $\frac{5}{4}$ **G** $\frac{8}{9}$ ▒ $\frac{9}{10}$ **L** $\frac{15}{16}$ ▒ $\frac{3}{4}$ **Q** $\frac{2}{5}$ ▒ $\frac{15}{40}$

C $\frac{1}{5}$ ▒ $\frac{5}{20}$ **H** $\frac{4}{5}$ ▒ $\frac{8}{15}$ **M** $\frac{3}{10}$ ▒ $\frac{1}{3}$ **R** $\frac{2}{3}$ ▒ $\frac{3}{4}$

D $\frac{3}{4}$ ▒ $\frac{8}{12}$ **I** $\frac{7}{8}$ ▒ $\frac{6}{7}$ **N** $\frac{1}{6}$ ▒ $\frac{3}{20}$ **S** $\frac{5}{6}$ ▒ $\frac{4}{5}$

E $\frac{5}{8}$ ▒ $\frac{15}{32}$ **J** $\frac{6}{5}$ ▒ $\frac{6}{7}$ **O** $\frac{7}{12}$ ▒ $\frac{5}{9}$ **T** $\frac{13}{16}$ ▒ $\frac{9}{12}$

2. Which number in each group is the largest?

A $\frac{1}{2}, \frac{1}{4}, \frac{1}{5}$ **D** $\frac{5}{2}, \frac{3}{2}, \frac{7}{2}$ **G** $\frac{5}{4}, \frac{5}{7}, \frac{5}{12}$ **J** $\frac{2}{3}, \frac{3}{3}, \frac{4}{3}$

B $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}$ **E** $\frac{4}{9}, \frac{5}{10}, \frac{6}{11}$ **H** $\frac{5}{8}, \frac{3}{4}, \frac{7}{16}$ **K** $\frac{9}{10}, \frac{10}{10}, \frac{10}{11}$

C $\frac{3}{5}, \frac{3}{7}, \frac{3}{8}$ **F** $\frac{7}{8}, \frac{3}{4}, \frac{9}{12}$ **I** $\frac{9}{4}, \frac{7}{3}, \frac{11}{5}$ **L** $\frac{3}{10}, \frac{33}{100}, \frac{1}{3}$

SET 23

1. Find the sums. Give your answers in lowest terms.

A $\frac{2}{3}+\frac{1}{6}$	**D** $\frac{2}{3}+\frac{2}{9}$	**G** $\frac{2}{7}+\frac{1}{3}$	**J** $\frac{5}{8}+\frac{2}{3}$	**M** $\frac{5}{6}+\frac{1}{5}$
B $\frac{2}{5}+\frac{3}{10}$	**E** $\frac{3}{4}+\frac{1}{8}$	**H** $\frac{5}{6}+\frac{2}{9}$	**K** $\frac{3}{5}+\frac{3}{7}$	**N** $\frac{4}{27}+\frac{1}{3}$
C $\frac{1}{3}+\frac{1}{2}$	**F** $\frac{2}{3}+\frac{4}{15}$	**I** $\frac{3}{4}+\frac{2}{3}$	**L** $\frac{1}{8}+\frac{4}{5}$	**O** $\frac{3}{10}+\frac{5}{6}$

2. Find the sums. Give your answers in lowest terms.

A $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$	**D** $\frac{1}{4}+\frac{3}{8}+\frac{5}{16}$	**G** $\frac{4}{5}+\frac{2}{3}+\frac{1}{2}$	**J** $\frac{1}{10}+\frac{1}{25}+\frac{1}{50}$
B $\frac{2}{3}+\frac{1}{2}+\frac{5}{6}$	**E** $\frac{3}{10}+\frac{1}{4}+\frac{2}{5}$	**H** $\frac{5}{6}+\frac{1}{3}+\frac{7}{10}$	**K** $\frac{7}{8}+\frac{1}{3}+\frac{5}{6}$
C $\frac{5}{8}+\frac{3}{4}+\frac{1}{2}$	**F** $\frac{1}{6}+\frac{5}{9}+\frac{2}{3}$	**I** $\frac{11}{12}+\frac{7}{8}+\frac{2}{3}$	**L** $\frac{4}{15}+\frac{1}{3}+\frac{6}{25}$

3. Find the differences. Give your answers in lowest terms.

A $\frac{5}{6}-\frac{1}{2}$	**D** $\frac{11}{12}-\frac{2}{3}$	**G** $\frac{17}{15}-\frac{5}{6}$	**J** $\frac{9}{8}-\frac{4}{5}$	**M** $\frac{9}{11}-\frac{2}{5}$
B $\frac{3}{4}-\frac{5}{8}$	**E** $\frac{6}{7}-\frac{2}{3}$	**H** $\frac{7}{8}-\frac{2}{3}$	**K** $\frac{15}{16}-\frac{2}{3}$	**N** $\frac{8}{9}-\frac{5}{6}$
C $\frac{7}{9}-\frac{1}{3}$	**F** $\frac{4}{5}-\frac{3}{4}$	**I** $\frac{7}{10}-\frac{1}{4}$	**L** $\frac{5}{7}-\frac{2}{5}$	**O** $\frac{11}{15}-\frac{3}{12}$

SET 24

1. Write a mixed numeral for each improper fraction.

A $\frac{4}{3}$	**D** $\frac{17}{3}$	**G** $\frac{25}{9}$	**J** $\frac{10}{4}$	**M** $\frac{14}{12}$	**P** $\frac{42}{20}$	**S** $\frac{75}{30}$	**V** $\frac{72}{27}$
B $\frac{8}{5}$	**E** $\frac{13}{2}$	**H** $\frac{61}{31}$	**K** $\frac{22}{10}$	**N** $\frac{35}{25}$	**Q** $\frac{50}{8}$	**T** $\frac{49}{21}$	**W** $\frac{150}{40}$
C $\frac{21}{10}$	**F** $\frac{32}{7}$	**I** $\frac{9}{6}$	**L** $\frac{15}{9}$	**O** $\frac{30}{12}$	**R** $\frac{26}{6}$	**U** $\frac{36}{16}$	**X** $\frac{210}{150}$

2. Write an improper fraction for each mixed numeral.

A $1\frac{1}{7}$	**D** $4\frac{1}{10}$	**G** $8\frac{5}{6}$	**J** $10\frac{5}{6}$	**M** $57\frac{1}{2}$	**P** $12\frac{3}{8}$	**S** $8\frac{3}{20}$	**V** $24\frac{1}{3}$
B $1\frac{3}{5}$	**E** $3\frac{2}{3}$	**H** $3\frac{5}{12}$	**K** $25\frac{1}{4}$	**N** $33\frac{1}{3}$	**Q** $19\frac{9}{10}$	**T** $3\frac{15}{16}$	**W** $1\frac{43}{45}$
C $2\frac{1}{3}$	**F** $4\frac{1}{2}$	**I** $16\frac{2}{3}$	**L** $11\frac{1}{9}$	**O** $6\frac{4}{15}$	**R** $3\frac{8}{25}$	**U** $47\frac{3}{4}$	**X** $8\frac{13}{19}$

SET 25

1. Find the sums.

A $1\frac{1}{2} + \frac{2}{3}$ **B** $5\frac{1}{3} + 3\frac{1}{4}$ **C** $2\frac{7}{10} + 5\frac{1}{2}$ **D** $8\frac{5}{6} + 1\frac{1}{12}$ **E** $4\frac{2}{9} + 1\frac{5}{6}$ **F** $6\frac{4}{5} + 3\frac{2}{3}$

G $8\frac{1}{6} + 3\frac{3}{4}$ **H** $12\frac{4}{7} + 2\frac{1}{3}$ **I** $23\frac{1}{8} + 7\frac{2}{3}$ **J** $17\frac{5}{6} + 4\frac{3}{5}$ **K** $31\frac{4}{9} + 8\frac{3}{4}$ **L** $65\frac{3}{20} + 13\frac{5}{6}$

2. Find the differences.

A $3\frac{1}{2} - 1\frac{1}{6}$ **B** $8\frac{3}{4} - 5\frac{1}{3}$ **C** $9\frac{1}{2} - 1\frac{7}{8}$ **D** $4\frac{1}{3} - 3\frac{4}{5}$ **E** $12\frac{4}{9} - 8\frac{5}{6}$ **F** $11\frac{3}{10} - 3\frac{3}{4}$

G $27\frac{1}{8} - 19\frac{1}{3}$ **H** $73\frac{1}{3} - 50\frac{5}{7}$ **I** $9\frac{1}{100} - 4\frac{1}{10}$ **J** $33\frac{3}{8} - 13\frac{5}{6}$ **K** $173\frac{5}{8} - 41\frac{4}{5}$ **L** $457\frac{1}{7} - 125\frac{1}{6}$

SET 26

1. Find the products.

A $\frac{1}{2} \cdot \frac{1}{4}$ **B** $\frac{1}{4} \cdot \frac{1}{3}$ **C** $\frac{1}{2} \cdot \frac{1}{2}$ **D** $\frac{1}{3} \cdot \frac{1}{5}$ **E** $\frac{1}{6} \cdot \frac{1}{2}$ **F** $\frac{1}{4} \cdot \frac{1}{4}$ **G** $\frac{1}{5} \cdot \frac{1}{6}$ **H** $\frac{1}{10} \cdot \frac{1}{7}$ **I** $\frac{1}{2} \cdot \frac{1}{5}$ **J** $\frac{1}{4} \cdot \frac{1}{6}$ **K** $\frac{1}{2} \cdot \frac{1}{8}$ **L** $\frac{1}{10} \cdot \frac{1}{5}$ **M** $\frac{1}{100} \cdot \frac{1}{4}$ **N** $\frac{1}{30} \cdot \frac{1}{2}$ **O** $\frac{1}{60} \cdot \frac{1}{50}$

2. Find the products. Give your answers in lowest-terms.

A $2 \cdot \frac{1}{3}$ **B** $4 \cdot \frac{1}{7}$ **C** $6 \cdot \frac{1}{5}$ **D** $9 \cdot \frac{1}{2}$ **E** $5 \cdot \frac{1}{10}$ **F** $6 \cdot \frac{1}{4}$ **G** $8 \cdot \frac{1}{12}$ **H** $10 \cdot \frac{1}{5}$ **I** $15 \cdot \frac{1}{9}$ **J** $1 \cdot \frac{1}{16}$ **K** $5 \cdot \frac{1}{5}$ **L** $32 \cdot \frac{1}{12}$ **M** $0 \cdot \frac{1}{12}$ **N** $42 \cdot \frac{1}{9}$ **O** $100 \cdot \frac{1}{4}$ **P** $36 \cdot \frac{1}{16}$ **Q** $\frac{1}{5}$ of 10 **R** $\frac{1}{3}$ of 25 **S** $\frac{1}{4}$ of 30 **T** $\frac{1}{10}$ of 46

SET 27

1. Find the products. Give your answers in lowest terms.

A $\frac{1}{2} \cdot \frac{3}{5}$	**E** $\frac{5}{9} \cdot \frac{3}{5}$	**I** $\frac{5}{8} \cdot \frac{2}{3}$	**M** $\frac{8}{9} \cdot \frac{15}{16}$	**Q** $\frac{5}{16} \cdot \frac{3}{20}$
B $\frac{3}{4} \cdot \frac{3}{4}$	**F** $\frac{4}{5} \cdot \frac{3}{8}$	**J** $\frac{6}{7} \cdot \frac{3}{20}$	**N** $\frac{6}{6} \cdot \frac{17}{21}$	**R** $\frac{13}{27} \cdot \frac{9}{10}$
C $\frac{5}{6} \cdot \frac{5}{7}$	**G** $\frac{2}{7} \cdot \frac{5}{4}$	**K** $\frac{4}{3} \cdot \frac{3}{8}$	**O** $\frac{9}{10} \cdot \frac{25}{36}$	**S** $\frac{12}{35} \cdot \frac{7}{24}$
D $\frac{3}{10} \cdot \frac{7}{8}$	**H** $\frac{4}{15} \cdot \frac{3}{8}$	**L** $\frac{6}{11} \cdot \frac{7}{8}$	**P** $\frac{24}{25} \cdot \frac{5}{6}$	**T** $\frac{48}{49} \cdot \frac{7}{16}$

2. Find the products. Give your answers in lowest terms.

A $4 \cdot \frac{2}{3}$	**E** $4\frac{1}{2} \cdot 6$	**I** $6\frac{2}{5} \cdot \frac{3}{4}$	**M** $1\frac{1}{2} \cdot 3\frac{1}{3}$	**Q** $6\frac{2}{3} \cdot 1\frac{7}{10}$
B $\frac{5}{6} \cdot 10$	**F** $8 \cdot 7\frac{3}{4}$	**J** $4\frac{3}{10} \cdot \frac{5}{8}$	**N** $2\frac{1}{5} \cdot 1\frac{1}{4}$	**R** $3\frac{3}{5} \cdot 8\frac{1}{3}$
C $1\frac{1}{2} \cdot 7$	**G** $\frac{4}{9} \cdot 1\frac{1}{8}$	**K** $\frac{4}{15} \cdot 2\frac{2}{9}$	**O** $1\frac{2}{3} \cdot 8\frac{1}{2}$	**S** $4\frac{4}{9} \cdot 6\frac{3}{10}$
D $3 \cdot 1\frac{5}{6}$	**H** $3\frac{1}{3} \cdot \frac{2}{5}$	**L** $7\frac{1}{5} \cdot \frac{7}{12}$	**P** $4\frac{1}{5} \cdot 1\frac{5}{6}$	**T** $12\frac{3}{4} \cdot 1\frac{11}{21}$

SET 28

1. Find the products. Think of the distributive principle.

A $8\frac{1}{2}$ $\times 4$	**B** $3\frac{1}{3}$ $\times 6$	**C** $7\frac{1}{10}$ $\times 5$	**D** 18 $\times 1\frac{1}{3}$	**E** 4 $\times 3\frac{1}{8}$
F $15\frac{1}{2}$ $\times 2\frac{1}{3}$	**G** $24\frac{1}{5}$ $\times 10\frac{1}{2}$	**H** $9\frac{4}{5}$ $\times 5\frac{1}{3}$	**I** $35\frac{3}{4}$ $\times 8\frac{1}{5}$	**J** 45 $\times 8\frac{4}{9}$

2. Find the products using the distributive principle.

A $3 \cdot 7\frac{1}{3}$	**E** $5 \cdot 8\frac{1}{15}$	**I** $\frac{1}{12} \cdot 60\frac{1}{2}$	**M** $\frac{1}{2} \cdot 4\frac{3}{5}$	**Q** $\frac{2}{3} \cdot 33\frac{2}{3}$
B $\frac{1}{4} \cdot 8\frac{1}{2}$	**F** $8 \cdot 3\frac{1}{16}$	**J** $\frac{2}{3} \cdot 9\frac{1}{2}$	**N** $\frac{3}{10} \cdot 20\frac{1}{2}$	**R** $18\frac{1}{7} \cdot \frac{1}{6}$
C $20 \cdot 6\frac{1}{4}$	**G** $\frac{1}{6} \cdot 6\frac{2}{7}$	**K** $45 \cdot 2\frac{1}{9}$	**O** $4 \cdot 5\frac{1}{16}$	**S** $150 \cdot 3\frac{1}{10}$
D $21 \cdot 3\frac{1}{3}$	**H** $\frac{1}{2} \cdot 100\frac{1}{5}$	**L** $6 \cdot 8\frac{5}{12}$	**P** $\frac{2}{7} \cdot 35\frac{1}{4}$	**T** $\frac{4}{9} \cdot 90\frac{3}{4}$

SET 29

1. Find the quotients.

A $\frac{1}{3} \div \frac{2}{5}$ **E** $6 \div \frac{9}{10}$ **I** $\frac{7}{8} \div \frac{7}{16}$ **M** $\frac{7}{12} \div 3$ **Q** $\frac{2}{27} \div \frac{10}{27}$

B $\frac{3}{4} \div \frac{3}{4}$ **F** $\frac{3}{4} \div \frac{1}{2}$ **J** $\frac{9}{20} \div \frac{9}{10}$ **N** $\frac{4}{15} \div \frac{4}{5}$ **R** $\frac{4}{25} \div \frac{6}{5}$

C $\frac{2}{3} \div \frac{9}{6}$ **G** $\frac{4}{9} \div \frac{2}{3}$ **K** $4 \div \frac{1}{5}$ **O** $\frac{13}{16} \div \frac{1}{32}$ **S** $\frac{4}{9} \div \frac{8}{27}$

D $10 \div \frac{5}{12}$ **H** $\frac{6}{7} \div \frac{7}{6}$ **L** $\frac{8}{9} \div 4$ **P** $\frac{15}{32} \div \frac{5}{32}$ **T** $\frac{5}{36} \div 15$

2. Find the quotients.

A $1\frac{1}{4} \div \frac{2}{3}$ **E** $6\frac{2}{3} \div 5$ **I** $1 \div 4\frac{5}{6}$ **M** $4\frac{1}{2} \div 18$ **Q** $35 \div 3\frac{3}{4}$

B $2\frac{1}{2} \div 3$ **F** $3\frac{1}{3} \div \frac{3}{10}$ **J** $1\frac{1}{3} \div 1\frac{1}{2}$ **N** $10 \div 8\frac{3}{5}$ **R** $3\frac{1}{3} \div 16\frac{2}{3}$

C $8\frac{2}{5} \div 6$ **G** $3 \div 1\frac{1}{5}$ **K** $4\frac{1}{2} \div 1\frac{7}{8}$ **O** $4\frac{7}{8} \div 4\frac{7}{8}$ **S** $3\frac{9}{10} \div 4\frac{1}{3}$

D $4\frac{1}{6} \div \frac{1}{3}$ **H** $25 \div 8\frac{1}{3}$ **L** $1\frac{5}{6} \div 3\frac{1}{2}$ **P** $9\frac{1}{5} \div 6\frac{3}{5}$ **T** $6\frac{7}{8} \div 9\frac{2}{7}$

SET 30

1. Write each complex fraction as a quotient of two fractional numbers.

A $\dfrac{4}{\frac{3}{5}}$ **B** $\dfrac{\frac{5}{6}}{10}$ **C** $\dfrac{\frac{4}{3}}{\frac{1}{2}}$ **D** $\dfrac{\frac{1}{2}}{3\frac{1}{2}}$ **E** $\dfrac{4\frac{1}{3}}{\frac{9}{10}}$ **F** $\dfrac{21\frac{1}{3}}{8\frac{7}{10}}$

G $\dfrac{1\frac{3}{5}}{6}$ **H** $\dfrac{8}{2\frac{1}{2}}$ **I** $\dfrac{5\frac{1}{3}}{\frac{1}{4}}$ **J** $\dfrac{10\frac{2}{3}}{4\frac{7}{8}}$ **K** $\dfrac{1}{\frac{2}{5}}$ **L** $\dfrac{6\frac{3}{5}}{\frac{1}{100}}$

2. Simplify each complex fraction.

A $\dfrac{\frac{1}{2}}{4}$ **B** $\dfrac{2}{\frac{3}{4}}$ **C** $\dfrac{\frac{1}{8}}{\frac{3}{4}}$ **D** $\dfrac{\frac{5}{6}}{\frac{1}{2}}$ **E** $\dfrac{2}{\frac{4}{9}}$ **F** $\dfrac{\frac{3}{10}}{\frac{5}{6}}$

G $\dfrac{1\frac{1}{2}}{8}$ **H** $\dfrac{1\frac{4}{15}}{\frac{1}{3}}$ **I** $\dfrac{8\frac{1}{6}}{7}$ **J** $\dfrac{9\frac{3}{4}}{\frac{3}{8}}$ **K** $\dfrac{4\frac{1}{3}}{1\frac{2}{3}}$ **L** $\dfrac{7\frac{3}{5}}{3\frac{4}{5}}$

SET 31

1. Find the sums.

A $4.36 + 1.53$

B $23.7 + 16.5$

C $0.65 + 0.39$

D $1.376 + 0.881$

E $1.5 + 2.3 + 9.4$

F $8.71 + 14.3 + 0.9$

G $13.7 + 1.09 + 8$

H $0.9 + 0.16 + 0.027$

I $854.1 + 3.46 + 14.007$

J $4.93 + 0.667 + 12.359$

K $235 + 18.7 + 0.46$

L $6.009 + 21.73 + 4.654$

M $0.83 + 0.965 + 0.3566$

N $5008 + 47.03 + 751.9$

O $90.2 + 9.02 + 0.902$

P $6793 + 4.788 + 54.17$

Q $5432 + 543.2 + 5.432$

R $0.909 + 0.009 + 0.999$

2. Find the differences.

A $3.78 - 1.45$

B $73.9 - 59.3$

C $4.35 - 1.99$

D $0.981 - 0.347$

E $3.25 - 2.8$

F $18.6 - 3.33$

G $21.7 - 6.45$

H $5 - 1.3$

I $0.09 - 0.031$

J $10 - 4.862$

K $45.1 - 0.394$

L $8.03 - 4.009$

M $155 - 79.6$

N $99 - 0.9$

O $10 - 0.91$

P $100 - 98.375$

Q $25 - 0.015$

R $9 - 0.0009$

S $40 - 0.001$

T $1000 - 0.001$

SET 32

1. Find the products.

A 3×0.9

B 0.5×0.7

C 0.03×0.6

D 0.6×0.007

E 0.05×0.09

F 0.02×0.008

G 0.006×0.4

H 0.05×0.0008

2. Find the products.

A 6.5×7

B 0.37×8

C 1.38×0.8

D 43.6×0.5

E 0.358×0.9

F 53×1.7

G 7.3×46

H 15.3×8.2

I 1.27×53

J 85.4×6.5

K 0.357×4.2

L 1.68×0.77

M 935×0.08

N 47.9×0.65

O 0.036×0.05

P 435×0.007

Q 67.3×2.17

R 8.55×4.03

S 0.999×6.75

T 1.078×0.937

SET 33

1. Find the products.

A 10×7.1	**F** 100×3.1	**K** 1000×32	**P** 10×0.0036
B 10×0.38	**G** 100×8.73	**L** 1000×8.7	**Q** 100×0.809
C 10×1.25	**H** 100×13.9	**M** 1000×1.39	**R** 100×0.0023
D 10×0.904	**I** 100×0.375	**N** 1000×0.765	**S** 1000×0.038
E 10×0.003	**J** 100×0.004	**O** 1000×0.069	**T** 1000×0.70004

2. Find the products.

A 0.1×26	**F** 0.01×412	**K** 0.001×2567	**P** 0.1×0.097
B 0.1×4.9	**G** 0.01×11.9	**L** 0.001×453.1	**Q** 0.01×0.033
C 0.1×124	**H** 0.01×1.635	**M** 0.001×85.7	**R** 0.01×0.5503
D 0.1×1.69	**I** 0.01×97.04	**N** 0.001×3.76	**S** 0.001×0.018
E 0.1×0.43	**J** 0.01×0.67	**O** 0.001×0.96	**T** 0.0001×23.45

SET 34

1. Find the quotients.

A $3\overline{)25.59}$	**D** $31\overline{)1.209}$	**G** $1.7\overline{)14.62}$	**J** $0.92\overline{)0.05796}$
B $8\overline{)4.256}$	**E** $0.6\overline{)3.96}$	**H** $0.68\overline{)0.884}$	**K** $3.21\overline{)205.44}$
C $23\overline{)11.04}$	**F** $0.8\overline{)75.2}$	**I** $0.53\overline{)0.4717}$	**L** $0.801\overline{)0.34443}$

2. Find the quotients. Annex zeros when necessary.

A $6\overline{)0.882}$	**F** $4\overline{)5.1}$	**K** $9.2\overline{)30.82}$	**P** $96\overline{)7.2}$
B $0.8\overline{)18.48}$	**G** $0.8\overline{)2.6}$	**L** $0.44\overline{)134.2}$	**Q** $3.2\overline{)3}$
C $5\overline{)25.7}$	**H** $0.5\overline{)6}$	**M** $0.09\overline{)36}$	**R** $0.99\overline{)4554}$
D $0.19\overline{)1.197}$	**I** $0.12\overline{)6.6}$	**N** $6.5\overline{)1976}$	**S** $0.82\overline{)3095.5}$
E $8.1\overline{)2.835}$	**J** $0.36\overline{)9}$	**O** $1.6\overline{)7}$	**T** $0.104\overline{)4.68}$

SET 35

1. Give the number for n. Then give the correct decimal.

A $\frac{2}{5} = \frac{n}{10} = ?$

B $\frac{9}{20} = \frac{n}{100} = ?$

C $\frac{37}{50} = \frac{n}{100} = ?$

D $\frac{12}{25} = \frac{n}{100} = ?$

E $\frac{37}{200} = \frac{n}{1000} = ?$

F $\frac{49}{500} = \frac{n}{1000} = ?$

G $\frac{93}{250} = \frac{n}{1000} = ?$

H $\frac{71}{125} = \frac{n}{1000} = ?$

I $\frac{7}{40} = \frac{n}{1000} = ?$

J $\frac{124}{125} = \frac{n}{1000} = ?$

K $\frac{5}{8} = \frac{n}{1000} = ?$

L $\frac{1}{400} = \frac{n}{10\,000} = ?$

2. Write a decimal for each fraction.

A $\frac{1}{2}$	**E** $\frac{1}{20}$	**I** $\frac{3}{5}$	**M** $\frac{9}{25}$	**Q** $\frac{1}{40}$
B $\frac{1}{4}$	**F** $\frac{1}{25}$	**J** $\frac{3}{4}$	**N** $\frac{37}{50}$	**R** $\frac{1}{200}$
C $\frac{1}{5}$	**G** $\frac{1}{50}$	**K** $\frac{3}{10}$	**O** $\frac{93}{100}$	**S** $\frac{1}{250}$
D $\frac{1}{10}$	**H** $\frac{1}{100}$	**L** $\frac{17}{20}$	**P** $\frac{79}{200}$	**T** $\frac{1}{500}$

3. Find the quotients. Continue dividing until the remainder is zero.

A $\frac{3}{4} \rightarrow 4\overline{)3} = ?$

B $\frac{4}{5} \rightarrow 5\overline{)4} = ?$

C $\frac{1}{8} \rightarrow 8\overline{)1} = ?$

D $\frac{17}{20} \rightarrow 20\overline{)17} = ?$

E $\frac{19}{25} \rightarrow 25\overline{)19} = ?$

F $\frac{11}{40} \rightarrow 40\overline{)11} = ?$

G $\frac{5}{16} \rightarrow 16\overline{)5} = ?$

H $\frac{17}{80} \rightarrow 80\overline{)17} = ?$

I $\frac{9}{16} \rightarrow 16\overline{)9} = ?$

J $\frac{37}{40} \rightarrow 40\overline{)37} = ?$

K $\frac{63}{80} \rightarrow 80\overline{)63} = ?$

L $\frac{5}{32} \rightarrow 32\overline{)5} = ?$

4. Write a decimal for each fraction.

A $\frac{4}{5}$	**E** $\frac{5}{8}$	**I** $\frac{15}{4}$	**M** $\frac{325}{200}$	**Q** $\frac{7}{200}$
B $\frac{7}{20}$	**F** $\frac{3}{16}$	**J** $\frac{93}{50}$	**N** $\frac{26}{80}$	**R** $\frac{15}{16}$
C $\frac{8}{25}$	**G** $\frac{43}{250}$	**K** $\frac{27}{25}$	**O** $\frac{21}{125}$	**S** $\frac{17}{32}$
D $\frac{17}{50}$	**H** $\frac{11}{40}$	**L** $\frac{87}{500}$	**P** $\frac{90}{150}$	**T** $\frac{1}{64}$

SET 36

1. Write a mixed decimal numeral for each number in **tenths**.

A $\frac{2}{3}$ **B** $\frac{3}{8}$ **C** $\frac{1}{6}$ **D** $\frac{11}{15}$ **E** $\frac{7}{40}$ **F** $\frac{65}{75}$

2. Write a mixed decimal numeral for each number in **hundredths**.

A $\frac{5}{8}$ **B** $\frac{1}{3}$ **C** $\frac{7}{16}$ **D** $\frac{7}{12}$ **E** $\frac{1}{15}$ **F** $\frac{3}{40}$

3. Write a mixed decimal numeral for each number in **thousandths**.

A $\frac{1}{6}$ **B** $\frac{7}{11}$ **C** $\frac{11}{12}$ **D** $\frac{27}{80}$ **E** $\frac{4}{15}$ **F** $\frac{1}{600}$

4. Write a fraction for each mixed decimal numeral.

A $0.1\frac{1}{2}$ **B** $0.2\frac{1}{4}$ **C** $0.25\frac{1}{2}$ **D** $0.06\frac{2}{3}$ **E** $0.333\frac{1}{3}$ **F** $0.007\frac{1}{2}$

SET 37

1. Represent each number in scientific notation.

A 600	**G** 72 000	**M** 4 735 000	**S** 1 050 000
B 2000	**H** 10 500	**N** 125 000 000	**T** 948 000 000
C 40 000	**I** 600 000	**O** 5 100 000	**U** 915 000 000
D 8100	**J** 210 000	**P** 9 750 000	**V** 4 069 000 000
E 620	**K** 8 300 000	**Q** 2 603 000	**W** 6 705 000 000
F 4370	**L** 23 000 000	**R** 48 900 000	**X** 400 800 000 000

2. Write each numeral in the usual way.

A 3×10^2	**E** 3.8×10^3	**I** 3.75×10^2	**M** 1.53×10^7
B 9×10^4	**F** 4.7×10^4	**J** 6.03×10^3	**N** 4.006×10^9
C 6×10^5	**G** 1.1×10^6	**K** 9.99×10^5	**O** 3.49×10^8
D 1.2×10^2	**H** 7.09×10^5	**L** 1.106×10^8	**P** 8.3×10^{12}

SET 38

1. Solve the following proportions for n.

A $\frac{2}{3} = \frac{n}{18}$

B $\frac{4}{5} = \frac{20}{n}$

C $\frac{1}{6} = \frac{7}{n}$

D $\frac{2}{n} = \frac{8}{20}$

E $\frac{5}{1} = \frac{n}{17}$

F $\frac{6}{n} = \frac{16}{24}$

G $\frac{6}{15} = \frac{n}{40}$

H $\frac{n}{12} = \frac{10}{15}$

I $3:4 = n:36$

J $5:6 = 20:n$

K $2:n = 10:35$

L $6:1 = n:13$

M $n:8 = 21:28$

N $7:10 = 140:n$

O $n:5 = 320:200$

P $4:3 = 180:n$

3. Write a proportion for each problem and then solve it.

A Baseball player: 3 hits for every 8 times at bat. How many hits for 96 times at bat?

B Car: 6 kilometres driven per litre of gasoline. How many kilometres driven on 37 litres?

C Basketball player: 7 free throws out of every 9 attempts. How many free throws in 135 attempts?

D Picnic: 2 hot dogs per person. How many hot dogs needed to feed 126 people?

E Class: ratio of girls to boys is 7 to 9. If there are 56 girls, how many boys are there?

F Population: ratio of City X to City Y is 6 to 5. If 3000 people live in City Y, how many in City X?

G Scale on map: 1 cm = 35 km. What is the actual distance if the distance on the map is 4.9 cm?

H Recipe: 5 grams salt per litre of water. How many grams of salt needed for 1 kilolitre of water?

SET 39

1. Write a decimal for each percent.

A 17%	**E** 34%	**I** 10%	**M** 100%	**Q** 105%	**U** 700%
B 28%	**F** 40%	**J** 9%	**N** 200%	**R** 5%	**V** 70%
C 85%	**G** 11%	**K** 2%	**O** 20%	**S** 50%	**W** 7%
D 31%	**H** 99%	**L** 1%	**P** 2%	**T** 15%	**X** 0.7%

2. Write a percent for each decimal.

A 0.29	**E** 0.12	**I** 0.6	**M** 0.3	**Q** 0.18	**U** 0.2
B 0.66	**F** 0.08	**J** 1.2	**N** 0.03	**R** 1.8	**V** 10
C 0.13	**G** 0.02	**K** 1	**O** 3	**S** 18	**W** 0.43
D 0.47	**H** 0.01	**L** 2	**P** 30	**T** 1.08	**X** 0.05

3. Write a lowest-terms fraction for each percent.

A 57%	**E** 25%	**I** 5%	**M** 15%	**Q** 80%	**U** 140%
B 50%	**F** 20%	**J** 4%	**N** 16%	**R** 8%	**V** 175%
C 30%	**G** 10%	**K** 2%	**O** 17%	**S** 800%	**W** 44%
D 31%	**H** 6%	**L** 1%	**P** 18%	**T** 88%	**X** 135%

4. Write a percent for each fraction.

A $\frac{37}{100}$	**E** $\frac{7}{10}$	**I** $\frac{1}{5}$	**M** $\frac{3}{5}$	**Q** $\frac{7}{20}$	**U** $\frac{21}{20}$
B $\frac{9}{100}$	**F** $\frac{1}{10}$	**J** $\frac{1}{50}$	**N** $\frac{1}{25}$	**R** $\frac{9}{25}$	**V** $\frac{4}{25}$
C $\frac{7}{100}$	**G** $\frac{1}{2}$	**K** $\frac{3}{4}$	**O** $\frac{7}{50}$	**S** $\frac{13}{50}$	**W** $\frac{7}{5}$
D $\frac{1}{100}$	**H** $\frac{1}{4}$	**L** $\frac{1}{20}$	**P** $\frac{49}{50}$	**T** $\frac{24}{25}$	**X** $\frac{11}{4}$

SET 40

1. Find the percent of each number.

A 45% of 100	**E** 10% of 340	**I** 1% of 43	**M** 400% of 35
B 50% of 64	**F** 29% of 200	**J** 5% of 840	**N** 125% of 240
C 25% of 80	**G** 43% of 540	**K** 3% of 90	**O** 93% of 682
D 20% of 45	**H** 11% of 950	**L** 110% of 950	**P** 81% of 1209

2. Find the products.

A 28% of 450	**E** 125% of 200	**I** 43.5% of 400	**M** $7\frac{1}{2}$% of 400
B 83% of 50	**F** 12.5% of 200	**J** 8.7% of 65	**N** $15\frac{1}{10}$% of 360
C 300% of 79	**G** 1.25% of 200	**K** 11.8% of 90	**O** $6\frac{1}{4}$% of 200
D 175% of 48	**H** 0.125% of 200	**L** 4.35% of 340	**P** $5\frac{3}{4}$% of 600

3. Solve the problems.

A Price of stereo: $75
Sales tax: 4%
Total price of stereo?

B Earnings: $135 per week
Deductions: 28%
How much take-home pay?

C Amount of money: $350
Rate of interest: $6\frac{1}{2}$%
Amount of interest?

D Monthly bill: $46.10
Service charge: $1\frac{1}{2}$%
What is the total amount of bill?

SET 41

1. Find the number for each ▒

A 7 is ▒% of 10	**F** 16 is ▒% of 20	**K** 8 is ▒% of 24
B 5 is ▒% of 20	**G** 9 is ▒% of 15	**L** 25 is ▒% of 40
C 38 is ▒% of 100	**H** 45 is ▒% of 150	**M** 10 is ▒% of 16
D 9 is ▒% of 12	**I** 18 is ▒% of 90	**N** 120 is ▒% of 80
E 28 is ▒% of 56	**J** 35 is ▒% of 140	**O** 8 is ▒% of 200

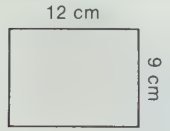
2. Find the percent the first number is of the second.
Round your answers to the nearest tenth of a percent.

A 4, 33	**C** 19, 21	**E** 18, 13	**G** 18, 75	**I** 83, 99
B 9, 26	**D** 15, 27	**F** 25, 28	**H** 98, 150	**J** 92, 101

SET 42

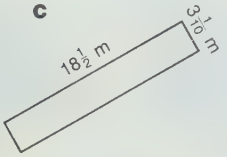
1. Find the area of each rectangular region. Use the formula $A = l \cdot w$.

A



B

C

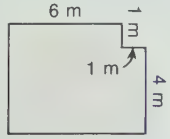


D $l = 600$ mm
$w = 150$ mm

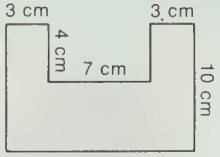
E $l = 8\frac{1}{2}$ km
$w = 4\frac{7}{10}$ km

2. Find the area of each region.

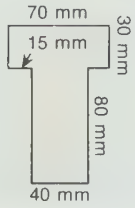
A



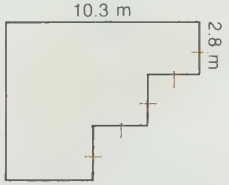
B



C

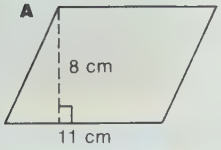


★ **D**

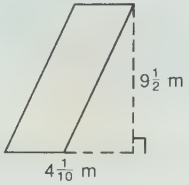


3. Find the area of each parallelogram. Use the formula $A = b \cdot h$.

A



B



C

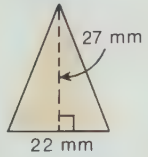
D $b = 18$ m
$h = 3\frac{1}{2}$ m

E $b = 15\frac{3}{10}$ m
$h = 8\frac{1}{5}$ m

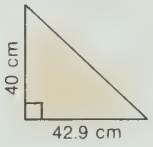
SET 43

1. Find the area of each triangular region. Use the formula $A = \frac{1}{2} \cdot b \cdot h$.

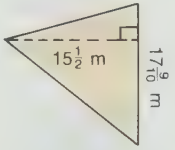
A



B



C



D

E $b = 101$ cm
$h = 46$ cm

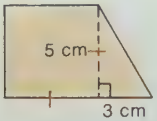
F $b = 3.8$ m
$h = 5.2$ m

G $b = 15\frac{7}{10}$ cm
$h = 3\frac{1}{2}$ cm

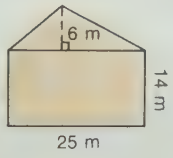
H $b = 450$ mm
$h = 860$ mm

2. Find the area of each region.

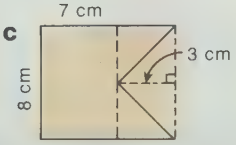
A



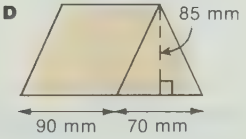
B



C

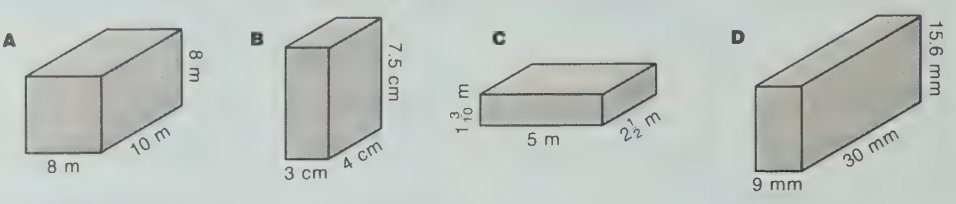


D



SET 44

1. Find the volume of each rectangular solid. Use the formula $V = l \cdot w \cdot h$.



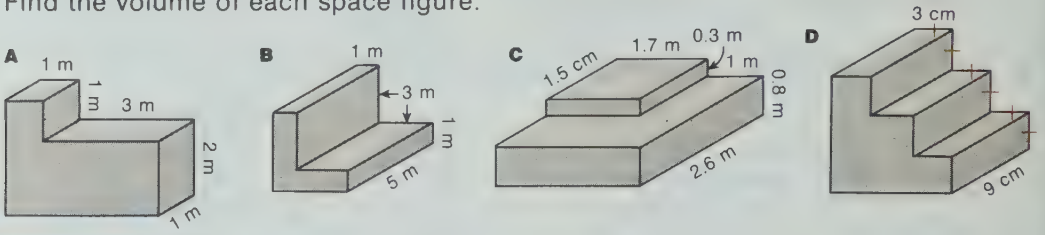
E $l = 9$ cm
$w = 5$ cm
$h = 12$ cm

F $l = 40$ m
$w = 40$ m
$h = 25$ m

G $l = 8.3$ cm
$w = 4.6$ cm
$h = 6.23$ cm

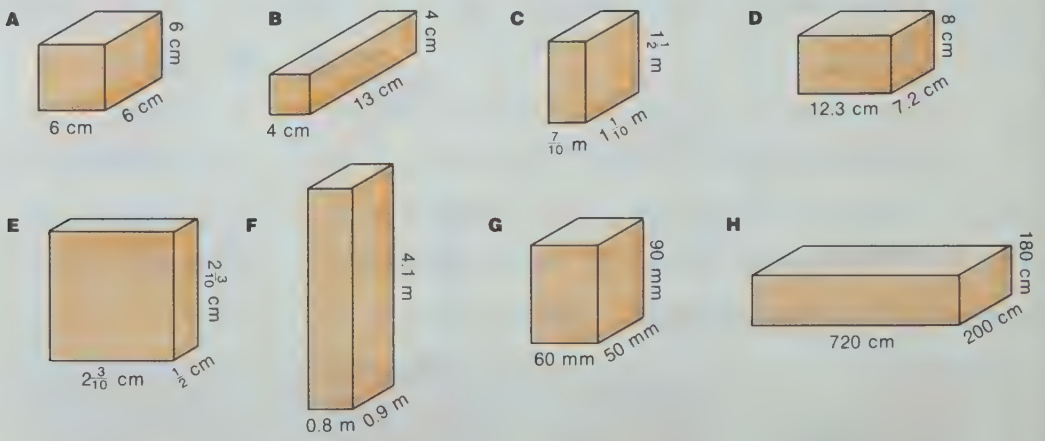
H $l = 8\frac{7}{10}$ m
$w = 10$ m
$h = 4\frac{1}{5}$ m

2. Find the volume of each space figure.



SET 45

1. Find the surface area of each space figure.



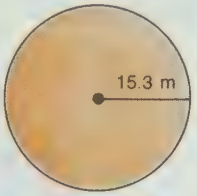
2. Find the surface area of each space figure in Exercise 2 in Set 44 above.

SET 46

1. Find the circumference of each circle. Use the formula $C = \pi \cdot d$ and $\pi \approx 3.14$.

A

B



C $d = 105$ mm

D $r = 45$ cm

E $d = 8.35$ m

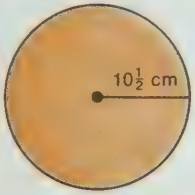
F $r = 166$ mm

G $r = 0.9$ km

H $d = 4.26$ m

2. Use the fact $\pi \approx 3\frac{1}{7}$ to find the circumference of each circle.

A



B

C $r = 5$ cm

D $d = 70$ mm

E $r = 17\frac{1}{2}$ m

F $d = 490$ mm

G $d = 4\frac{9}{10}$ m

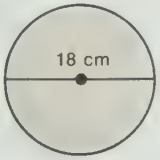
H $d = 6\frac{3}{10}$ cm

SET 47

1. Find the area of each circular region. Use the formula $A = \pi \cdot r^2$ and $\pi \approx 3.14$.

A

B



C

D $r = 30$ mm

E $d = 1.6$ m

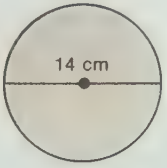
F $r = 41$ cm

G $d = 210$ mm

2. Find the area of each circular region. Use the fact $\pi \approx 3\frac{1}{7}$.

A

B



C

D $r = 4$ cm

E $d = 20$ m

F $r = \frac{6}{10}$ km

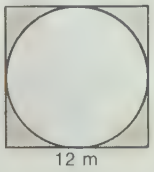
G $r = 5\frac{1}{2}$ m

3. Find the area of the shaded portion of each region.

A

B

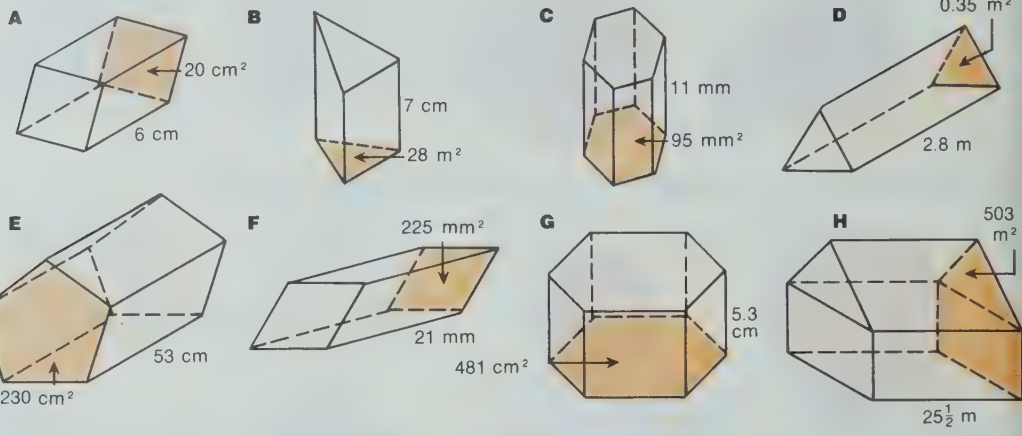
★ **C**



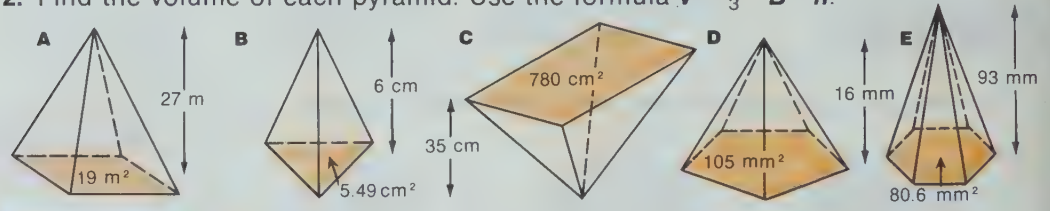
★ **D**

SET 48

1. Find the volume of each prism. Use the formula $V = B \cdot h$.

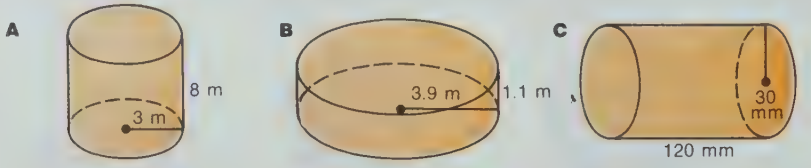


2. Find the volume of each pyramid. Use the formula $V = \frac{1}{3} \cdot B \cdot h$.



SET 49

1. Find the volume of each circular cylinder. Use the formula $V = \pi \cdot r^2 \cdot h$.

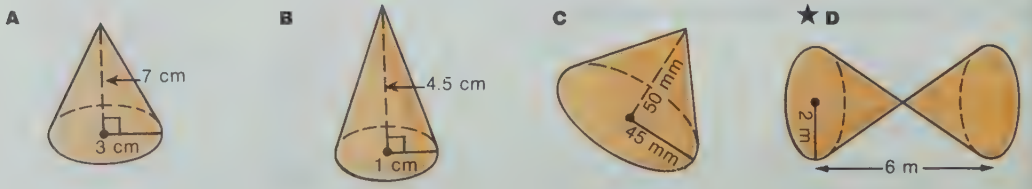


D $r = 50$, $h = 37$

E $r = 0.1$, $h = 0.6$

F $r = \frac{7}{10}$, $h = 1\frac{1}{2}$

2. Find the volume of each circular cone. Use the formula $V = \frac{1}{3} \cdot \pi \cdot r^2 \cdot h$.



SET 50

1. Find the sums.

A $7 + {}^{-}5$

B ${}^{-}8 + {}^{-}6$

C ${}^{-}9 + 3$

D $12 + {}^{-}15$

E ${}^{-}10 + 4$

F $0 + {}^{-}13$

G $21 + {}^{-}16$

H $21 + 16$

I ${}^{-}21 + 16$

J ${}^{-}21 + {}^{-}16$

K $21 + {}^{-}8 + 8$

L ${}^{-}21 + {}^{-}8 + {}^{-}8$

M $75 + {}^{-}53$

N $49 + {}^{-}53$

O $75 + {}^{-}49$

P ${}^{-}75 + 53$

Q ${}^{-}53 + 49$

R $53 + {}^{-}49$

S $125 + {}^{-}99$

T ${}^{-}82 + 18$

2. Find the sums.

A $5 + 5 + 5$

B $5 + 5 + {}^{-}5$

C $5 + {}^{-}5 + {}^{-}5$

D ${}^{-}5 + {}^{-}5 + {}^{-}5$

E $27 + {}^{-}13 + 7$

F $100 + ({}^{-}23 + 43)$

G $(45 + {}^{-}18) + {}^{-}25$

H $35 + {}^{-}15 + 23$

I ${}^{-}35 + {}^{-}15 + 23$

J ${}^{-}35 + 15 + {}^{-}23$

K ${}^{-}35 + {}^{-}15 + {}^{-}23$

L $35 + {}^{-}15 + {}^{-}23$

M $18 + 17 + {}^{-}15 + 23$

N ${}^{-}18 + 17 + 15 + {}^{-}23$

O $185 + {}^{-}37 + {}^{-}8$

P ${}^{-}250 + ({}^{-}126 + 106)$

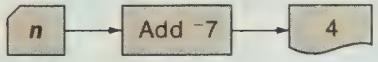
Q $467 + ({}^{-}310 + {}^{-}25)$

R ${}^{-}755 + (825 + {}^{-}57)$

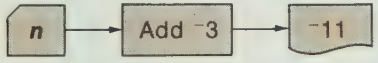
SET 51

1. Find the number n in each flow chart.

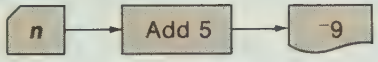
A



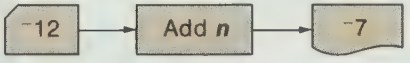
B



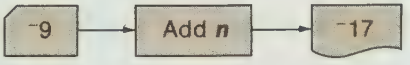
C



D



E



F

2. Solve the equations.

A $n + {}^{-}5 = 15$

B $n + {}^{-}14 = 0$

C $n + 5 = {}^{-}7$

D $n + 6 = 3$

E $n + 8 = {}^{-}8$

F $n + 6 = 12$

G $n + {}^{-}6 = {}^{-}12$

H $n + 6 = {}^{-}12$

I $n + {}^{-}6 = 12$

J $n + {}^{-}6 = 0$

K $14 + n = 9$

L $18 + n = 23$

M ${}^{-}15 + n = 2$

N $n + {}^{-}25 = {}^{-}28$

O ${}^{-}13 + n = {}^{-}34$

P $n + {}^{-}53 = 26$

Q $79 + n = {}^{-}10$

R $n + 46 = {}^{-}12$

S ${}^{-}55 + n = 100$

T $n + 43 = {}^{-}43$

SET 52

1. Solve the equations. Think about missing addends.

S A A	S A A	S A A	S A A
A $0 - 5 = n$	D $0 - {}^{-}5 = n$	G $^{-}3 - {}^{-}5 = n$	J $5 - {}^{-}5 = n$
B $3 - 5 = n$	E $3 - {}^{-}5 = n$	H $^{-}8 - {}^{-}5 = n$	K $^{-}5 - {}^{-}5 = n$
C $8 - 5 = n$	F $8 - {}^{-}5 = n$	I $5 - 5 = n$	L $^{-}5 - 5 = n$

2. Find the differences.

A $15 - 8$	G $0 - 27$	M $38 - {}^{-}42$	S $401 - 189$
B $8 - 15$	H $0 - {}^{-}13$	N $^{-}56 - 4$	T $529 - {}^{-}125$
C $15 - {}^{-}8$	I $43 - 77$	O $^{-}73 - {}^{-}27$	U $^{-}630 - {}^{-}575$
D $^{-}15 - 8$	J $24 - {}^{-}24$	P $44 - {}^{-}45$	V $193 - {}^{-}87$
E $^{-}15 - {}^{-}8$	K $^{-}24 - 24$	Q $^{-}65 - 3$	W $^{-}153 - 153$
F $8 - {}^{-}15$	L $^{-}24 - {}^{-}24$	R $99 - 199$	X $495 - 495$

SET 53

1. Find the products.

A $5 \cdot {}^{-}9$	E $^{-}1 \cdot {}^{-}9$	I $^{-}10 \cdot {}^{-}43$	M $21 \cdot {}^{-}21$	Q $^{-}125 \cdot 1000$
B $9 \cdot {}^{-}5$	F $^{-}3 \cdot 11$	J $10 \cdot {}^{-}67$	N $^{-}21 \cdot {}^{-}21$	R $^{-}1001 \cdot 1$
C $0 \cdot {}^{-}9$	G $5 \cdot {}^{-}18$	K $100 \cdot {}^{-}6$	O $^{-}100 \cdot 23$	S $29 \cdot {}^{-}36$
D $^{-}1 \cdot 9$	H $^{-}10 \cdot 17$	L $14 \cdot {}^{-}100$	P $^{-}100 \cdot {}^{-}56$	T $^{-}53 \cdot {}^{-}18$

2. Find the products.

A $2 \cdot 5 \cdot {}^{-}4$	I $10 \cdot {}^{-}10 \cdot 10$	Q $12 \cdot {}^{-}3 \cdot {}^{-}4$
B $2 \cdot {}^{-}5 \cdot {}^{-}4$	J $^{-}10 \cdot {}^{-}10 \cdot {}^{-}10$	R $^{-}21 \cdot {}^{-}3 \cdot {}^{-}3$
C $^{-}2 \cdot {}^{-}5 \cdot {}^{-}4$	K $^{-}10 \cdot {}^{-}10 \cdot 10$	S $^{-}1 \cdot 12 \cdot 0 \cdot {}^{-}8$
D $3 \cdot 15 \cdot {}^{-}1$	L $^{-}10 \cdot {}^{-}10 \cdot {}^{-}100$	T $4 \cdot 10 \cdot {}^{-}1 \cdot {}^{-}10$
E $8 \cdot {}^{-}6 \cdot {}^{-}1$	M $^{-}4 \cdot {}^{-}3 \cdot 8$	U $^{-}63 \cdot {}^{-}2 \cdot 1 \cdot {}^{-}1$
F $12 \cdot {}^{-}5 \cdot 2$	N $^{-}7 \cdot {}^{-}11 \cdot {}^{-}1$	V $^{-}6 \cdot ({}^{-}15 \cdot {}^{-}8 \cdot {}^{-}1)$
G $^{-}1 \cdot 10 \cdot 10$	O $5 \cdot {}^{-}25 \cdot 2$	W $({}^{-}7 \cdot {}^{-}9) \cdot ({}^{-}1 \cdot {}^{-}5)$
H $^{-}1 \cdot 10 \cdot {}^{-}10$	P $^{-}1 \cdot 43 \cdot {}^{-}10$	X $^{-}1 \cdot ({}^{-}5 \cdot {}^{-}2) \cdot ({}^{-}4 \cdot 3)$

SET 54

1. Solve the equations.

A $^{-}3 \cdot n = {}^{-}12$

B $5 \cdot n = {}^{-}35$

C $n \cdot 6 = {}^{-}24$

D $n \cdot {}^{-}5 = 45$

E $4 \cdot n = 44$

F $n \cdot 6 = {}^{-}54$

G $n \cdot {}^{-}8 = 72$

H $^{-}5 \cdot n = {}^{-}100$

I $5 \cdot n = {}^{-}100$

J $^{-}5 \cdot n = 100$

K $83 \cdot n = {}^{-}83$

L $^{-}5 \cdot n = {}^{-}65$

M $10 \cdot n = {}^{-}140$

N $n \cdot {}^{-}8 = 120$

O $^{-}11 \cdot n = 88$

P $4 \cdot {}^{-}3 \cdot n = 60$

Q $^{-}1 \cdot 14 \cdot n = 70$

R $^{-}2 \cdot {}^{-}5 \cdot n = 190$

S $n \cdot {}^{-}3 \cdot 8 = 120$

T $n \cdot n = 49$

2. Find the quotients.

A $^{-}21 \div 7$

B $^{-}27 \div {}^{-}3$

C $^{-}42 \div {}^{-}6$

D $54 \div {}^{-}9$

E $^{-}48 \div {}^{-}16$

F $^{-}12 \div {}^{-}12$

G $0 \div {}^{-}9$

H $^{-}45 \div {}^{-}3$

I $^{-}25 \div {}^{-}1$

J $63 \div {}^{-}7$

K $^{-}72 \div 6$

L $43 \div {}^{-}43$

M $81 \div {}^{-}9$

N $60 \div {}^{-}4$

O $36 \div {}^{-}9$

P $^{-}88 \div {}^{-}11$

Q $^{-}75 \div {}^{-}25$

R $90 \div {}^{-}18$

S $100 \div {}^{-}10$

T $^{-}300 \div 15$

U $^{-}144 \div {}^{-}6$

V $256 \div {}^{-}8$

W $^{-}128 \div 16$

X $^{-}216 \div 216$

Y $^{-}579 \div {}^{-}3$

SET 55

1. Find the average (arithmetic mean) of each set of numbers.

A 23, 26, 29

B 63, 57, 72

C 87, 82, 69, 98

D 53, 68, 79, 64

E 879, 904, 946, 891

F 125, 99, 132, 120, 124

G 4301, 4078, 3996, 4125

H 209, 205, 199, 241, 238, 216

I 10.4, 9.9, 7.7, 8.2, 9.1, 9.5

J 1.59, 1.52, 1.76, 1.62, 1.56

2. Find the average of each set of number to the nearest whole number.

A 38, 41, 37

B 50, 49, 61, 55

C 93, 97, 85, 104

D 208, 210, 212, 214, 215

E 398, 406, 421

F 1537, 1302, 1496, 1388

G 1001, 1003, 1004, 999, 1011

H 250, 260, 180, 190, 360, 250

I 697, 588, 634, 596, 619

J 4873, 5370, 5093, 4961

SET 56

1. Find the median of each set of numbers.

A 27, 29, 35, 37, 37

B 26, 29, 35, 37

C 43, 49, 53, 51, 54

D 397, 405, 399, 401

E 87, 88, 89, 92, 94, 93

F 5, 6, 6, 7, 8, 9, 9, 10, 12, 12, 15

G 43, 45, 48, 50, 53, 54, 54, 57, 58

H 17, 20, 21, 19, 18, 17, 24, 26, 22

I 39, 40, 41, 39, 38, 43, 45, 47, 51, 46

J 3.83, 4.61, 3.52, 4.89, 4.05, 3.99

2. Solve the problems.

A Test scores:
79, 88, 93, 87, 100, 86, 94
What is the median score?

B Annual incomes:
$9100, $7200, $10 000,
$8500, $9600
What is the median income?

C Refer to the chart.
What is the median height
of the girls?
of the boys?
of all students?

Student Heights (metres)			
Girls		Boys	
1.50	1.61	1.55	1.67
1.51	1.54	1.60	1.74
1.48	1.64	1.57	1.82
1.56	1.51	1.64	1.77
1.54	1.57	1.73	1.69

SET 57

1. Find the mode(s) of each set of numbers.

A

Die Tossing Outcomes			
6	3	4	6
5	4	3	5
4	2	5	4
5	5	4	3
6	1	3	4

B

Test Scores		
85	93	79
79	91	86
78	73	85
86	80	89
93	86	93

C

Students' Ages			
13	14	14	14
12	15	12	14
14	12	13	13
13	13	12	12
14	14	12	15

2. Refer to the chart to answer parts **A**, **B**, and **C**.

A Find the mode.

B Find the median.

C Find the arithmetic mean.

Men's Masses (kilograms)			
72.6	68.1	93.1	79.4
74.9	79.4	99.8	86.3
70.4	86.3	74.9	90.8
81.7	88.5	83.9	74.9
74.9	79.4	90.8	72.6

Mathematical Symbols

$=$	is equal to
$\neq$	Is not equal to
$>$	Is greater than
$<$	Is less than
$\leq$	Is less than or equal to
$\approx$	Is approximately
$\cong$	Is congruent to
$\sim$	Is similar to
$\{a, b, c\}$	Set with elements a, b, c
ϕ	Empty set
$\cup$	Union
$\cap$	Intersection

$\overleftrightarrow{AB}$	Line through points A and B
$\overrightarrow{AB}$	Ray AB with endpoint A
$\overline{AB}$	Segment with endpoints A and B
$\angle ABC$	Angle ABC with vertex B
$m\angle ABC$	Measure of angle ABC
$\triangle ABC$	Triangle ABC
$\overset{\frown}{RS}$	Arc with endpoints R and S
$35^\circ\ 14'\ 20''$	Thirty-five *degrees*, fourteen *minutes*, twenty *seconds*
%	Percent
π	Pi

Metric System Prefixes

kilo-	one thousand
hecto-	one hundred
deca-	ten
deci-	one tenth
centi-	one hundredth
milli-	one thousandth

Formulae

$P = a + b + c$	Perimeter of triangle
$P = 2 \cdot l + 2 \cdot w$	Perimeter of rectangle
$A = l \cdot w$	Area of rectangle
$A = b \cdot h$	Area of parallelogram
$A = \frac{1}{2} \cdot b \cdot h$	Area of triangle
$a^2 + b^2 = c^2$	Pythagorean Theorem
$V = l \cdot w \cdot h$	Volume of rectangular solid
$C = \pi \cdot d$	Circumference of circle
$A = \pi \cdot r^2$	Area of circle
$V = \frac{4}{3} \cdot \pi \cdot r^3$	Volume of sphere
$V = \pi \cdot r^2 \cdot h$	Volume of cylinder
$V = \frac{1}{3} \cdot \pi \cdot r^2 h$	Volume of cone
$V = B \cdot h$	Volume of prism (B = base area)
$V = \frac{1}{3} \cdot B \cdot h$	Volume of pyramid

GLOSSARY

abacus A device used for calculating, usually involving sliding beads or counters along a wire.

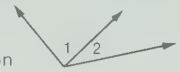
acute angle An angle smaller than a right angle.

addend Any one member of a set of numbers to be added. In the equation 7 + 9 = 16, 7 and 9 are addends.

addition An operation that combines a first number (*addend*) and a second number (*addend*) to give another number (this number is called the *sum* of the two addends).

additive inverse Each of two numbers whose sum is zero is said to be the additive inverse of the other.

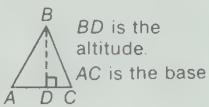
adjacent angles Two angles with a common vertex, a common side, and no common interior points. In the figure, $\angle 1$ and $\angle 2$ are adjacent angles.



adjacent sides Two sides of a polygon with a common vertex.

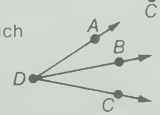
algorithm (algorism) A computation procedure. For example, the algorithm for finding the quotient of two numbers.

altitude (of a triangle) The segment from any vertex perpendicular to the opposite side. Also, see *height*.



angle. The union of two rays with a common endpoint (vertex).

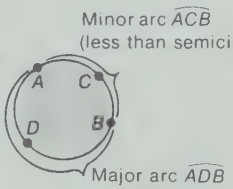
angle bisector The ray which divides an angle into two congruent angles. In the figure $\overrightarrow{DB}$ bisects $\angle ADC$.



approximation A number that is suitably "close" to another number. For example, a decimal approximation for $\frac{2}{9}$ is 0.22. The symbol "$\approx$" denotes "is approximately"; for example $\pi \approx 3.14$.

are (pronounced "air") A unit of area. 100 square metres

arc A part of a circle. Note that three points, instead of two, are used to denote a specific arc since $\overset{\frown}{AB}$ could denote a *minor arc* $\overset{\frown}{AB}$ or a *major arc* $\overset{\frown}{ADB}$.



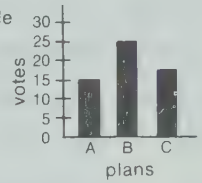
area The measure of a plane region in terms of a chosen unit, usually a square.

arithmetic mean The quotient of the sum of numbers in a set divided by the number of numbers in a set. (Often called the "average".)

associative (grouping) principle The principle which states that the sum (or product) of three or more numbers is the same regardless of grouping:
$(a + b) + c = a + (b + c)$
or $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

average See *arithmetic mean*.

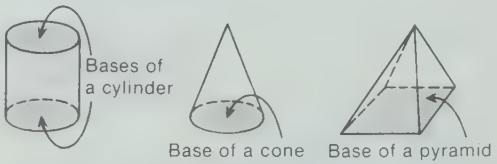
bar graph A diagram made of graduated markings and rectangular bars to show statistical information, usually about specifically stated quantities.



base (of numeration) The term "base" refers to the type of grouping involved in a system of numeration. For example, in base eight: $25_{(8)}$ means 2 eights and 5 or $346_{(8)}$ means 3 sixty-fours, 4 eights and 6.

base (of a polygon) Any side of a polygon may be referred to as a base. See figure for *altitude of a triangle*.

base (of a space figure) See examples below.



bisect To divide into two congruent parts.

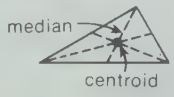
centre A (fixed) point of symmetry for a geometric figure. See the figure for *circle*.

centi- A prefix meaning *one hundredth*.

centimetre A unit of length that is $\frac{1}{100}$ of a metre.

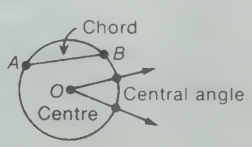
central angle An angle with its vertex at the centre of the circle. See the figure for *circle*.

centroid (of a triangle) The point of intersection of the three medians of a triangle.

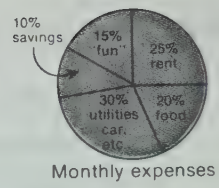


chord A segment with both endpoints on a circle. See the figure for *circle*.

circle The set of all points in a plane which are a specified distance from a fixed point called the *centre*.



circle graph A diagram used to show statistical information, particularly to show how a given quantity is divided into fractional parts.



circumcentre The point of intersection of the perpendicular bisectors of the sides of a triangle. It is the centre of the circle circumscribing the triangle.

circumference The distance around a circle. For each circle, the number is the product of its diameter and the number π. The formula is $C = \pi d$.

circumscribed circle A circle drawn through the vertices of a polygon. A circle can be circumscribed about any triangle but only about certain other polygons.

circumscribed polygon If the sides of a polygon are tangent to a circle, then the polygon is circumscribed about the circle. Also, see *inscribed circle*.

closed curve Intuitively, a curve is closed if you can begin at any point and trace the curve until you return to the starting point.

collinear (points) Two or more points lying on the same line.

common factor A number that is a factor of each of two or more given numbers. For example, 3 is a common factor of 6 and 9.

common multiple A number that is a multiple of each of two or more given numbers. For example, 18 is a common multiple of 2, 3, and 9.

commutative (order) principle The principle which states that the sum (or product) of any two numbers is the same regardless of the order in which they are added (or multipled):
$a \cdot b \quad b \cdot a$ or $a \cdot b \quad b \cdot a$

complementary angles Two angles whose measures have a sum of 90.

composite number Any whole number greater than 1 which is not prime. Composite numbers have more than two factors.

complex fraction A fraction that has a fraction for the numerator or the denominator, or both.

concave figure See *convex figure*.

cone A space figure formed by a closed plane curve and all line segments from a point not in the plane of the curve to all the points of the curve.

congruent angles Angles which have the same measure.

congruent figures Two figures are congruent if one of the figures is the image of the other by one of the *rigid motions* (reflection, rotation, translation) or by any combination of motions.

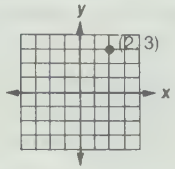
congruent segments Segments whose endpoints are equally far apart; segments which have the same length.

convex figure A figure is convex if every segment joining any two points on the boundary of the figure contains only points on the figure or in its interior.

convex figure

non-convex or **concave** figure

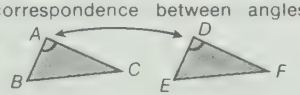
co-ordinate axes Two intersecting perpendicular number lines used for graphing ordered number pairs (*co-ordinates*).



co-ordinates An ordered number pair matched with a point in the co-ordinate plane. See the figure for *co-ordinate axes*.

correspondence A matching of elements in one set with those of another set.

corresponding angles Two angles matched by any one-to-one correspondence between angles of two triangles. For example, $\angle A$ and $\angle D$.

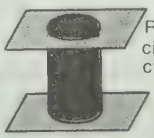


corresponding points Two points matched by any one-to-one correspondence between points of two triangles. For example, point C corresponds to point F above.

corresponding segments Two segments matched by any one-to-one correspondence between segments of two figures. For example, $\overline{AB}$ and $\overline{DE}$ are corresponding segments above.

cross section The intersection of a space figure and a plane.

cylinder The space figure formed by two congruent curves in parallel planes and the parallel segments connecting corresponding points of the curves.



Right circular cylinder

cube (geometry) A regular polyhedron each of whose six faces are squares.

cube (numeration) A number raised to the third power. 8 is the cube of 2 because $2^3 = 8$. Also, to raise a number to the third power.

deca- A prefix meaning *ten*.

decagon A 10-sided polygon.

decametre A unit of length that is 10 metres.

deci- A prefix meaning *one tenth*.

decimetre A unit of length that is $\frac{1}{10}$ of a metre, or 10 centimetres.

decimal (numeral) Any base-ten numeral written using a decimal point.

decimal point The dot that is used in a decimal numeral.

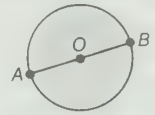
decimal system of numeration A system of numeration in which powers of ten and place value is used.

degree A unit angle that is $\frac{1}{90}$ of a right angle.

denominator For each fraction $\frac{a}{b}$, b is the denominator.

diagonal A segment connecting two nonconsecutive vertices of a polygon.

diameter Any chord that contains the centre of a circle.



difference The number resulting from subtraction.

digits The basic symbols used in a place-value system of numeration. In base ten, the symbols are 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.

distributive principle The fundamental principle connecting addition and multiplication. $a \cdot (b + c) = a \cdot b + a \cdot c$.

dividend The number to be divided in a division problem.

$$\begin{array}{r} 3 \leftarrow \text{Quotient} \\ 5\overline{)17} \leftarrow \text{Dividend} \\ \text{Divisor} \rightarrow 15 \\ \hline 2 \leftarrow \text{Remainder} \end{array}$$

divisible A given number is divisible by a second number if the remainder is zero. If $a = b \cdot c$ with $b \neq 0$ and $c \neq 0$, then a is divisible by either b or c.

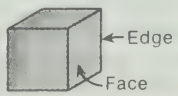
division The inverse of the operation of multiplication.

$$\begin{array}{ccc} F & F & P \\ 3 & \cdot\ 4 & 12 \end{array} \quad \begin{array}{l} P\ \ F\ \ F \\ 12 \div 4 = 3 \\ 12 \div 3 = 4 \end{array}$$

divisor See *dividend*.

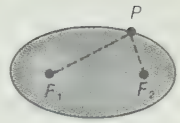
dodecahedron A polyhedron which has 12 faces. A regular dodecahedron has 12 congruent pentagonal faces.

edge A segment that is a side of any face of a space figure. A cube has 12 edges.



element A member of a set.

ellipse A closed plane curve generated by a point (P) moving in such a way that the sum of the distances from two fixed points (F_1 and F_2) is constant.



empty set The set containing no elements. It is denoted by "{ }" or the symbol ϕ.

equal sets Sets containing the same elements. They may be listed in different order.

equality The relation of identity. A statement of equality asserts that two symbols name the same object or number.

equally likely outcomes Outcomes or results that have the same chance of occurring.

equation A mathematical sentence using the equality symbol (=). $7 + n = 9$ is an equation.

equiangular triangle A triangle all of whose angles have the same measure.

equilateral triangle A triangle having all sides the same length.

equivalent decimals Decimals which represent the same number.

equivalent fractions Fractions which represent the same number such as $\frac{1}{2}$, $\frac{5}{10}$ or $\frac{50}{100}$.

equivalent ratios Two ratios which represent the same fractional number. For example, the ratio 5:10 is the same as 1:2.

equivalent sets Sets having the same number of elements.

estimate An approximation for a given number. Often used in the sense of a rough calculation.

even number A whole-number multiple of 2. The set of even numbers is $\{0, 2, 4, 6, \ldots\}$.

expanded notation A representation of a number as a sum of powers of ten such as: $3425 = 3 \cdot 10^3 + 4 \cdot 10^2 + 2 \cdot 10^1 + 5 \cdot 10^0$.

exponent A numeral written above and to the right of a mathematical expression to indicate how many times a number is to be used as a factor. For example, in the expression 5^4, the 4 indicates that 5 is to be used as a factor 4 times.

extremes The first and last terms in a proportion written in the form $a : b = c : d$. The numbers a and d are extremes and b and c are called the *means*.

face Any one of the bounding polygonal regions of a space figure. See the figure for *edge*.

factor Any one member of a set of numbers to be multiplied. In the equation $5 \cdot 7 = 35$, 5 and 7 are factors.

factor tree A diagram suggestive of a tree showing the prime factorization of a number.

flow chart A diagram which gives instructions in a logical order.

focal points See *ellipse*.

formula A general fact or rule expressed by using symbols. For example, the area any parallelogram with base b and height h is given by $b \cdot h$. The formula is usually presented with the equation $A = b \cdot h$.

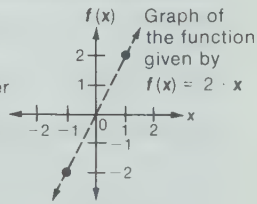
fraction A symbol for a fractional number of the form $\frac{a}{b}$ such that $b \neq 0$.

fractional number The set of non-negative rational numbers.

function A correspondence that associates each element in a first set with a unique element of a second set. Different elements in a first set may possibly be matched with the same element of the second set.

gram A unit of mass. The mass of one cubic centimetre of water at 4°C.

graph A set of points associated with a given set of numbers or number pairs showing a relation or function.

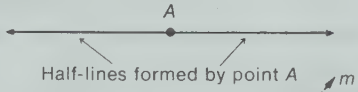


graph (statistical) A picture used to illustrate a given collection of data. It may be in the form of a bar graph, a circle graph, a line-segment graph, or a pictograph.

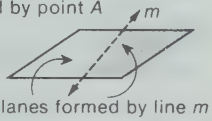
greatest common factor (GCF) The largest, or greatest, number that is a factor of each of a set of numbers.

greatest possible error Half of the basic unit in which a measurement is given. For example, if a length is given as 18 centimetres to the nearest centimetre, then the basic unit is the centimetre and the greatest possible error is 0.5 centimetre.

half-line Each of two sides of a point on a line.



half-plane The region on each of two sides of a line in a plane.



hectare A unit of area that is 10 000 square metres.

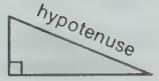
hecto- A prefix meaning *one hundred*.

height (of a prism) The perpendicular distance between the planes of the parallel faces.

heptagon A 7-sided polygon.

hexagon A 6-sided polygon.

hypotenuse The side of a right triangle opposite the right angle.



icosahedron A polyhedron of 20 faces. A regular icosahedron has equilateral triangles for its faces.

identity element For addition, zero is the identity element since $a + 0 = a$ for any number a. Similarly, for multiplication, 1 is the identity element since $a \cdot 1 = a$.

identity principles For any number a, $a + 0 = a$ and $a \cdot 1 = a$.

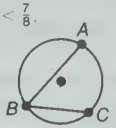
image The figure obtained after a rigid motion on a given figure. From a functional viewpoint, y is the image of x if there is a function f such that y is the output when x is the input; that is, $f(x) = y$.

improper fraction A fraction whose numerator is greater than or equal to its denominator.

incentre The point of intersection of the angle bisectors of each angle of a triangle. The centre of the circle inscribed in a triangle.

inequality A mathematical statement using either of the symbols $<$ or $>$ such as $\frac{3}{4} < \frac{7}{8}$.

inscribed angle An angle that contains three points of the circle and has its vertex on the circle.



inscribed polygon See *circumscribed circle*.

integers The set consisting of the natural numbers. {1, 2, 3, . . .}, the negatives of the natural numbers {−1, −2, −3, . . .}, and zero.

intersection The intersection of two sets is the set of all elements common to both sets.

inverse (operation) Operations which are opposite in effect such as addition and subtraction or multiplication and division.

isosceles triangle A triangle with at least two congruent sides.

kilo- A prefix meaning *one thousand*.

kilogram The basic unit of mass in the metric system, 1000 grams.

kilometre A unit of length that is 1000 metres.

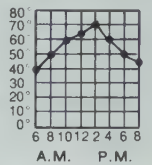
least common denominator The least common multiple of the denominators of two or more fractions. For example, the l.c.d. of $\frac{5}{6}$ and $\frac{3}{4}$ is 12.

least common multiple (LCM) The smallest non-zero number that is a multiple of each of two or more given numbers. The LCM of 4 and 6 is 12.

legs (of a right triangle) The perpendicular sides of a right triangle.

length The measure of a segment (or curve) in terms of a chosen unit.

line-segment graph A diagram used to show statistical information, particularly that of a continuous nature.



litre The unit of capacity in the metric system. 1000 cubic centimetres.

lowest-terms (simplest) fraction A fraction is in lowest terms if the numerator and denominator of the fraction have no common factor other than 1.

mean See *arithmetic mean*.

means The second and third terms in a proportion written in the form $a : b = c : d$. See *extremes*.

median (of a set) The middle number of a set containing an odd number of elements that are arranged in order. If a set contains an even number of elements, the average of the two middle numbers is the median.

median (of a triangle) The segment from the vertex to the midpoint of the opposite side of a triangle.

metre The basic unit of length in the metric system approximately 1650763.73 times the wave length of the orange-red spectral emission line of krypton-86.

metric system A system of measurement in which the metre, the litre, and the kilogram are the basic units of length, capacity, and mass.

midpoint (middle point) A point which divides a segment into two congruent segments.

milli- A prefix meaning *one thousandth*.

minute A unit of angular measure that is $\frac{1}{60}$ of a degree.

mixed decimal numeral A combination of a decimal and a fraction such as $0.4\frac{1}{3}$ and read as "four and one-third tenths."

mixed numeral A numeral such as $4\frac{2}{3}$ indicating the sum of a whole number and a fractional number; $4\frac{2}{3} = 4 + \frac{2}{3}$.

mode The most frequently occurring element of a set. There may be more than one mode of a set.

multiple A first number is a multiple of a second number if there is a whole number that multiplies by the second number to give the first number. For example, 24 is a multiple of 4 and 6 since $4 \times 6 = 24$.

multiplication An operation that combines a first number (*factor*) and a second number (*factor*) to give another number (this result is called the *product* of the two factors).

multiplicative inverse See *reciprocal*.

negative integers The set of numbers $\{-1, -2, -3, \ldots\}$.

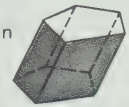
nonagon A 9-sided polygon.

number pair An ordered pair of numbers such as (3, 4) often co-ordinates of a point in a plane.

numeral A symbol for a number.

numerator For each fraction $\frac{a}{b}$, a is the numerator.

oblique prism A prism in which the base is not perpendicular to its lateral edges.



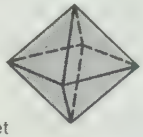
Oblique prism

Right prism

obtuse angle An angle greater than a right angle and smaller than a straight angle.

octagon An 8-sided polygon.

octahedron A polyhedron having eight faces. Each face of a regular octahedron is an equilateral triangle.



odd number Any number in the set $\{1, 3, 5, 7, \ldots\}$.

one principle For any number a, $a \cdot 1 = a$. See *identity principle* for multiplication.

one-to-one correspondence Two sets S and T are in one-to-one correspondence if there is a matching so that each element in S is matched with exactly one element in T, and vice versa.

operation A binary operation like "multiplication" associates each order paired of numbers with one number. Usually we speak of the four fundamental operations: addition, subtraction, multiplication, and division.

opposites principle For any integer a, $a + {}^{-}a = 0$.

origin The intersection of the co-ordinate axes; the point associated with the number pair (0, 0).

outcome A possible result in a probability experiment.

parallel lines Lines in the same plane which do not intersect.

parallelogram A quadrilateral whose opposite sides are parallel.

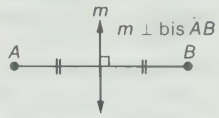
pentagon A 5-sided polygon.

percent Literally means "per hundred". A symbol such as 3% is an abbreviation for "3 per 100" or "$\frac{3}{100}$".

perimeter The distance around a polygon.

period Each set of three digits in a numeral. In the numeral 421 365 761 409, there are 421 billions, 365 millions, 761 thousands, and 409 units.

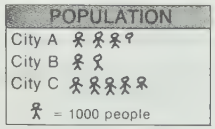
perpendicular bisector A line which bisects a segment as well as being perpendicular to it.



perpendicular lines Two intersecting lines that form right angles.

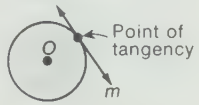
pi (π) The ratio of the circumference of a circle to its diameter. $\pi \approx 3.14$.

pictograph A diagram used to show statistical information with the use of pictures.



plane figure A set of points in one plane.

point of tangency If a line and circle intersect in just one point, the intersection is the point of tangency.



polygon A closed plane figure whose boundary is made up of segments.

polygonal path A path consisting of a union of segments which are connected end to end.

polyhedron A space figure each of whose faces is a polygonal region.

positive integer Any number in the set $\{1, 2, 3, \ldots\}$.

power In the statement $a = b^n$, a is the n^{th} power of b. For example, for $1000 = 10^3$, 1000 is the third power of 10.

precision The *more precise* of two measures is the measure with the smaller relative error.

prime factorization An expression of a composite number as a product of prime factors.

prime number A whole number greater than 1 whose only factors are itself and 1.

prism A 3-dimensional (space) figure whose bases are congruent polygonal regions in parallel planes and whose faces are parallelograms.

probability The ratio of the number of times a certain outcome can occur to the number of total possible outcomes.

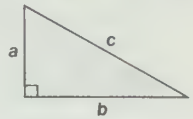
product The number $a \cdot b$ which results from applying the multiplication operation to the numbers a and b.

proportion An equation stating that two ratios are equal: $a : b = c : d$ or $\frac{a}{b} = \frac{c}{d}$

protractor An instrument for measuring angles.

pyramid A 3-dimensional (space) figure with a polygonal base and triangular lateral faces.

Pythagorean Theorem In any right triangle, the sum of the areas of the squares on the legs is equal to the area of the square on the hypotenuse.



$a^2 + b^2 = c^2$

quadrilateral A 4-sided polygon.

quotient The number $a \div b$ (or $b \div a$) which results from applying the division operation to the numbers a and b.

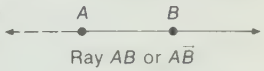
radian A unit of angular measure. A central angle of a circle subtended by an arc whose length is equal to the radius; approximately 57.3°; exactly $(180 \div \pi)°$.

radius Any segment from the centre of a circle to a point on the circle.

ratio The ratio of two numbers a and b is their quotient, $\frac{a}{b}$.

rational number The quotient of two integers, the divisor not being zero.

ray A half-line together with the point determining it (endpoint).



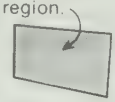
reciprocal If $a \cdot b = 1$, each of the numbers a and b is the reciprocal (multiplicative inverse) of the other. Each non-zero real number has a unique reciprocal. Zero has no reciprocal.

rectangle A parallelogram with four right angles.

reflection A rigid motion that maps the points of a plane onto itself. If point P is reflected in a line with a reflection image P_1, then the line is the perpendicular bisector of PP_1.

region All the points in the part of a plane bounded by a simple closed curve.

The interior of a quadrilateral is a region.



This curve forms two regions.

regular polygon A polygon that is equiangular and equilateral; for example, a square.

relative error The ratio of the greatest possible error of a measurement to the measurement itself. For example, if a measurement is given as 10 centimetres to the nearest centimetre and the greatest possible error is 0.5 centimetre, then the relative error is the ratio of 0.5 to 10 or 5%.

remainder In whole-number division, if dividend D is divided by divisor d and the relation $D = (q \cdot d) + r$ is obtained where $0 < r < d$, the r is called the remainder. Note that $r = 0$ if d is a factor of D (see *dividend*). Also, the difference in a subtraction problem.

repeating decimal A decimal whose digits can be grouped from some point on in repeating periods. Examples: 0.333 . . .; 0.2727 . . .; 4.273149149 . . . . In these examples, the repeating decimals have (respectively) periods of 1, 2, and 3 digits and can be written as 0.3, 0.27, and 4.273149.

rhombus An equilateral parallelogram which is not a square.

right angle An angle whose measure is 90°.

rigid motion A transformation of the points of the plane that preserves all distances.

Roman numerals The numerals I, V, X, L, C, D, M, and combinations of the numerals used in the Roman numeration system.

rotation A rigid motion in which the points of the plane are turned about a fixed point.

rounding A process of replacing a number by an approximation (another number) with fewer significant digits. For example, rounding 456 789 to the nearest thousand would be 457 000.

sample space The set of all possible outcomes of an experiment.

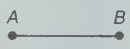
scale drawing A drawing of an object made so that distances in the drawing are proportional to actual distances. A scale of 1:10 indicates that distances in the drawing are $\frac{1}{10}$ of the actual distances.

scalene triangle A triangle with no pair of congruent sides.

scientific notation A notation used for writing any number as the product of a power of ten and a number between 1 and 10 (including 1).

second A unit of angular measure that is $\frac{1}{60}$ of a minute.

segment A set of points consisting of two points A and B and all points between them.



Segment AB or $\overline{AB}$

semicircle An arc that is exactly half a circle. One of two arcs cut off by a diameter.

set A group or collection of objects.

skew lines Lines that are not in the same plane. They do not intersect, yet they are not parallel.

space The set of all points. This term usually denotes a 3-dimensional quality. We also refer to a plane as 2-dimensional space and a line as a 1-dimensional space.

space figure A geometric figure whose points are not all in the same plane.

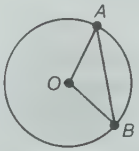
sphere The set of all points in space at a fixed distance from a given point.

statistics The science of analyzing numerical information.

straightedge An unmarked ruler.

subset Set A is a subset of set B if every element of A belongs to B.

subtend Literally means "to be opposite to". For example, central angles and inscribed angles in a circle are said to subtend the arcs they cut off. Also, chords and arcs in a circle subtend the angles of a circle.



$\overset{\frown}{AB}$ subtends $\angle AOB$ or $\overline{AB}$ is subtended by $\angle AOB$

subtraction The operation that is the inverse of the addition operation.

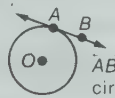
sum The number $a + b$ which results when the operation of addition is applied to the numbers a and b.

supplementary angles Two angles whose measures have a sum of 180°.

surface area The total area of the polygonal regions (faces) of a polyhedron.

symmetric figure A plane figure which can be divided into two congruent parts by a line (of symmetry).

tangent A line which intersects a circle in just one point.



$\overleftrightarrow{AB}$ is tangent to circle O at point A.

terminating decimal A decimal such as 0.5, 1.0, 1.24, 0.0307, and so on, that represents the quotient of a whole number and a power of 10:

$$0.5 = \frac{5}{10},\ 1.24 = \frac{124}{10^2},\ 0.0307 = \frac{307}{10^4}$$

tessellation A repeated pattern of geometric figures which will completely "cover a plane" without any gaps or overlapping.

tetrahedron A polyhedron having 4 faces.

translation A rigid motion in which each point of the plane is moved the same distance and in the same direction.

trapezoid A quadrilateral with a pair of parallel sides.

bases, or parallel sides

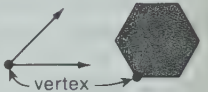
triangle A 3-sided polygon.

union If A and B are sets, then the union of A and B ($A \cup B$) is the set consisting of all elements that belong to at least one of the two sets.

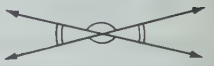
unit The object adopted as a standard of measurement. For length, we choose a *unit* segment; for area, we choose a *unit* square; and for volume, a *cube*.

variable A symbol, usually a letter, used to represent any number in a given set. For example, we may say: "If x is a whole number, then $2 \cdot x$ is an even number."

vertex A point that two rays of an angle have in common. Also, the common point of any two sides of a polygon.



vertical angles Two pairs of angles formed by two intersecting lines.



volume The measure of a space figure in terms of a chosen unit.

whole numbers Any number in the set $\{0, 1, 2, 3, \ldots\}$.

zero principle The principle which states that for any number a, $a + 0 = a$. Also, see *identity element* for addition.

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Answers

ANSWERS

UNIT A: MODULE 1

A-2 DISCUSSING THE IDEAS

1. $\{1, 3\}$ **2. A** No **B** Answers may vary. Example: the set of women premiers. **3** A set is a subset of another set when all elements of the first set are elements of the second set. **4.** Answers may vary. Example: A = $\{2, 4, 6\}$, B = $\{1, 2, 3\}$

A-3 USING THE IDEAS

1. $\{d, e, f\}$, $\{d, e\}$, $\{e, f\}$, $\{d, f\}$, $\{d\}$, $\{e\}$, $\{f\}$, $\varnothing$
2. A $\{1, 2, 3, 4, 5, 6, 7\}$
B $\{4999, 5000, 5001, 5002, 5003\}$
C $\{T, U, V, W, X, Y, Z\}$ **D** $\{March, May\}$
E $\{1, 2, 3, 4, 6, 12\}$ **3. A** $\varnothing$ **B** $\{February\}$ **C** $\varnothing$
D $\varnothing$ **4. A** 23 **B** Yes **5. A** Yes **B** Yes.
Think ART, THE, ARM, AT, IT, MET, I, ME, RITE, TIC, A, AH, AM, RIM, AIM, ATE, ARE, ARC, HE, RIME

A-4 DISCUSSING THE IDEAS

1. A 3 **B** 8 **C** 2 **2. A** $\{a, b, c, d, e, f, g, h\}$
B $\{d, g\}$ **3. A** $\{1, 2, 3, 4, 5, 6, 7\}$ **B** $\{3, 4\}$
4. $\{m, n, o, x, y, z\}$ **5. A** $\{1, 2, 3, 4\}$ The union is M.
B $\{2, 3, 4\}$ The intersection is N.

A-5 USING THE IDEAS

1. A $\{a, b, c, d, e\}$ **B** $\{b, c, d, e, f\}$
C $\{a, b, c, d, e, f, g\}$ **D** $\{c, d, e\}$ **E** $\{b\}$ **F** $\{f\}$
G $\{a, c, d, e, f\}$ **H** $\{e\}$ **I** $\{b, e, f, g\}$ **J** $\{e\}$ **2. A** T
B F **C** T **D** T **E** F **F** F **G** F **H** F **3. A** Yes
B Yes **4. A** m **B** m **C** m **D** $\varnothing$
5. A $\{1, 2, 3, 4, 5, 6\}$ **B** $\{3\}$ **C** $\{2, 3\}$ **D** $\{2, 3, 4, 5, 6\}$
E $\{1, 2, 3, 4, 5\}$ **F** $\{1, 2, 3, 4, 5, 6\}$ **G** $\{2, 3\}$ **H** $\{3\}$
I $\{1, 2, 3, 4, 5\}$ **6.** A, F; E, I; B, H; C, G **7. A** 9 **B** 5
Think **1.** 12 **2.** 3; 12 **3.** 4

A-6 DISCUSSING THE IDEAS

1. 10 **2.** (9×1); (3×10); (8×100); (7×1000)
3. $(2 \times 10\,000) + (3 \times 1000) + (7 \times 100) + (4 \times 10) + 6$
4. A 2; 9 **B.** Five hundred forty seven billion, two hundred eight million, seven hundred ninety three thousand, one hundred sixty five.

A-7 USING THE IDEAS

1. B ten thousands; 10 000 **C** hundred millions; 100 000 000 **D** hundreds; 100 **E** thousands; 1000
F ten millions; 10 000 000 **2. A** $(3 \times 100) + (5 \times 10) + 8$
B $(4 \times 1000) + (5 \times 100) + (6 \times 10) + 2$ **C** $(5 \times 10\,000) + (3 \times 1000) + (7 \times 100) + (2 \times 10) + 9$ **D** $(7 \times 10\,000) + (3 \times 100) + (6 \times 10) + 4$ **E** $(4 \times 100\,000) + (5 \times 10\,000) + (8 \times 1000) + (2 \times 100) + (9 \times 10) + 7$ **F** $(6 \times 1\,000\,000) + (2 \times 100\,000) + (8 \times 100) + (7 \times 10) + 6$
G $(2 \times 10\,000\,000) + (4 \times 1\,000\,000) + (3 \times 100\,000) + (6 \times 10\,000) + (4 \times 1000) + (2 \times 100)$ **H** $(6 \times 100\,000\,000) + (4 \times 10\,000\,000) + (7 \times 1\,000\,000) + (8 \times 100\,000) + (7 \times 10\,000)$ **3. A** 974 **B** 8675 **C** 53 427 **D** 305 894
4. A 300 000 000 **B** 7 000 000 000 000 000 000 000
C 140 000 000 000 000 000 **5. A** 136 153 324
B 412 114 194 **C** 405 030 317

A-8 DISCUSSING THE IDEAS

1. 100; The number of times the digit 0 appears is equal to the number of tens which would have to be multiplied together in order to make a googol. **2. A** 4; 10 000 **B** 5; 100 000 **C** 6; 1 000 000 **3. A** second **B** third
C fourth **D** one hundredth

A-9 USING THE IDEAS

1. A 10^3 **B** 10^7 **C** 10^8 **2. A** 3 **B** 6 **C** 19 **D** 100
3. A 10^2 **B** 10^3 **C** 10^4 **D** 10^5 **E** 10^6 **F** 10^{13}
G 10^{17} **H** 10^{23} **4. A** 100 **B** 1000 **C** 10 000
D 100 000 **E** 1 000 000 **F** 10 000 000 **G** 100 000 000
H 1 000 000 000 **I** 10 000 000 000 **J** 100 000 000 000
5. A 1000; 10^3 **B** 10 000; 10^4 **C** 100 000; 10^5
6. A 10 000 **B** 1 000 000 **C** 1 000 000 **D** 100 000 000
E 900 **F** 42 000 **G** 870 000 **H** 6 800 000 **I** 300 000
J 470 000 000 **7. A** 10; 10^1 **B** 1000; 10^3 **C** 1000; 10^3
8. 1

A-10 DISCUSSING THE IDEAS

1. Carol; 6000 has 3 zeros **2.** 10^0 **3. A** $9 \cdot 10^3$
B $8 \cdot 10^1$ **C** $2 \cdot 10^6$ **D** $3 \cdot 10^5$ **4. A** 60 000 **B** 9 000
C 7 000 000 **D** 4

A-11 USING THE IDEAS

1. A $9 \cdot 10^2$ **B** $5 \cdot 10^2$ **C** $4 \cdot 10^3$ **D** $3.2 \cdot 10^4$ **E** $9 \cdot 10^4$
F $1 \cdot 10^7$ **G** $7 \cdot 10^8$ **H** $3 \cdot 10^{10}$ **I** $8 \cdot 10^{30}$ **2. A** 60 000
B 30 000 000 **C** 40 000 000 000 **D** 700 000
E 100 000 000 **F** 6 **G** 2 000 000 **H** 900 **I** 5000
J 8 000 000 000 **K** 6 000 000 000 000 000
L 300 000 000 000 000 000 000 000
3. A $3 \cdot 10^6$ **B** $6 \cdot 10^{12}$ **C** $5 \cdot 10^1$ **D** $5 \cdot 10^{15}$
E $2 \cdot 10^{21}$ **F** $4 \cdot 10^9$ **G** $9 \cdot 10^{18}$ **H** $6 \cdot 10^{15}$ **I** $2 \cdot 10^8$
Think **1.** Answers may vary (approximately 72)
2. A 10 days **B** 26 years

A-12 DISCUSSING THE IDEAS

1. Answers may vary **2.** 300 000 **3.** 5000 **4.** The 4 in the hundred's place would give an answer of 5000.

A-14 DISCUSSING THE IDEAS

1. Answers may vary. Example: the arrow always points to the smallest number. **2.** The 5 and the 3 are in the same place. Since $5 > 3$, then $B > A$. **3.** 999 **4.** 100 **5. A** $<$
B $>$ **C** $<$ **D** $<$ **E** $<$ **F** $>$

A-15 USING THE IDEAS

1. A $>$ **B** $<$ **C** $<$ **D** $>$ **E** $<$ **F** $>$ **G** $<$ **H** $<$
I $<$ **2. A** $73 < 75$ **B** $39 < 93$ **C** $452 > 442$
D $3574 > 3534$ **E** $8878 < 8888$ **F** $6491 < 6492$
G $4672 > 4592$ **H** $1000 > 999$ **I** $0 < 364$ **3. A** $<$
B $=$ **C** $>$ **D** $<$ **E** $=$ **F** $<$ **4. A** 999 900
B 999 000 **C** 900 000 **D** 990 000 **E** 999 999 899

F 500 000 000 **G** 52 000 846 **H** 8 000 000 059
I 7 087 346 **J** 456 472 863 **K** 27 137 **5.** 114; older

Think **1.** F **2.** 19 years

A-16 DISCUSSING THE IDEAS

1. 0, 1, 2, 3, 4, 5, 6 **2.** ones, sixes, six-sixes **3.** 1, 9
4. A $14_{(6)}$ means 1 six and 4 **B** $27_{(8)}$ means 2 eights and 7
C $30_{(5)}$ means 3 fives **D** $101_{(ten)}$ means 1 hundred and 1
E $121_{(3)}$ means 1 nine, 2 threes and 1

A-17 USING THE IDEAS

1. A 12 **B** 21 **C** 5 **D** 11 **E** 3 **F** 23 **G** 30
H 26 **I** 10 **J** 20 **2. A** $4_{(8)}$ **B** $5_{(8)}$ **C** $6_{(8)}$ **D** $12_{(8)}$
E $13_{(8)}$ **F** $14_{(8)}$ **3. A** $32_{(8)}$ **B** $135_{(6)}$ **C** $571_{(8)}$ **D** $233_{(4)}$
E $302_{(5)}$ **F** $110_{(2)}$ **4.** 56

A-18 DISCUSSING THE IDEAS

1. A ten thousands **B** four-thousand-ninety-sixes
2. A 1 **B** 3 **C** 1 **D** $131_{(8)}$ **3. A** $(3 \cdot 8) + 7$
B $(1 \cdot 8^2) + (3 \cdot 8) + 7$ **C** $(4 \cdot 8^2) + (1 \cdot 8) + 6$ **D** $(5 \cdot 8^2) + 6$

A-19 USING THE IDEAS

1. A $56_{(8)}$ **B** $124_{(8)}$ **C** $200_{(8)}$ **2. A** ones **B** eights
C sixty fours **3. A** 3 **B** 3 **C** 7 **D** 6 **E** 5 **F** 0
4. A $23_{(8)}$ **B** $36_{(8)}$ **C** $72_{(8)}$ **D** $76_{(8)}$ **E** $55_{(8)}$ **F** $60_{(8)}$
5. A $115_{(8)}$ **B** $146_{(8)}$ **C** $143_{(8)}$ **D** $225_{(8)}$ **E** $327_{(8)}$
F $777_{(8)}$ **6. A** $(3 \cdot 8^1) + 5$ **B** $(1 \cdot 8^2) + (2 \cdot 8) + 5$
C $(7 \cdot 8^2) + (4 \cdot 8) + 0$ **D** $(3 \cdot 8^2) + (4 \cdot 8) + 6$
E $(4 \cdot 8^2) + (0 \cdot 8) + 7$ **F** $(6 \cdot 8^2) + (2 \cdot 8) + 3$ **7. A** $29_{(10)}$
B $85_{(10)}$ **C** $480_{(10)}$ **D** $230_{(10)}$ **E** $263_{(10)}$ **F** $403_{(10)}$
8. A $60_{(8)}$ **B** $130_{(8)}$ **C** $300_{(8)}$ **D** $7100_{(8)}$ **E** $101\,000_{(8)}$
9. A 1802 **B** 2970 **C** 2235 **D** 1377 **E** 4029 **F** 4095

A-20 DISCUSSING THE IDEAS

1. $15_{(8)}$ **2. A** 75 **B** 110 **3. A** $6_{(8)}$ **B** $7_{(8)}$ **C** $1_{(8)}$
D $6_{(8)}$

A-21 USING THE IDEAS

1. A $17_{(8)}$ **B** $20_{(8)}$ **C** $22_{(8)}$ **D** $27_{(8)}$ **E** $30_{(8)}$ **F** $31_{(8)}$
G $57_{(8)}$ **H** $60_{(8)}$ **I** $63_{(8)}$ **J** $77_{(8)}$ **K** $102_{(8)}$ **L** $100_{(8)}$
M $110_{(8)}$ **N** $353_{(8)}$ **O** $471_{(8)}$ **P** $1375_{(8)}$ **Q** $1230_{(8)}$
R $4233_{(8)}$ **2. A** $11_{(8)}$ **B** $17_{(8)}$ **C** $20_{(8)}$ **D** $30_{(8)}$ **E** $20_{(8)}$
F $26_{(8)}$ **3. A** $7_{(8)}$ **B** $5_{(8)}$ **C** $3_{(8)}$ **D** 4 **E** $10_{(8)}$ **F** $11_{(8)}$
G $7_{(8)}$ **H** $7_{(8)}$ **I** $6_{(8)}$ **J** $6_{(8)}$ **K** $1_{(8)}$ **L** $7_{(8)}$ **4. A** $13_{(8)}$
B $31_{(8)}$ **C** $34_{(8)}$ **D** $25_{(8)}$ **E** $37_{(8)}$ **F** $1_{(8)}$ **G** $12_{(8)}$
H $246_{(8)}$

Think

$15_{(8)}$	$6_{(8)}$	$13_{(8)}$
$8_{(8)}$	$12_{(8)}$	$14_{(8)}$
$9_{(8)}$	$16_{(8)}$	$7_{(8)}$

A-22 DISCUSSING THE IDEAS

1. $5_{(8)} \cdot 4_{(8)} = 24_{(8)}$ **2.** $34_{(8)}$ **4. A** $3_{(8)}$ **B** $6_{(8)}$ **5. A** $5_{(8)}$
B $4_{(8)}$ **C** $4_{(8)}$ **D** $6_{(8)}$

A-23 USING THE IDEAS

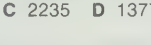
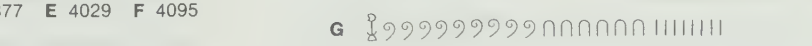
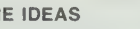
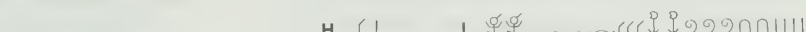
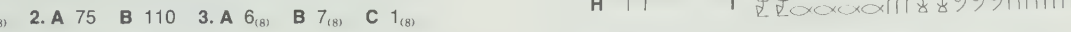
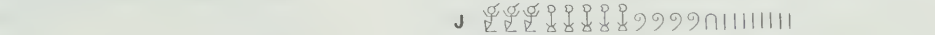
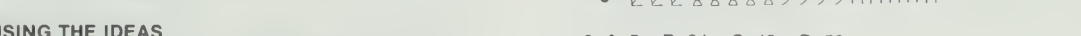
1. A $36_{(8)}$ **B** $50_{(8)}$ **C** $62_{(8)}$ **D** $71_{(8)}$ **E** $70_{(8)}$ **F** $132_{(8)}$
G $133_{(8)}$ **H** $252_{(8)}$ **I** $74_{(8)}$ **J** $121_{(8)}$ **K** $432_{(8)}$ **L** $416_{(8)}$
2. A $3_{(8)}$ **B** $4_{(8)}$ **C** $5_{(8)}$ **D** $7_{(8)}$ **E** $4_{(8)}$ **F** $6_{(8)}$ **G** $5_{(8)}$
H $6_{(8)}$ **3.** $40_{(8)}$; $100_{(8)}$; $200_{(8)}$; $400_{(8)}$; $1000_{(8)}$; $2000_{(8)}$; $4000_{(8)}$; $10\,000_{(8)}$ **4.** Less **5. A** $367_{(8)}$ **B** $1004_{(8)}$ **C** 3650
D $32\,032_{(8)}$ **6. A** $75_{(8)}$ **B** $262_{(8)}$ **C** $21_{(8)}$ **D** $57_{(8)}$

Think **1.** Base 3

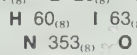
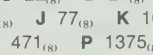
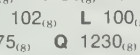
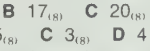
A-24 ROMAN NUMERALS

1. A 6 **B** 51 **C** 15 **D** 16 **E** 60 **F** 150 **G** 110
H 105 **I** 600 **J** 511 **K** 1100 **L** 1500 **2. A** 4 **B** 9
C 40 **D** 90 **E** 400 **F** 900 **3. A** III **B** VI **C** XI
D XXXIV **E** XXXVIII **F** XIX **G** XXXXIX **H** CXXVI
I DXXIX **J** CCCXL **K** MCCCLXXII **L** MMCMLXVIII
4. A 262 **B** 545 **C** 444 **D** 437 **E** 959 **F** 2051
G 1492 **H** 1976 **5. A** I, II, III, IV, V, VI, VII, VIII, IX, X
B XV, XVI, XVII, XVIII, XIX, XX **C** XL, XLI, XLII, XLIII, XLIV, XLV, XLVI, XLVII, XLVIII, XLIX, L **D** CXX, CXXI, CXXII, CXXIII, CXXIV, CXXV, CXXVI, CXXVII, CXXVIII, CXXIX, CXXX **E** CCCXCV, CCCXCVI, CCCXCVII, CCCXCVIII, CCCXCIX, CD **F** CMXC, CMXCI, CMXCII, CMXCIII, CMXCIV, CMXCV, CMXCVI, CMXCVII, CMXCVIII, CMXCIX, M **6. A** CXXVII **B** MCD **C** CCXXXIII **7. A** XXI
B DXVII **C** CDI **8.** MMCCCXL

A-25 OTHER ANCIENT NUMERALS

1. A 3 **B** 42 **C** 111 **D** 2325 **E** 31 230
F 3 420 300 **G** 1 010 101 **H** 203 076
2. A **B**  **C** 
D **E**  **F** 
G 
H **I** 
J 

3. A 5 **B** 24 **C** 48 **D** 59

4. A  **B**  **C**  **D**
E **F** 

5. A 9 **B** 11 **C** 18 **D** 22 **E** 39

A-26 REVIEWING THE IDEAS

1. A $\{0, 1, 2, 3, 4, 5, 6, 8\}$ **B** $\{2, 4\}$ **2.** $(7 \times 1000) + (3 \times 100) + (6 \times 10) + 9$ **3. A** 9 **B** 2 **C** 3 **D** 6 **E** 9
F 4 **4.** 98 537 **5.** E **6. A** 4 **B** 10 000 **7.** 10^5
8. 100 000; 10^5 **9. A** 4000 **B** 7 000 000 000 **C** 5
10. 4×10^5 **11. A** 7 856 500 **B** 7 856 000 **C** 7 900 000
D 8 000 000 **12. A** $<$ **B** $=$ **C** $>$ **13.** 999 999
14. A $20_{(10)}$ **B** $24_{(8)}$ **C** $40_{(5)}$ **15. A** 5 **B** 3 **C** 53
16. A $12_{(8)}$ **B** $4_{(8)}$ **C** $62_{(8)}$ **D** $5_{(8)}$ **17. A** $14_{(8)}$ **B** $4_{(8)}$
C $137_{(8)}$ **D** $50_{(8)}$ **18. A** 167 **B** 1709 **19. A** DCXXIX
B MCMLXXXIV

UNIT A: MODULE 2

A-30 DISCUSSING THE IDEAS

1. A multiply by 6 **B** subtract 4

2. 4 → +1 → ·5 → 25

3. A replace the 3 with 0; 240 **B** add 1 to the tens' digit; 520 **C** rounds to nearest ten.

A-31 USING THE IDEAS

1. B 59 **C** 10000 **D** 41 **E** 5 **2.** 76

3. A 0 → +8 → ·9 → +12 → ÷6

B 0 → +12 → ÷6 → ·9 → +8

C 0 → +12 → +8 → ·9 → ÷6

D 0 → ·9 → +12 → ÷6 → +8

E 0 → +8 → ·9 → ÷6 → +12

F 0 → ÷6 → +8 → +12 → ·9

A-32 DISCUSSING THE IDEAS

1. yes **2. A** sum; 10 **B** addend; 4 **C** addend; 4 **D** addend; 5 **3. A** factor; 4 **B** product; 35 **C** factor; 6 **D** product; 56

A-33 USING THE IDEAS

1. A $a = 9$ **B** $n = 1$ **C** $b = 9$ **D** $m = 12$ **E** $a = 8$ **F** $r = 13$ **G** $y = 8$ **H** $q = 15$ **I** $f = 0$ **J** $n = 12$ **K** $d = 3$ **L** $r = 8$ **2. A** 7 **B** 6 **C** 9 **D** 8 **E** 8 **F** 7 **G** 1 **H** 6 **I** 7 **3. A** 12 **B** 5 **C** 27 **D** 7 **E** 6 **F** 7 **G** 3 **H** 15 **I** 2 **J** 9 **K** 8 **L** 7 **4. A** 6 **B** 4 **C** 6 **D** 1 **E** 7 **F** 5 **G** 9 **H** 0 **I** 1 **5. A** 15 **B** 9 **C** 36 **D** 4

A-34 USING THE IDEAS

1. A 11 **B** 16 **C** 13 **D** 7 **E** 17 **2. A** 11 **B** 10 **C** 14 **D** 7 **E** 12 **3. A** 3 **B** 4 **C** 0 **D** 7 **E** 2 **4.** Using a number as an addend 3 times gives the same result as multiplying the number by 3. **5.** Dividing n into 3 parts ($n \div 3$) is the same as finding how many 3's you can subtract from n.

A-35 USING THE IDEAS

1. A 45 **B** 27 **C** 63 **D** 9 **E** 90 **F** 18 **G** 0 **H** 900 **2. A** 16 **B** 4 **C** 24 **3. A** 7 **B** 4 **C** 10 **D** 13 **4. A** 8 **B** 2 **C** 9 **D** 60 **E** 5 **5.** $f(n) = n + 4$ **6.** $f(n) = 2 \cdot n$ **7.** $f(n) = n \div 7$ **8.** $f(n) = n_2 + n$

Think

1	2^0
1 1	2^1
1 2 1	2^2
1 3 3 1	2^3
1 4 6 4 1	2^4
1 5 10 10 5 1	2^5
1 6 15 20 15 6 1	2^6
1 7 21 35 35 21 7 1	2^7

A-36 DISCUSSING THE IDEAS

1. $(9 - 3) + 2 = 8$; $9 - (3 + 2) = 4$ **2.** no **3.** no **4. A** 21 **B** 11 **C** 3 **D** 24 **E** 27 **F** 19

A-37 USING THE IDEAS

1. A 24 **B** 1 **C** 6 **D** 15 **E** 18 **F** 3 **2. A** 12 **B** 17 **C** 7 **D** 7 **E** 14 **F** 13 **G** 9 **H** 10 **I** 36 **J** 13 **K** 0 **L** 32 **M** 6 **N** 6 **O** 6 **P** 7 **Q** 2 **R** 5 **3. A** 33 **B** 4 **C** 10 **D** 31 **E** 63 **F** 43 **4. A** 4 **B** 13 **C** 22 **D** 28 **E** 19 **F** 18 **G** 14 **H** 8 **I** 6 **J** 0 **K** 4 **L** 2 **M** 6 **N** 9 **O** 5 **5. A** 5 **B** 2 **C** 10 **D** 2 **E** 5 **F** 7

A-38 DISCUSSING THE IDEAS

1. A Answers may vary. Example: the order in which numbers are added (or multiplied) does not alter the answer. **B** $12 + 34 = 34 + 12 = 46$ **C** $9 \cdot 12 = 12 \cdot 9 = 108$ **2. A** yes **B** $(5 \cdot 4) \cdot 2 = 5 \cdot (4 \cdot 2) = 40$ **3.** $(5 \cdot 20) \cdot 39 = 3900$ **4.** $a + 0 = a$; $a - 0 = a$; $a - a = 0$ **5.** $a \cdot 1 = a$; $a + 1 = a$; $a + a = 1$ **6.** If a is not zero, $0 \cdot a = 0$ then $0 \div a = 0$. If we say $a \div 0 = b$ then $0 \cdot b = 0$ and not a, so we cannot divide by zero.

A-39 USING THE IDEAS

1. A commutative **B** commutative **C** associative **D** commutative **E** commutative **F** commutative **2. A** 26 **B** 51 **C** 2 **D** 8 **E** 3264 **F** 6 **3. A** 147 **B** 179 **C** 1356 **D** 1653 **E** 130 **F** 4100 **G** 3700 **H** 9700 **I** 93000 **4. A** 0 **B** 92 **C** 0 **D** 1 **E** 92 **F** 1 **G** 12 **H** 12 **I** 0 **5.** No; no **6.** B; C **7. A** At least one of the whole numbers is zero. **B** Both numbers are 1 (or −1)

Think 201; 38

A-40 DISCUSSING THE IDEAS

1. $20 \cdot (34 + 9) = (20 \cdot 34) + (20 \cdot 9)$ **2.** 4.5 is the product of the number of rows and the number of columns. The five columns can be grouped into two sections of 2 and 3, and the same result is found by finding the sum of $4 \cdot 2$ and $4 \cdot 3$; $n = 4$ **3. A** find the sum of $6 + 3$; $n = 9$ **B** find the number that gives 13 when added to 6; $n = 7$

A-41 USING THE IDEAS

1. A C **B** D **C** A **D** F **E** B **F** E

2. A 2 **B** 3 **C** 6 **3. A** 6 **B** 7 **C** 13 **D** 17 **E** 35 **F** 78 **G** 150 **H** 619 **I** 438 **J** 999 **4.** 4000 cubes

5. A $a \cdot c$ **B** $s \cdot r$ **C** n **D** m **E** y **F** $(f \cdot g) + (f \cdot h)$ **G** $a \cdot (n + m)$ **H** $b \cdot (s + s)$ or $(b + b) \cdot s$

A-42 DISCUSSING THE IDEAS

1. When adding two numbers, the most that can be carried over is 2 (9 + 9 = 18) 7 + 9 = 16 or 7 + 9 + 1 = 17 **2.** When adding three numbers, the most that can be carried over is 2(9 + 9 + 9 = 27) 8 + 5 + 3 = 21 or 8 + 5 + 8 + 1 = 22 or 8 + 5 + 8 + 2 = 23 **3.** 7 − 4 = 3 or if a 10 was borrowed: 6 − 4 = 2 **4.** 7 or 6 **5.** commutative and associative principles **6. A** subtract **B** addition

A-43 USING THE IDEAS

1 A 89 **B** 123 **C** 790 **D** 1001 **E** 10163 **F** 784 **G** 1774 **H** 3383 **I** 19663 **J** 180201 **K** 194 **L** 6506 **M** 31704 **2. A** 52 **B** 24 **C** 82 **D** 85 **E** 358 **F** 2563 **G** 2904 **H** 3662 **I** 3854 **J** 666556 **3. A** 9774 km **B** 3948 km **C** 2814 km **D** 3035 km **4.** A; 5184 km **5.** 13890 km

Think 37; 19

A-44 NORTH AMERICA

1. 628430 km² **2.** 30354 km² **3.** 26632000 people **4. A** 7771000 **B** 782000 **5.** 21927399 **6.** 300041000 **7.** 110955000

A-45 MONEY PROBLEMS

1. $4.55 **2.** $4.31 **3.** $339 **4.** $14368 **5.** $669.95 **6.** $57.65 **7.** $2.54 **8.** $3.65

Think No

A-46 DISCUSSING THE IDEA

1. A add 1 zero **B** add 2 zeros; add 3 zeros **2.** $a = 10$; $b = 100$; $c = 1200$ **3.** 1200 **4.** 80; 20

A-47 USING THE IDEAS

1. A 50 **B** 70 **C** 800 **D** 20 **E** 600 **F** 300 **G** 90 **H** 40 **I** 700 **J** 500 **K** 9000 **L** 40000 **2. A** 100; 700 **B** 700; 700 **C** 1000; 4000 **D** 4000; 4000 **3. A** 400 **B** 300 **C** 6000 **D** 7000 **E** 800 **F** 1500 **G** 1200 **H** 2400 **I** 2000 **J** 2400 **K** 60000 **L** 2100 **M** 4000 **N** 27000 **O** 48000 **4. A** 60 **B** 40 **C** 70 **D** 80 **E** 50 **5. A** 20 **B** 60 **C** 30 **D** 30 **E** 200 **F** 4 **6. A** 50 **B** 2000 **C** 20 **D** 30 **E** 90 **F** 70 **7. A** 6×10^7 **B** 2×10^{10} **C** 4×10^3 **D** 3×10^0 **E** 1×10^7 **F** 5×10^5 **G** 3×10^2 **H** 2×10^0 **I** 5×10^9 **J** 1×10^2 **K** 2×10^7 **L** 4×10^{15}

A-48 DISCUSSING THE IDEAS

1. 329 **4.** 50 and 742; 37100; 635068

A-49 USING THE IDEAS

1. A 4320 **B** 103680 **C** 37843200 **D** 2649024000 **2.** 1080 **3.** 93750 **4.** 2750 **5.** 3300 **6.** 4080 **7.** 70 **8. A** 1376 **B** 1728 **C** 1566 **D** 16192 **E** 21588 **F** 10672 **G** 217422 **H** 155975 **I** 187488 **J** 303064 **K** 2781592 **L** 40673656

9. A

$$\begin{array}{r} 71 \\ 54 \\ \hline 284 \\ 355 \\ \hline 3834 \end{array}$$

B

$$\begin{array}{r} 99 \\ 23 \\ \hline 297 \\ 198 \\ \hline 2277 \end{array}$$

C

$$\begin{array}{r} 359 \\ 77 \\ \hline 2513 \\ 2513 \\ \hline 27643 \end{array}$$

D

$$\begin{array}{r} 476 \\ 999 \\ \hline 4284 \\ 4284 \\ \hline 475524 \end{array}$$

A-50 DISCUSSING THE IDEAS

1. Answers may vary **2. A** 80 **B** 30 **C** 200 **D** 600 **E** 100 **F** 900 **G** 6000 **H** 6000 **3. B** 24000 **C** 120000 **4. A** 490 **B** 1600 **C** 5600 **D** 12000

A-51 USING THE IDEAS

1. A 320 **B** 720 **C** 180 **D** 400 **E** 540 **2. A** 4800 **B** 6000 **C** 2800 **D** 4500 **E** 6300 **3. A** 800 **B** 4000 **C** 1400 **D** 4000 **E** 4800 **4. A** 720 **B** 4500 **C** $12.00 **D** 5200 **E** 1800 **F** 18000 **5. A** 50 **B** 80 **C** 60 **D** 20 **E** 70 **F** 50 **G** 90 **H** 70 **6. A** 4 **B** 6 **C** 5 **D** 8 **E** 5 **F** 7 **G** 8 **H** 3 **I** 6

Think The result will be a nine-digit number, each digit of which is the selected number. Thus, if 7 is the selected number and 123456789 is multiplied by 9 × 7 or 63 the product is 777777777.

A-52 DISCUSSING THE IDEA

1. A 200 **B** 30; 6 **C** 236 **D** smaller **E** The quotient was multiplied by the divisor. **2. A** 415, R1 **B** 776, R5 **C** 471 **D** 986, R3

A-53 USING THE IDEAS

1. A 48, R2 **B** 34, R8 **C** 742, R3 **D** 729, R1 **E** 456, R1 **F** 2641, R1 **G** 6823 **H** 143 **I** 4115 **J** 20406 **K** 255, R5 **L** 17692, R3 **M** 7325, R3 **N** 1792143 **O** 12345679 **P** 62000, R1 **2.** 54 **3.** 96 **4.** 111; R1 **5. A** 6; 68908 **B** 2; 4269 **C** 4; 5204 **D** 3; 38657 **6. A** 2, 8 **B** 2, 6 **C** 0, 6 **D** 2, 9 **7.** 5313 ÷ 23 = 231

Think $a = 9$, $b = 1$, $c = 0$

A-54 DISCUSSING THE IDEAS

1. A 2, 3 **C** 23 **D** 53 **E** multiply 23 × 67 then add 53 **2.** 24, R27 **3. A** 4 **B** 3 **C** 7 **D** 2 **4. A** 46(A); 33, R13(B); 69, R37(C); 30, R11(D) **B** perhaps in C **C** perhaps in D **5. A** 2 **B** 8

A-55 USING THE IDEAS

1. A 29, R13 **B** 152, R18 **C** 27, R73 **D** 1451, R38 **E** 60, R7 **F** 928, R37 **G** 33, R27 **H** 795, R41 **I** 91, R52 **J** 84 **K** 68, R1 **L** 30, R17 **2. A** 46, R297 **B** 68, R156 **C** 42, R690 **D** 143 **E** 229, R323 **F** 728, R186 **G** 1226, R390 **H** 75, R541 **I** 271, R184

Short Stories

1. 44 **2.** 45 **3.** 83 **4.** 35 **5.** 8 **6.** 420000 **7.** 9 **8.** 5

A-56 REVIEWING THE IDEAS

1. 0 **2.** $13 - 6 = 7$ **3.** $104 \div 8 = 13$; $104 \div 13 = 8$
4. A 40 **B** 56 **C** 72 **D** 18 **E** 48 **F** 4
5. A [5] **B** [3] **C** [4] **D** [1]
E [7] **F** [2] **G** [6]
6. A 152 **B** 81 **C** 56 **7. A** 1660 **B** 282 464
8. A 824 **B** 14 787 **9. A** 140 **B** 3000 **C** 60 **D** 90
10. A 3200 **B** 18 000 **11. A** 4602 **B** 156 384
12. A 186, R18 **B** 146, R52 **13. A** 6154 **B** 43
C 555 **D** 413

UNIT A: MODULE 3

A-60 DISCUSSING THE IDEAS

1. A y **B** x **C** t **2.** c **3. A** $4 \cdot n = 28$
B $3 \cdot n - 1 = 11$

A-61 USING THE IDEAS

1. A $n + 17 = 31$ **B** $t - 6 = 42$ **C** $r \cdot 9 = 72$
D $56 \div q = 8$ **E** $z \cdot 2 \cdot 3 = 24$ **F** $p + 7 - 7 = 43$
G $6 \cdot 8 - c = 18$ **H** $17 \cdot x - 2 = 14$ **2. A** $n + 7 = 19$
B $n - 38 = 29$ **C** $n \cdot 8 = 96$ **D** $2 \cdot n + 17 = 35$
3. A $(n - 4) \cdot 5 = 15$ **B** $(9 + n) \cdot 6 = 120$
C $n - (19 + 12) = 11$

A-62 DISCUSSING THE IDEAS

1. The instructions for flow chart A are the opposite of those for B. **2.** Flow chart B starts with the results of flow chart B and ends with the beginning input of A.
3. Substitute the output number for n in the equation $12 \cdot n + 15 = 111$ **4.** $(18 - 11) \cdot 2 = n$ **5.** 14

A-63 USING THE IDEAS

1. $1500 \rightarrow -24 \rightarrow \div 36 \rightarrow n$

2. $2 \rightarrow +20 \rightarrow \cdot 7 \rightarrow n$

3. $245 \rightarrow \div 9 \rightarrow -14 \rightarrow +48 \rightarrow n$

4. ex. 1 $n \cdot 36 + 2 \cdot 4 = 1500$; $n = 41$
ex. 2 $n - 20 \div 7 = 2$; $n = 34$
ex. 3 $(n - 48 + 14) \cdot 7 = 245$; $n = 69$
5. A 63 **B** 3 **C** 3 **D** 22

6. A $n \rightarrow +427 \rightarrow \cdot 12 \rightarrow 5760$

B $5760 \rightarrow \div 12 \rightarrow -427 \rightarrow n$; $n = 53$

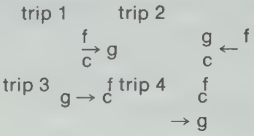
A-64 WRITING AND SOLVING EQUATIONS

1. B $x \cdot 3 = 24$; $x = 8$ **C** $y + 35 = 71$, $y = 36$
D $z \cdot 13 = 130$; $z = 10$ **E** $(r + 6) \cdot 7 = 49$; $r = 1$
F $(t - 9) \div 4 = 3$; $t = 21$ **G** $(w + 7) \cdot 3 - 2 = 25$; $w = 2$
2. A 36 **B** 7 **C** 111 **D** 6 **E** 3 **F** 8 **G** 8 **H** 16
I 3 **J** 6
Think **A** + **B** ÷

A-65 SOLVING WORD PROBLEMS WITH EQUATIONS

2. $n + 19 = 153$; $n = 134$ **3.** $63 \cdot n = 441$; $n = 7$
4. $10 \cdot n - 10 = 100$; $n = 11$ **5.** $2 \cdot n + 1 = 7$; $n = 3$
6. $4n = 23 - 3$; $n = 5$ **7.** $(n + 6) \cdot 4 = 40$; $n = 4$
8. $4 \cdot n - 47 = 1$; $n = 12$ **9.** $2 \cdot n + 2 \cdot 12 = 40$; $n = 8$
10. $n \div 2 - 30 = 14$; $n = 88$

Think



A-66 DISCUSSING THE IDEAS

1. A 1 **B** It uses a litre of gasoline every 6 kilometres
2. A 2 **B** How much money Kathy had before she went shopping **3. A** How many students the bus will carry.
B How many adult and / or student tickets were sold
4. A There is only one penny in the set. **B** The original number is an odd number less than 10.

A-67 USING THE IDEAS

1. 12 years **2.** 1497 km **3.** 25 765 **4.** 42 **5.** 1 m 74 cm **6.** not enough information **7.** 1818 **8.** not enough information **9.** 8 **10.** not enough information
11. 128 159 176
Think 5 dimes, 1 nickel, 4 pennies (or) 1 quarter, 1 dime, 4 nickels, 4 pennies

A-68 DISCUSSING THE IDEAS

1. About 700 000 **2.** Estimate the time to be 5 hr and the distance 4000 km; 800 km/h. **3.** $8.00; Find the number halfway between 59c and $1.00 and multiply it by 10.
4. Estimations help to speed up calculations and to check work. **5.** $5700

A-69 USING THE IDEAS

1. 1100 **2.** 90 **3.** 400 **4.** 5000 **5.** 6 **6.** 30
Think 4662; 162 192

A-70 DISCUSSING THE IDEAS

1. The only drawing showing 4 parts is B. **2.** The drawings help to visualize the problem and to check the answer. **3. A** $12 + 12 + xxx = 40$; $x = 8$ cm **B** 20 cm
C $20 - 12 = 8$ cm **D** $2n + 2 \cdot 12 = 40$; $n = 8$ **4. A** The distance between city A and city C is 275 km, and the distance between city A and city B is 169 km. **B** Example: find the distance between city B and city C; 106 km.

A-71 USING THE IDEAS

1. 84 cm **2.** 17 m **3.** 4 cm **4.** 9 **5.** 30 km / h; 20 km / h **6** to **11** Answers will vary. Suggestions follow:
6. How much did the groceries cost? $6.73 **7.** Find the width of the box. 32 cm **8.** How much does the x-marked box weigh? 12 kg **9.** How long are pieces A and B? A 34

cm, B 29 cm **10.** How wide is each quarter? 28 cm
11. Where will the cars meet? 180 km from A

A-72 DISCUSSING THE IDEAS

1. 9 hours is the sum of the time spent working alone.
2. It will be more than 3 but less than 6 hours. **3. A** 7
B 576 **C** 60

A-73 USING THE IDEAS

1. C **2.** A **3.** C **4.** C **5.** 42 km **6.** 18 **7.** $9
8. 14 min
Think 36

A-74 REVIEWING THE IDEAS

1. $(n \div 4) - 7 = 0$

2. $0 \rightarrow +7 \rightarrow \cdot 4 \rightarrow n$

3. 28 **4.** $(n - 3) \cdot 7 = 56$; $n = 11$ **5. A** 4 **B** 15 **C** 14
6. 388 **7.** C **8.** How long will it take to drive from A to B?
8 hours **9. A** 800 g **B** 900 000 **C** $270
10. length $= n + 4$ or 16 cm; $w = 12$ cm **B** 12 m; 24 m
11. 65 m **12.** 64

UNIT A: MODULE 4

A-78 DISCUSSING THE IDEAS

1. A **2.** yes **3.** If a number can be evenly divided by 2 it is even; if not it is odd. **4.** 9, 12, 15, 18, 24, 27 **5.** yes; $3 \cdot 17 = 51$ **6.** 0, 5, 10, 15; 0, 7, 14, 21

A-79 USING THE IDEAS

1. A odd **B** even **C** odd **D** odd **E** even **F** even
G odd **H** odd **I** even **J** even **2. A** even **B** even
C odd **D** odd **E** even **F** even **3. A** 3, 6, 9, 12, 15, 18, 21, 24, 27, 30 **B** yes **C** yes **4. A** 5, 10, 15, 20, 25, 30, 35, 40, 45, 50 **B** 15 **5. A** yes **B** yes **C** no **D** yes
E yes **F** no

A-80 DISCUSSING THE IDEAS

1. A 1, 7 **B** 1, 2, 4, 8 **C** 1, 3, 9 **D** 1, 2, 5, 10
E 1, 11 **F** 1, 13 **2.** 2; 5 **3.** 2 **4.** 1, 2, 3, 4, 6, 8, 12, 16, 24, 48 are all the factors of 48 **5.** yes; $136 \div 8 = 17$, R0

A-81 USING THE IDEAS

1. A 1, 5, 25 **B** 1, 7, 11, 77 **C** 1, 11, 13, 143 **D** 1, 3, 9, 27 **E** 1, 2, 3, 4, 6, 8, 12, 24 **F** 1, 2, 4, 5, 10, 20, 25, 50, 100
2. Answers will vary. **A** 2 **B** 3 **C** 2, 5, 10 **D** 5, 2
E 5, 2, 3 **F** 2, 3, 2, 2, 6, 4 **G** 2, 3, 2, 3, 6, 6 **3. A** 1, 2, 3, 4, 6, 12 **B** 1, 3, 5, 15 **C** 1, 2, 4, 5, 8, 10, 20, 40 **D** 1, 2, 3, 6, 9, 18 **E** 1, 2, 13, 26 **F** 1, 2, 3, 6 **G** 1, 2, 3, 5, 6, 10, 15, 30
H 1, 2, 4, 8, 16 **I** 1, 3, 9, 27 **J** 1, 2, 5, 10, 25, 50 **K** 1, 29
L 1, 2, 4, 8, 32, 64 **4.** 2, 3, 5, 7 **5. A** yes **B** yes **C** no
D yes **E** no **F** yes
Think **A** abundant **B** deficient **C** perfect
D abundant **E** deficient **F** deficient **G** deficient

A-82 DISCUSSING THE IDEAS

1. A 1 **B** 6 **2.** 2, 3, 5, 7, 11, 13, 17, 19 **3.** yes; 1
4. A 4 **B** composite **5. A** neither; it has only one factor and it is not greater than 1. **B** neither; it has more than 2 factors but is less than 1.

A-83 USING THE IDEAS

1. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97 **2. A** 4 **B** 73 **3.** 1 **4.** 2, 3
5. Answers will vary. 5, 7; 11, 13; 17, 19; 29, 31; 41, 43; 71, 73 **6. A** 2, 5 **B** No, since the difference of 3 must be obtained from an even and an odd number and 2 is the only even prime. **7. A** 2nd number **B** 289, 151 **8. A** 11, 13, 17, 23, 31, 41, 53, 67, 83, 101 **B** prime **C** no
Think **1.** $22 = 11 + 11$, $24 = 11 + 13$, $26 = 13 + 13$, $28 = 11 + 17$, $30 = 11 + 19$, $32 = 3 + 29$, $34 = 17 + 17$, $36 = 13 + 23$, $38 = 19 + 19$, $40 = 11 + 29$, $42 = 11 + 31$ ($42 = 5 + 37$ and $42 = 13 + 29$ and $42 = 19 + 23$), $44 = 13 + 31$, $46 = 23 + 23$, $48 = 11 + 37$, $50 = 13 + 37$
Each sum is not necessarily unique. See "42."
2. $200 = 197 + 3$

A-84 DISCUSSING THE IDEAS

1. They are prime numbers. **2. A** $2 \cdot 2 \cdot 3 \cdot 5$
B $2 \cdot 2 \cdot 2 \cdot 3$ **3. A** 126 **B** 297 **C** 1001

A-85 USING THE IDEAS

1. A 2 **B** 3, 2, 2 **C** 3, 3, 9 **D** 5, 3, 3 **E** 11, 2, 3, 11
F 2, 5, 3, 3, 9 **G** 2, 3, 2, 7, 6 **2. A** 5 **B** 2 **C** 7 **D** 2
E 3 **F** 5 **G** 2 **H** 3 **I** 7 **3. A** $3 \cdot 11$ **B** $5 \cdot 7$
C $2 \cdot 2 \cdot 3 \cdot 3$ **D** $2 \cdot 19$ **E** $2 \cdot 2 \cdot 2 \cdot 5$ **F** $2 \cdot 3 \cdot 7$ **G** $2 \cdot 5 \cdot 5$
H $3 \cdot 3 \cdot 3 \cdot 3$ **I** $2 \cdot 2 \cdot 5 \cdot 5$ **J** $2 \cdot 5 \cdot 5 \cdot 5$ **K** $3 \cdot 7 \cdot 11$
L $2 \cdot 3 \cdot 3 \cdot 7$ **4. A** 100 **B** 42 **C** 108 **D** 455 **E** 4114
F 405 **G** 899 **H** 78 **I** 3125 **J** 1311 **K** 4514
L 44 100 **5.** 2^{10}
Think **1.** 2, 4 **2.** 3, 6 **3.** 4, 8 **4.** 6, 12 **5.** 4, 6
6. 4, 6 **7.** 200

A-86 DISCUSSING THE IDEAS

1. The top row are always prime numbers. **2.** $32 = 2^5$; $81 = 3^4$; $343 = 7^3$ **3. A** 3^3 **B** $2^2 \cdot 5^2$ **C** $2 \cdot 3^2 \cdot 5$ **D** 2^6
E $5^2 \cdot 3^2$

A-87 USING THE IDEAS

1. A 4^2 **B** 9^5 **C** 10^3 **D** 15^6 **E** $2^2 \cdot 3^4$ **F** $4^3 \cdot 5^4$
2. A $16 \cdot 16 = 256$ **B** $4 \cdot 4 \cdot 4 \cdot 4 = 256$
C $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 256$
D $9 \cdot 9 = 81$ **E** $3 \cdot 3 \cdot 3 \cdot 3 = 81$ **F** 17
G $13 \cdot 13 \cdot 13 = 2197$ **H** $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 4 = 288$
I $1 \cdot 1 \cdot 1 \cdot 1 = 1$ **J** $4 \cdot 4 \cdot 4 \cdot 3 \cdot 3 \cdot 2 \cdot 1 = 1152$ **3. A** $2^4 \cdot 5$
B $2^2 \cdot 3^3$ **C** 3^5 **D** $2^3 \cdot 5^3$ **E** $2^4 \cdot 3^2$ **4. A** 125 **B** 27
C 3125 **D** 64 **E** 169 **F** 1024 **G** 729 **H** 4096
I 65 536 **J** 1 **K** 23 **L** 6 **5. A** 72 **B** 2500 **C** 189
D 128 **E** 100 000 **F** 900 **G** 1001 **H** 7744 **6. A** no; $2^3 \cdot 3^2$ **B** no; $2^3 \cdot 3^3$ **C** yes **D** no; $3 \cdot 11 \cdot 19$ **E** no; 11^3
F no; $2^3 \cdot 3^2 \cdot 5$ **G** no; $7 \cdot 11 \cdot 13$ **H** yes **7. A** 70
B 1300 **C** 4500 **D** 2200 **E** 6000 **F** 147 000

G 230 000 **H** 142 000 **I** 505 000 **J** 1 540 000
K 210 000 **L** 9 000 000 **8. A** 7007 **B** 11 011
C 13 013 **D** 77 077 **E** 17 017 **F** 53 053 **G** 253 253
H 1 002 001

A-88 DISCUSSING THE IDEAS

1. no **2.** 6 **3.** 35

A-89 USING THE IDEAS

1. A 1, 2, 3, 4, 6, 12 **B** 1, 2, 4, 5, 10, 20 **C** 1, 2, 4 **D** 4
2. A 1, 2, 4, 7, 14, 28 **B** 1, 5, 7, 35 **C** 1, 7 **D** 7 **3. B** 10
C 5 **D** 6 **E** 10 **F** 3 **G** 6 **H** 6 **I** 10 **J** 30 **K** 15
L 35 **4. A** 5 **B** 6 **C** 1 **D** 12 **E** 20 **F** 1 **G** 2
H 5 **I** 2 **J** 4 **K** 1 **L** 7 **M** 20 **N** 25 **O** 24 **P** 17
Q 9 **R** 30 **S** 30 **T** 90 **5.** 56 **6.** 1 **7.** 10 **8.** 12, 60 (or) 60, 300 (or) . . .

Think 3, 7, 11, 15, 19, 23, 27, 31, 35, 39, 43, 47, 51, 55, 59, 63, 67, 71, 75, 79, 99; 12

A-90 DISCUSSING THE IDEAS

1. 56 **2. A** 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36 **B** 4, 8, 12, 16, 20, 24, 28, 32, 36 **C** 12, 24, 36 **D** 12 **3. A** 18 **B** 28 **C** 60 **D** 24 **E** 35

A-91 USING THE IDEAS

1. A 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24 **B** 3, 6, 9, 12, 15, 18, 21, 24 **C** 6, 12, 18, 24 **D** 6 **2. A** 15 **B** 20
C 14 **D** 12 **E** 24 **F** 24 **G** 30 **H** 15 **I** 70 **J** 110
K 75 **L** 48 **M** 12 **N** 1200 **O** 60 **3. A** 4 **B** 6
C 10 **D** 24 **E** 24 **F** 15 **G** 40 **H** 16 **I** 48 **J** 60
4. A 30 **B** 12 **C** 12 **D** 8 **E** 12 **F** 6 **G** 24 **H** 30
I 60 **J** 120 **5. A** 30 **B** 60 **C** 84 **D** 72 **E** 36
6. A $2^2 \cdot 3^2$ **B** $3^3 \cdot 5^2$ **C** $2^3 \cdot 3^2$ **D** $2^2 \cdot 3 \cdot 5^2$ **E** $2^2 \cdot 3^2 \cdot 5$
F $3^3 \cdot 5^3 \cdot 7 \cdot 11$ **7. A** 240 s or 4 min **B** 2400 s or 40 min

A-92 REVIEWING THE IDEAS

1. A Any number that is 2 times a whole number is an even number. **B** Any number that is 1 more than an even number is an odd number. **2. A** 1, 2, 7, 14 **B** 1, 2, 3, 4, 5, 6, 10, 15, 20, 30, 60 **C** 1, 2, 3, 5, 6, 7, 10, 30, 35, 42, 70, 105, 210 **3.** $5808 \div 48 = 121$, R0 **4. A** 2 **B** composite
C no **D** no **5. A** yes **B** yes **C** yes **D** no **E** no
F no **G** no **H** no **I** no **J** yes **K** no **L** no
6. A 3, 3, 5; 5 **B** 13, 3, 2; 6 **C** 2, 5, 7, 2; 14
D 5, 5, 3, 3, 9 **7. A** $2^3 \cdot 3^2$ **B** $2^4 \cdot 3^2$ **C** $2^2 \cdot 5^2$ **D** $5^2 \cdot 3^3$
8. $2^2 \cdot 3 \cdot 7$ **9. A** 2^4 **B** 23^2 **C** 7^5 **D** $6^3 \cdot 4^2$ **E** 10^8
10. A 3^3 **B** $2^2 \cdot 3 \cdot 5$ **C** $2^4 \cdot 3$ **D** $2^3 \cdot 3 \cdot 5$ **E** $2^4 \cdot 5^2$
F $2^3 \cdot 3^2$ **11. A** 1, 3, 5, 9, 15, 27, 45, 135 **B** 1, 5, 7, 25, 35, 175 **C** 1, 5 **12. A** 8 **B** 1 **C** 1 **D** 7 **E** 6 **F** 26
13. A 5 **B** 2^2 **C** 11 **D** 3^2 **14. A** 0, 6, 12, 18, 24 **B** 0, 8, 16, 24, 32 **C** 24 **15. A** 6 **B** 18 **C** 24 **D** 12
E 30 **16. A** 24 **B** 30 **C** 36 **D** 60

A-94 CUMULATIVE REVIEW

1. A $\{0, 1, 2, 3, 5, 7, 8, 11\}$ **B** $\{0, 5\}$ **2. A** 9 **B** 7
C 8 **3.** 9351 **4.** 3 **5.** 100 000 **6.** 8.0×10^6
7. A 6 746 300 **B** 6 750 000 **8. A** $<$ **B** $>$ **9. A** 14
B $16_{(8)}$ **C** $24_{(5)}$ **10. A** $13_{(8)}$ **B** $7_{(8)}$ **11.** CXXVI
12. 74 **13.** 6 **14.** $n - 8 = 14$; $n - 14 = 8$
15. $n = \frac{27}{9}$; $9 = \frac{27}{n}$
16. A

n	$f(n)$
2	12
1	7
$\frac{22}{5}$	24
0	2
$\frac{48}{5}$	50

B $f(n) = 2n + 1$
17. A 2 **B** 5 **C** 4 **D** 7 **E** 6 **18. A** 5 **B** 35 **C** 7
19. A 1487 **B** 576 **20. A** 370 **B** 72 000 **C** 50
D 80 **21.** B **22. A** 1817 **B** 1 963 590 **23. A** 22, R25
B 39, R103 **24.** $n \cdot 4 + 8 = 40$

25. 40 $\rightarrow$ -8 $\rightarrow$ $\div 4$ $\rightarrow$ n

26. 8 **27.** $8 \cdot n - 17 = 39$, $n = 7$ **28.** 1 500 000
29. 130 cm **30.** 0, 4, 8, 12, 16 **31.** 1, 2, 3, 4, 6, 8, 12, 24
32. 221 **33.** $\{2, 13, 59\}$ **34.** $2 \times 3 \times 2 \times 5$; 10
35. $2^2 \cdot 3^2 \cdot 5^3$ **36.** $3^2 \cdot 5$ **37.** 720 000 **38. A** 3 **B** 8
39. A 45 **B** 104 **C** 60

UNIT B: MODULE 1

B-2 DISCUSSING THE IDEAS

1. A 30 **B** 1, 2, 3, 7 **2.** $\frac{13}{30}$ **3. A** $\frac{3}{4}$ **B** $\frac{1}{4}$
4. A proper **B** improper **C** improper **D** proper
E proper **F** improper **G** improper **H** improper

B-3 USING THE IDEAS

1. A $\frac{5}{7}$ **B** $\frac{1}{4}$ **C** $\frac{3}{12}$ **D** $\frac{1}{4}$ **2. A** $\frac{4}{9}$ **B** $\frac{3}{6}$ **C** $\frac{4}{3}$ **D** $\frac{9}{6}$
E $\frac{3}{8}$ **3. A** $\frac{1}{10}$ **B** $\frac{1}{100}$ **4. A** $\frac{1}{2}$ **B** $\frac{11}{16}$ **C** $\frac{1}{2}$ **5. A** $\frac{6}{25}$ **B** $\frac{3}{25}$
C $\frac{2}{25}$ **D** $\frac{13}{25}$ **E** $\frac{14}{25}$ **F** $\frac{9}{25}$

B-4 DISCUSSING THE IDEAS

1. A yes **B** yes **C** no **2. A** $\frac{6}{15}$ **B** $\frac{8}{20}$ **C** $\frac{10}{25}$ **D** $\frac{12}{30}$
3. equivalent

B-5 USING THE IDEAS

1. A $\frac{2}{3}, \frac{4}{6}$ **B** $\frac{6}{9}, \frac{12}{18}$ **C** $\frac{1}{4}, \frac{2}{8}$ **D** $\frac{3}{12}, \frac{4}{16}$ **2. A** $\frac{10}{12}$ **B** $\frac{15}{18}$
C $\frac{20}{24}$ **D** $\frac{25}{30}$ **E** $\frac{30}{36}$ **F** $\frac{4}{3}$ **G** $\frac{8}{6}$ **H** $\frac{12}{9}$ **I** $\frac{16}{12}$ **J** $\frac{20}{15}$ **K** $\frac{24}{18}$
3. A $\left\{\frac{10}{12}, \frac{12}{18}, \frac{14}{21}\right\}$ **B** $\left\{\frac{5}{20}, \frac{6}{24}, \frac{7}{28}\right\}$ **C** $\left\{\frac{15}{33}, \frac{18}{42}, \frac{21}{49}\right\}$
D $\left\{\frac{4}{24}, \frac{5}{30}, \frac{6}{36}\right\}$ **E** $\left\{\frac{12}{32}, \frac{15}{40}, \frac{18}{48}\right\}$ **F** $\left\{\frac{4}{36}, \frac{5}{45}, \frac{6}{54}\right\}$ **G** $\left\{\frac{3}{75}, \frac{4}{20}, \frac{5}{25}\right\}$
H $\left\{\frac{6}{21}, \frac{8}{28}, \frac{10}{35}\right\}$ **I** $\left\{\frac{14}{20}, \frac{21}{30}, \frac{28}{40}\right\}$ **J** $\left\{\frac{6}{4}, \frac{12}{8}, \frac{24}{16}\right\}$ **K** $\left\{\frac{8}{18}, \frac{16}{36}, \frac{32}{72}\right\}$
L $\left\{\frac{10}{10}, \frac{15}{15}, \frac{20}{20}\right\}$ **4. A** $\frac{8}{12}$ **B** $\frac{2}{3}$ **C** $\frac{12}{36}$ **D** $\frac{2}{3}$ **E** $\frac{10}{6}$ **F** $\frac{4}{18}$
Think $\frac{23}{60}$

B-6 DISCUSSING THE IDEAS

1. $\frac{1}{2}$ **2. A** $\frac{9}{16}$ **B** $\frac{4}{15}, \frac{9}{16}$ **3. A** $\frac{4}{5}$ **B** $\frac{5}{6}$ **C** $\frac{5}{2}$ **D** $\frac{2}{3}$ **E** 1
F $\frac{6}{20}$ **G** $\frac{3}{2}$

B-7 USING THE IDEAS

1. A $\frac{4}{5}$ **B** $\frac{6}{7}$ **C** $\frac{5}{8}$ **D** $\frac{3}{5}$ **E** $\frac{1}{1}$ **F** $\frac{9}{10}$ **G** $\frac{7}{10}$ **H** $\frac{3}{5}$ **I** $\frac{2}{9}$

J $\frac{4}{3}$ **K** $\frac{11}{14}$ **L** $\frac{3}{5}$ **M** $\frac{3}{8}$ **N** $\frac{2}{5}$ **2. A** no; $\frac{5}{6}$ **B** no; $\frac{5}{8}$
C yes **D** yes **E** no; $\frac{1}{3}$ **F** yes **G** no; $\frac{1}{4}$ **H** no; $\frac{3}{4}$
I yes **J** yes **K** no; $\frac{33}{33}$ **L** yes **M** yes **N** no; $\frac{2}{5}$
3. A $\frac{2}{3}$ **B** $\frac{1}{4}$ **C** $\frac{9}{10}$ **D** $\frac{5}{4}$ **E** $\frac{4}{3}$ **F** $\frac{2}{7}$ **4. A** $\frac{3}{5}$ **B** $\frac{2}{5}$ **C** $\frac{7}{13}$
D $\frac{23}{31}$ **E** $\frac{5}{7}$ **F** $\frac{11}{2}$ **G** $\frac{7}{13}$ **H** $\frac{3}{5}$ **I** $\frac{11}{31}$ **J** $\frac{5}{13}$ **K** $\frac{22}{13}$ **L** $\frac{23}{7}$
5. A 5, 12, 30, 20 **B** 6, 49, 21, 20 **C** 27, 40, 81, 64 **D** 15, 60, 48, 90 **E** 14, 121, 49, 143
Think 2 seconds

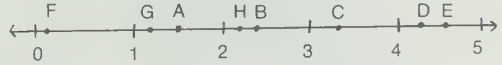
B-8 DISCUSSING THE IDEAS

1. both **2.** $x = \frac{3}{10}; y = \frac{2}{5}; z = \frac{1}{2}$ **3.** A = $\frac{3}{10}$; B = $\frac{1}{2}$; C = $1\frac{1}{10}$

B-9 USING THE IDEAS

1. A = $\frac{2}{3}$; B = $\frac{5}{3}$; C = $\frac{6}{3}$; D = $\frac{3}{6}$; E = $\frac{5}{6}$; F = $\frac{7}{16}$; G = $\frac{9}{6}$; H = $\frac{11}{6}$; I = $\frac{12}{6}$; **2.** A = $\frac{3}{4}$; B = $\frac{5}{4}$; C = $\frac{7}{4}$; D = $\frac{8}{4}$; E = $\frac{11}{4}$ **3.** A = $\frac{2}{5}$; B = $\frac{6}{5}$; C = $\frac{8}{5}$; D = $\frac{14}{5}$; E = $\frac{16}{5}$ **4.** R = $\frac{4}{3}$; V = $\frac{9}{7}$; S = $\frac{9}{3}$; T = $\frac{7}{2}$; U = $\frac{11}{3}$

5.



6. A = $\frac{5}{6}$; B = $\frac{4}{3}$; C = $\frac{11}{4}$; D = $\frac{18}{5}$; E = $\frac{33}{7}$

B-10 DISCUSSING THE IDEAS

1. yes **2. A** yes **B** no **C** yes **D** yes **E** no
3. A 15 **B** 3 **C** 5

B-11 USING THE IDEAS

1. B, D, G, H, I **2. A** = **B** = **C** = **D** ≠ **E** ≠ **F** ≠ **G** = **H** = **I** = **J** = **K** = **L** = **3. A** T **B** F **C** F **D** T **E** T **F** F **G** F **H** T **I** F **J** F **K** T **L** T **4. A** 10 **B** 14 **C** 8 **D** 2 **E** 20 **F** 9 **G** 15 **H** 5 **I** 4 **J** 6 **5. A** yes **B** no

Think It is impossible. Each letter in a shaded square must be moved to an unshaded square, but there are only four unshaded squares and five letters in the shaded squares. Thus, there will always be one letter (A, C, E, G, or I) which cannot be moved according to the rules.

B-12 DISCUSSING THE IDEAS

1. C **2. A** > **B** < **C** < **D** > **3. A** $\frac{3}{8}$, $\frac{1}{4}$ **B** $\frac{3}{8}$
4. A y **B** $x < y$ **5. A** $\frac{7}{8}$ **B** $\frac{7}{12}$ **C** $\frac{4}{5}$ **D** $\frac{12}{10}$ **E** $\frac{5}{2}$ **F** $\frac{1}{4}$

B-13 USING THE IDEAS

1. A 3, 4, $\frac{1}{4} < \frac{1}{3}$ **B** 10, 15, $\frac{5}{3} < \frac{5}{2}$ **C** 27, 28, $\frac{3}{4} < \frac{7}{6}$
D 14, 15, $\frac{2}{3} < \frac{3}{7}$ **E** 30, 28, $\frac{5}{8} > \frac{7}{12}$ **F** 35, 36, $\frac{7}{6} < \frac{6}{5}$
G 30, 33, $\frac{3}{10} < \frac{33}{100}$ **H** 30, 21, $\frac{1}{3} > \frac{7}{30}$ **2. A** < **B** > **C** < **D** < **E** > **F** < **G** > **H** < **I** < **J** > **K** > **L** < **M** < **N** < **O** > **3. A** $\frac{1}{3}$ **B** $\frac{3}{5}$ **C** $\frac{4}{5}$ **D** $\frac{3}{5}$ **E** $\frac{6}{5}$ **F** $\frac{1}{2}$ **G** $\frac{3}{4}$ **H** $\frac{2}{3}$ **I** $\frac{7}{8}$ **J** $\frac{7}{6}$ **K** $\frac{1}{9}$ **L** $\frac{4}{11}$ **M** $\frac{4}{19}$ **N** $\frac{7}{50}$
Think **1.** {70, 72, 74} **2.** {51, 53, 55, 57}
3. {71, 72, 73}

B-14 DISCUSSING THE IDEAS

1. yes, $\frac{61}{80}$ **2.** $\frac{115}{200}$ or $\frac{23}{40}$ **3.** yes **5. A** $\frac{5}{8}$ **B** $\frac{6}{8}$

B-15 USING THE IDEAS

1. $\frac{1}{2}$ **2. A** $\frac{4}{15}$ **B** smaller **C** $\frac{1}{5}$ **D** $\frac{9}{40}$ **3. A** $\frac{1}{2} < \frac{3}{4} < \frac{7}{8}$
B $\frac{1}{5} < \frac{3}{10} < \frac{3}{8}$ **C** $\frac{5}{4} < \frac{5}{3} < \frac{5}{2}$ **D** $\frac{1}{7} < \frac{1}{6} < \frac{1}{5}$ **E** $\frac{1}{3} > \frac{1}{6} > \frac{1}{12}$
F $\frac{2}{5} < \frac{3}{4} < \frac{7}{8}$ **4. A** 7 **B** 5 **C** 13 **D** 11 **E** 9 **F** 17
G 1 **H** 43 **5. A** $\frac{1}{32}$ **B** $\frac{1}{64}$, $\frac{1}{128}$ **6.** $\frac{1}{8}$ **7.** $\frac{1}{3}$ **8. A** $\frac{7}{8}$ **B** $\frac{4}{5}$
9. $\frac{15}{16}$, $\frac{7}{8}$, $\frac{7}{6}$, $\frac{9}{16}$, $\frac{3}{4}$, $\frac{2}{3}$, $\frac{5}{8}$, $\frac{3}{2}$ **10.** no
Think A = $\frac{59}{6}$ B = $\frac{61}{6}$

B-16 COMPARISONS IN THE METRIC SYSTEM

1. $\frac{1}{1000000}$ **2.** $\frac{1}{100}$ **3.** $\frac{1}{4}$ **4.** $\frac{1}{1000}$ **5.** $\frac{1}{1000}$ **6.** $\frac{1}{10000}$

B-17 FRACTIONAL NUMBER PROBLEMS

1. $\frac{2}{5}$ **2.** $\frac{10}{13}$ **3.** $\frac{3}{5}$ **4.** $\frac{5}{12}$ **5.** $\frac{3}{4}$ km **6.** B **7.** T-bone steak **8.** $\frac{2}{3}$

B-18 REVIEWING THE IDEAS

1. A $\frac{2}{8}, \frac{1}{4}$ **B** $\frac{5}{20}, \frac{1}{4}$ **C** $\frac{3}{5}, \frac{6}{10}$ **2. A** $\frac{8}{16}, \frac{12}{13}, \frac{16}{20}$ **B** $\frac{2}{12}, \frac{3}{18}, \frac{4}{24}$
C $\frac{10}{16}, \frac{15}{24}, \frac{20}{32}$ **D** $\frac{6}{20}, \frac{9}{30}, \frac{12}{40}$ **E** $\frac{10}{32}, \frac{15}{48}, \frac{20}{64}$ **3. A** 3 **B** 4 **C** 9
D 1 **E** 13 **F** 7 **G** 5 **H** 10 **4. A** $\frac{4}{5}$ **B** $\frac{5}{6}$ **C** $\frac{2}{3}$ **D** $\frac{2}{5}$
E $\frac{5}{6}$ **F** $\frac{4}{7}$ **G** $\frac{2}{7}$ **H** $\frac{2}{5}$ **I** $\frac{4}{3}$ **J** $\frac{12}{25}$ **K** $\frac{3}{10}$ **L** $\frac{5}{6}$ **5.** A = $\frac{3}{5}$, B = $\frac{7}{5}$, C = $\frac{14}{5}$, D = $\frac{1}{2}$, E = $\frac{7}{4}$, F = $\frac{5}{2}$, G = $\frac{1}{3}$, H = $\frac{5}{3}$, I = $\frac{17}{6}$
6. A 16 **B** 12 **C** 15 **D** 12 **E** 90 **F** 18 **7. A** < **B** > **C** = **D** < **E** > **F** < **8. A** $\frac{5}{6} > \frac{2}{3}$ **B** 19 **C** 9
9. $\frac{17}{24}$ **10.** 11 **11.** $\frac{6}{7}$ **12.** $\frac{7}{10}$

UNIT B: MODULE 2

B-22 DISCUSSING THE IDEAS

1. A Put the 0 on the top slide over $\frac{3}{8}$ then read the fraction under $\frac{7}{8}$ on slide B. **B** $\frac{5}{4}$ **2. A** $\frac{4}{8}, \frac{6}{8}$ **B** $\frac{5}{4}$ **3.** $\frac{1}{4}$
4. A $\frac{7}{8}$ **B** $\frac{5}{8}$

B-23 USING THE IDEAS

1. $\frac{3}{8} + \frac{4}{8} = \frac{7}{8}$ **2. A** $\frac{3}{4}$ **B** $\frac{3}{4}$ **C** 2 **D** $\frac{1}{8}$ **E** $\frac{5}{2}$ **F** 1 **G** 2
H 3 **I** $\frac{13}{16}$ **J** $\frac{7}{6}$ **K** $\frac{4}{5}$ **L** $\frac{2}{3}$ **3. A** 2, 5 **B** 10, 3 **C** 8, 13
4. A 4, 11 **B** 6, 5 **C** 18, 19 **D** 36, 29 **E** 8, 13
F 12, 21 **G** 6, 3 **H** 18, 13 **I** 22, 27 **5. A** $\frac{15}{8}$ **B** $\frac{9}{8}$
C $\frac{5}{8}$ **D** $\frac{3}{4}$ **E** $\frac{21}{8}$ **F** $\frac{3}{8}$ **G** $\frac{43}{8}$ **H** $\frac{17}{16}$ **I** $\frac{7}{16}$ **J** $\frac{9}{4}$ **K** $\frac{13}{2}$
L $\frac{23}{2}$ **6.** $(a + c) \div b$
Think **1.** 15, 21, 28, 36, 45 **2.** 25, 36, 49, 64, 81
2. The answer is a square number.

B-24 DISCUSSING THE IDEAS

1. 24 **2. A** 8, 16, 24, 32, 40, 48, 56, 64, 72, 80 **B** 14, 28, 42, 56, 70, 84, 98, 112, 126, 140 **C** 56 **D** 56
3. A $\frac{128}{210} = \frac{64}{105}$ **B** Find the prime factorization of 18 and 24, then find the LCM, $\frac{41}{72}$

B-25 USING THE IDEAS

1. A 80, 160 **B** 80 **C** 80 **D** $\frac{32}{80}$ **2. A** 9 **B** 24
C 30 **D** 24 **E** 36 **F** 56 **G** 15 **H** 100 **I** 48 **J** 22
K 36 **L** 48 **3. A** 135 **B** 135 **C** $\frac{648}{1350}$ **D** $\frac{2}{1350}$ **4. A** $\frac{17}{12}$
B $\frac{9}{8}$ **C** $\frac{7}{6}$ **D** $\frac{1}{24}$ **E** $\frac{7}{40}$ **F** $\frac{3}{16}$ **G** $\frac{1}{2}$ **H** $\frac{29}{30}$ **I** $\frac{3}{4}$ **J** $\frac{17}{30}$
K $\frac{7}{100}$ **L** $\frac{11}{10}$ **M** $\frac{59}{40}$ **N** $\frac{1}{6}$ **O** $\frac{5}{7}$ **5. A** $\frac{7}{8}$ **B** $\frac{15}{16}$ **C** $\frac{275}{16}$
D 3 **E** 5 **F** $\frac{8}{5}$ **G** 1 **H** $\frac{43}{28}$ **I** $\frac{10}{7}$ **J** $\frac{12}{5}$ **6. A** $\frac{5}{77}$
B $\frac{8}{35}$ **C** $\frac{38}{77}$ **D** $\frac{53}{72}$ **E** $\frac{47}{143}$ **F** $\frac{7}{45}$ **G** $\frac{3}{4}$
H $\frac{4}{5}$ **I** $\frac{1}{2}$ **J** $\frac{13}{30}$ **K** $\frac{31}{24}$ **L** $\frac{175}{120}$

B-26 DISCUSSING THE IDEAS

1. The numerator is equal to one added to twice the denominator of the preceding fraction, the denominator is equal to the sum of the numerator and denominator of the preceding fraction. **2. A** 5, 5, $1\frac{5}{12}$ **B** 1, 1, $7\frac{1}{3}$ **3. A** $2\frac{2}{5}$ **B** $5\frac{1}{2}$ **C** $5\frac{5}{8}$ **D** $33\frac{1}{3}$ **E** $4\frac{2}{9}$ **F** 1 **G** $4\frac{1}{5}$ **4. A** $\frac{5}{3}$ **B** $\frac{16}{5}$ **C** $\frac{10}{5}$ **D** $\frac{50}{8}$ **E** $\frac{99}{10}$ **F** $\frac{38}{7}$ **G** $\frac{87}{8}$

B-27 USING THE IDEAS

1. A $2\frac{1}{3}$ **B** $2\frac{3}{8}$ **C** $2\frac{7}{10}$ **D** $6\frac{1}{4}$ **E** $3\frac{11}{12}$ **F** $13\frac{1}{2}$ **G** $4\frac{7}{8}$ **H** $9\frac{3}{7}$ **I** $8\frac{2}{5}$ **J** $1\frac{15}{16}$ **K** $5\frac{11}{20}$ **L** $3\frac{13}{13}$ **M** $4\frac{9}{10}$ **N** $2\frac{23}{100}$ **O** $5\frac{287}{1000}$ **P** $7\frac{5}{8}$ **Q** $3\frac{1}{8}$ **R** $1\frac{1}{2}$ **2. A** $\frac{57}{8}$ **B** $\frac{17}{6}$ **C** $\frac{19}{5}$ **D** $\frac{26}{3}$ **E** $\frac{100}{3}$ **F** $\frac{65}{6}$ **G** $\frac{27}{4}$ **H** $\frac{109}{10}$ **I** $\frac{79}{8}$ **J** $\frac{75}{2}$ **K** $\frac{129}{6}$ **L** $\frac{10}{3}$ **M** $\frac{50}{5}$ **N** $\frac{80}{7}$ **O** $\frac{79}{4}$ **P** $\frac{121}{4}$ **Q** $\frac{125}{2}$ **R** $\frac{250}{3}$ **3. A** 3, 6, 9 **B** 21, 35, 49 **C** 40, 80, 100 **D** 13, 26, 39 **E** 63, 72, 81 **F** 0, 0, 0 **4. A** 8, 9 **B** 4, 5 **C** 3, 5 **D** 5, 9 **5. A** $\frac{1}{4}$ **B** $\frac{3}{8}$ **C** $\frac{1}{3}$ **D** $\frac{1}{8}$ **E** $\frac{3}{4}$ **F** $\frac{3}{16}$ **6. A** $14\frac{1}{4}$ **B** $11\frac{1}{2}$ **C** $27\frac{1}{2}$ **D** $2\frac{1}{3}$ **E** $11\frac{1}{5}$ **F** $5\frac{2}{5}$ **G** $33\frac{1}{4}$ **H** $12\frac{1}{6}$

Think $n = 73$; quotient is 4.

B-28 DISCUSSING THE IDEAS

1. A commutative principle **B** associative principle **2.** $\frac{3}{4}, \frac{5}{6}, \frac{1}{3}$; $\frac{3}{4}, \frac{1}{3}, \frac{2}{6}$; $\frac{1}{3}, \frac{5}{6}, \frac{3}{4}$; $\frac{5}{6}, \frac{3}{4}, \frac{1}{3}$ **3. A** zero principle **B** $\frac{23}{42}$ **4. A** associative principle $(\frac{1}{8} + \frac{7}{8}) + \frac{1}{4} = 1\frac{1}{4}$ **B** zero principle **C** associative and commutative principle $(2 + 6) + (\frac{1}{4} + \frac{3}{4}) = 9$ **5.** Rewrite each addend with a LCD, then add and reduce the sum to lowest terms. **6.** Rewrite each number with a LCD. Subtract the numbers, and write the answer in lowest terms.

B-29 USING THE IDEAS

1. A $1\frac{1}{4}$ **B** $1\frac{3}{8}$ **C** $1\frac{1}{3}$ **D** 1 **E** $1\frac{9}{10}$ **F** $1\frac{5}{9}$ **G** $\frac{8}{3}$ **H** $\frac{5}{7}$ **2. A** $1\frac{5}{12}$ **B** $1\frac{1}{2}$ **C** $1\frac{1}{24}$ **D** $1\frac{1}{72}$ **E** $1\frac{1}{15}$ **F** $1\frac{11}{36}$ **G** $56\frac{1}{10}$ **H** $28\frac{9}{20}$ **I** 100 **J** $82\frac{1}{5}$ **K** $74\frac{23}{24}$ **L** $40\frac{7}{18}$ **3. A** $4\frac{1}{8}$ **B** $3\frac{7}{8}$ **C** $6\frac{11}{12}$ **D** $\frac{3}{10}$ **E** $13\frac{1}{6}$ **F** $12\frac{3}{8}$ **G** $5\frac{2}{3}$ **H** $5\frac{5}{6}$ **I** $5\frac{3}{8}$ **J** $32\frac{1}{12}$ **K** $10\frac{1}{6}$ **L** $13\frac{3}{8}$ **4.** $2\frac{3}{4}$ **5.** $\frac{3}{4}$ block

Think **1.** $\frac{2}{3}$ **2.** $\frac{3}{4}$ **3.** $\frac{4}{5}$ **4.** $\frac{5}{6}$ **5.** $\frac{99}{100}$

B-30 PAPER PROBLEMS

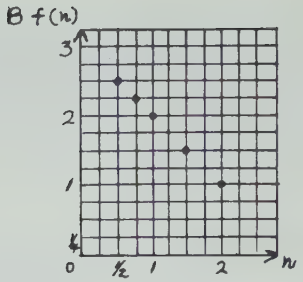
Measurements may vary. Example: **1.** 99 cm **2.** $35\frac{6}{10}$ cm **3.** $6\frac{1}{2}$ cm, $21\frac{1}{2}$ cm **4.** 91 cm **5.** 83 cm **6.** 75 cm; 67 cm **7.** less

B-31 1500-METRE OLYMPIC RECORDS

1. Arnold Jackson in 1912 **2.** 6.6 seconds **3.** Harry Larva ran the 1500 m in 0.4 seconds less than Paavo Nurmi. **4.** Lightbody ran 29.1 seconds slower than Vasala. **5.** Kipchoge Keino **6.** 1.7 seconds

B-32 DISCUSSING THE IDEAS

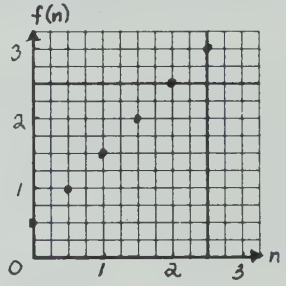
1. yes; a straight line **2. A** $f(\frac{1}{2}) = 2\frac{1}{2}$; $f(\frac{3}{4}) = 2\frac{1}{4}$; $(1) = 2$; $f(1\frac{1}{2}) = 1\frac{1}{2}$, $f(2) = 1$



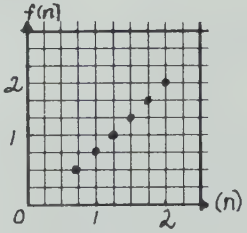
C yes, a straight line **3. A** $2\frac{5}{8}$, $2\frac{3}{8}$, $2\frac{1}{8}$ **B** yes

B-33 USING THE IDEAS

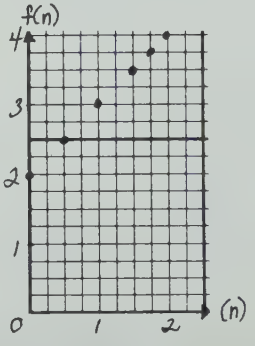
1.



2.



3.



4. A 1 **B** $1\frac{1}{4}$ **C** $1\frac{1}{2}$ **D** $1\frac{3}{4}$ **E** 2 **F** $2\frac{1}{4}$ **G** $2\frac{1}{2}$
5. A $1\frac{1}{2}$ **B** $1\frac{1}{4}$ **C** 1 **D** $\frac{3}{4}$ **E** $\frac{1}{2}$ **F** $\frac{1}{4}$ **G** 0
6. A 0 **B** $\frac{1}{2}$ **C** 1 **D** $1\frac{1}{2}$ **E** 2 **F** $2\frac{1}{2}$ **G** 3

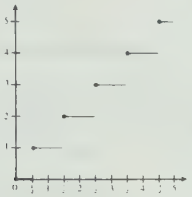
Think 9:00 am

B-34 INVESTIGATING THE IDEAS

1. yes **2.** yes **3.** rounds off to the next highest whole number **4. A** 3 **B** 7 **C** 4 **D** 17 **E** 1 **5. A** 6, 5 **B** In A you round off the sum only, whereas in B you round off the addends as well as the sum.

B-35 USING THE IDEAS

1. A 1 **B** 1 **C** 2 **D** 2 **E** 2 **F** 3 **2. A** 1 **B** 1 **C** 3 **D** 5 **E** 33 **F** 0 **3. A** 19 **B** 9 **C** 1 **D** 2 **E** 1100 **F** 0 **4.** 3 **5.** 4 **6.** 6 **7.** 5 **8. A** =, < **B** <, = **C** <, = **9. A** $1\frac{1}{2}$ **B** $2\frac{1}{2}$ **C** no **10.**



Think $[\overrightarrow{n}\overleftarrow{n}]_{(2)}$

B-36 REVIEWING THE IDEAS

1. A $4 + 5 = 9$ **B** $2 + 2\frac{1}{2} = 4\frac{1}{2}$ **C** $\frac{4}{9} + \frac{5}{9} = 1$ **D** $\frac{7}{24} + \frac{5}{24} = \frac{3}{8}$ **2. A** $\frac{2}{3}$ **B** $\frac{5}{7}$ **C** $\frac{2}{3}$ **D** $\frac{1}{3}$ **E** $\frac{2}{3}$ **F** 2 **G** $3\frac{2}{7}$ **H** 7 **3. A** 12 **B** 12 **4. A** $\frac{4}{15}$ **B** $\frac{13}{16}$ **C** $\frac{8}{33}$ **D** $\frac{13}{24}$ **5. A** $\frac{31}{3}$ **B** $\frac{73}{10}$ **C** $\frac{23}{8}$ **D** $\frac{11}{3}$ **E** $\frac{27}{4}$ **F** $\frac{35}{6}$ **6. A** $2\frac{2}{5}$ **B** $5\frac{2}{3}$ **C** $4\frac{5}{7}$ **D** $3\frac{7}{10}$ **E** $9\frac{1}{14}$ **F** $4\frac{41}{100}$ **7. A** $4\frac{3}{5}$ **B** $4\frac{2}{5}$ **C** $1\frac{5}{11}$ **D** $11\frac{4}{7}$ **8. A** $11\frac{5}{6}$ **B** $58\frac{5}{8}$ **9. A** $7\frac{5}{12}$ **B** $14\frac{7}{8}$ **10. A** $1\frac{1}{4}$ **B** $1\frac{1}{2}$ **C** $1\frac{3}{4}$ **D** $1\frac{5}{12}$ **11. A** 2 **B** 7 **C** 17 **12.** $11\frac{2}{3}$

UNIT B: MODULE 3

B-40 DISCUSSING THE IDEAS

1. $\frac{1}{8}$ **2.** $\frac{3}{5}$ **3. A** 2 **B** 2 **4.** $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$; $2 \cdot \frac{1}{3} = \frac{6}{3}$

B-41 USING THE IDEAS

1. A $\frac{1}{8}$ **B** $\frac{1}{12}$ **2. A** $5 \cdot \frac{1}{7} = 5$ **B** $4 \cdot \frac{1}{3} = 1\frac{1}{3}$ **C** $5 \cdot \frac{1}{2} = 2\frac{1}{2}$ **C** $8 \cdot \frac{1}{8} = 1$ **3. A** $\frac{1}{6}$ **B** $\frac{1}{20}$ **C** $\frac{1}{25}$ **D** $\frac{1}{8}$ **E** $\frac{1}{18}$ **F** $\frac{1}{20}$ **G** $\frac{1}{100}$ **H** $\frac{1}{1000}$ **I** $\frac{1}{32}$ **J** $\frac{1}{10000}$ **4. A** $\frac{3}{4}$ **B** $\frac{5}{8}$ **C** 1 **D** 2 **E** $\frac{1}{10}$ **F** $\frac{7}{1000}$ **G** 0 **H** $\frac{1}{8}$ **I** 6 **J** 50 **5. A** $\frac{1}{3} \cdot 30 = 10$ **B** $\frac{1}{10} \cdot 100 = 10$ **C** $\frac{1}{2} \cdot 14 = 7$ **D** $\frac{1}{6} \cdot 54 = 9$ **E** $\frac{1}{8} \cdot 32 = 4$ **F** $\frac{1}{3} \cdot 5 = 1$ **G** $\frac{1}{9} \cdot 54 = 6$ **H** $\frac{1}{4} \cdot 20 = 5$ **I** $\frac{1}{10} \cdot 20 = 2$ **J** $\frac{1}{25} \cdot 100 = 4$ **6.** $\frac{1}{12}$

Think end of the 7th day

B-42 DISCUSSING THE IDEAS

1. A $\frac{1}{10}$, $\frac{1}{10}$ **B** $\frac{1}{21}$, $\frac{1}{21}$ **C** $\frac{1}{72}$, $\frac{1}{72}$ **D** $\frac{1}{24}$, $\frac{1}{24}$ **E** $\frac{1}{30}$, $\frac{1}{30}$ **F** $\frac{1}{40}$, $\frac{1}{40}$ **2. A** associative **B** unit fraction **C** unit fraction **3.** associative and commutative **4. A** 1 **B** 0 **5. A** $\frac{5}{7}$ **B** $\frac{13}{25}$ **C** 0 **D** 0 **E** 0 **F** 1 **6. A** $\frac{4}{5}$ **B** $\frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8}$

B-43 USING THE IDEAS

1. A $\frac{2}{5}$ **B** $\frac{2}{7}$ **C** $\frac{9}{10}$ **D** $\frac{7}{25}$ **E** $\frac{5}{8}$ **F** $\frac{10}{3}$ **2. A** $\frac{1}{4}$ **B** $\frac{1}{8}$ **C** $\frac{5}{6}$ **D** $\frac{3}{5}$ **E** $2\frac{1}{2}$ **F** $\frac{9}{10}$ **G** $\frac{1}{9}$ **H** 0 **I** 0 **J** $\frac{7}{8}$ **3. A** $\frac{3}{8}$ **B** $\frac{3}{8}$ **C** $\frac{3}{8}$ **D** $\frac{5}{12}$ **E** $\frac{5}{12}$ **F** $\frac{5}{12}$ **G** $\frac{7}{10}$ **H** $\frac{7}{10}$ **4. A** $\frac{1}{3}$ **B** $\frac{1}{8}$ **C** 7 **5. A** $\frac{1}{2}$ **B** $\frac{7}{30}$ **C** $\frac{5}{8}$ **D** 1 **E** $\frac{7}{15}$ **F** $\frac{3}{100}$ **G** $\frac{23}{1000}$ **H** $\frac{5}{16}$ **I** $\frac{4}{21}$ **J** 1 **6. A** $\frac{3}{5}$ **B** $\frac{6}{7}$ **C** $\frac{17}{23}$ **D** $\frac{37}{45}$ **E** $\frac{2}{5}$ **F** $\frac{8}{9}$ **G** $\frac{10}{11}$ **H** 1 **7. A** 8 **B** 6 **C** 100 **D** $\frac{1}{20}$ **E** $\frac{1}{6}$ **F** 55 **G** $\frac{1}{20}$

Think 211

B-44 DISCUSSING THE IDEAS

1. $\frac{1}{2} \cdot 5$ **2.** $7 \cdot \frac{1}{3}$ **3.** $5 \cdot \frac{1}{2} \cdot \frac{1}{3}$ **4.** 1st step — unit fraction principle 2nd step — associative principle 3rd step — unit fraction principle 4th step — unit fraction principle **5. A** $\frac{3}{10}$ **B** $\frac{8}{35}$ **C** $\frac{3}{16}$ **D** $\frac{15}{32}$ **E** $\frac{35}{44}$ **F** $\frac{1}{4}$

B-45 USING THE IDEAS

1. A $\frac{9}{20}$ **B** $\frac{8}{15}$ **C** $\frac{5}{14}$ **D** $1\frac{1}{3}$ **E** $5\frac{5}{6}$ **F** $4\frac{3}{8}$ **G** $\frac{15}{16}$ **H** $\frac{1.5}{143}$ **I** $1\frac{535}{578}$ **J** $\frac{12}{133}$ **K** $\frac{12}{133}$ **L** $\frac{13}{19}$ **M** $1\frac{45}{256}$ **N** $\frac{35}{48}$ **O** $\frac{1}{2}$ **P** $\frac{1}{3}$ **Q** $1\frac{19}{100}$ **R** 1081/10000 **S** $\frac{1}{15}$ **T** $a \cdot c/b \cdot d$ **2. A** $5\frac{5}{6}$ **B** $2\frac{5}{8}$ **C** $3\frac{9}{32}$ **D** $34\frac{1}{2}$ **E** $54\frac{2}{3}$ **F** 10 **G** $29\frac{1}{6}$ **H** $10\frac{2}{8}$ **I** $5\frac{1}{7}$ **J** 30 **K** 16 **L** 170 **3. A** $\frac{1}{14}$ **B** $\frac{1}{6}$ **C** $\frac{4}{9}$ **D** $1\frac{7}{8}$ **E** $1\frac{1}{2}$ **F** $\frac{2}{3}$ **G** $\frac{5}{18}$ **H** $\frac{1}{8}$ **I** $\frac{32}{40}$ **J** $\frac{8}{9}$

Think 120

B-46 DISCUSSING THE IDEAS

1. Multiplying the 6 by the 4 and the $\frac{1}{3}$ by the 4 **2.** Add the results of $3 \cdot 5$ and $\frac{3}{4} \cdot \frac{1}{2}$. **3.** Find $1\frac{5}{4} \cdot \frac{1}{2}$ and change result to improper fraction.

B-47 USING THE IDEAS

1. A $48\frac{4}{5}$ **B** 6 **C** $28\frac{1}{6}$ **D** $76\frac{1}{4}$ **E** $26\frac{13}{32}$ **F** $91\frac{1}{2}$ **G** $209\frac{3}{8}$ **H** $1462\frac{2}{6}$ **I** $187\frac{1}{15}$ **J** $3042\frac{8}{13}$ **2. A** 25 **B** $17\frac{1}{2}$ **C** $12\frac{1}{2}$ **D** $5\frac{1}{8}$ **E** $1\frac{9}{16}$ **F** $5\frac{1}{6}$ **G** $6\frac{1}{2}$ **H** 83 **I** $6\frac{2}{3}$ **J** $6\frac{1}{2}$ **3.** 42 cm **4.** \$30 **5.** $18\frac{3}{4}$ hours **6.** $164\frac{3}{10}$ cm² **7.** \$6 **8.** 13 blocks **9.** \$3.57

B-48 DISCUSSING THE IDEAS

1. A 8 **B** $\frac{2}{3}$ **2.** No, division by zero is not defined in our number system. **3. A** $\frac{1}{3}$, $\frac{1}{2}$ **B** $\frac{1}{2}$, $\frac{2}{3}$ **C** 0 **D** 1 **E** 1 **4.** Invert the fraction. If it is a mixed fraction, change it to an improper fraction before inverting.

B-49 USING THE IDEAS

1. A $\frac{8}{3}$ **B** $\frac{5}{8}$ **C** $\frac{9}{1}$ **D** $\frac{1}{8}$ **E** $\frac{11}{4}$ **F** $\frac{8}{15}$ **G** $\frac{7}{37}$ **H** $\frac{100}{99}$ **2. A** $\frac{11}{3}$ **B** $\frac{6}{5}$ **C** $\frac{6}{7}$ **D** $\frac{7}{6}$ **E** $\frac{2}{7}$ **F** $\frac{1}{3}$ **G** 10 **H** $\frac{1}{100}$ **3. A** 4 **B** 7 **C** $3\frac{1}{2}$ **D** $\frac{1}{2}$ **E** $7\frac{1}{2}$ **F** 10 **4. A** 4 **B** 10 **C** 6 **D** 3 **E** 20 **F** $\frac{8}{3}$ **5. A** 2 **B** 5 **C** 20 **D** 8 **E** 3 **F** $\frac{7}{5}$ **6.** 2 **7. A** $1\frac{1}{2}$ **B** $4\frac{1}{2}$ **C** 1 **D** $1\frac{3}{4}$ **E** $\frac{1}{8}$ **F** 0

Think 9^9

B-50 DISCUSSING THE IDEAS

1. Divide the two numerators and the two denominators. **2.** $1 \div \frac{1}{3} = 3$ **3. A** $\frac{3}{4} \div \frac{3}{4} = \frac{1}{2}$; $\frac{3}{8} \div \frac{1}{2} = \frac{3}{4}$ **B** $\frac{2}{21} \div \frac{1}{3} = \frac{2}{7}$; $\frac{2}{21} \div \frac{2}{9} = \frac{1}{3}$ **C** $\frac{4}{15} \div \frac{2}{3} = \frac{3}{5}$; $\frac{4}{15} \div \frac{3}{5} = \frac{2}{3}$ **D** $\frac{9}{28} \div \frac{3}{4} = \frac{3}{7}$; $\frac{9}{28} \div \frac{3}{7} = \frac{3}{4}$ **4. A** $\frac{1}{2}$; $\frac{3}{4} \div \frac{3}{2} = \frac{1}{2}$ **B** $\frac{3}{2}$; $\frac{3}{10} \div \frac{1}{5} = \frac{3}{2}$ **C** $\frac{4}{5}$; $\frac{16}{35} \div \frac{4}{7} = \frac{4}{5}$ **D** $\frac{1}{4}$; $\frac{3}{20} \div \frac{3}{5} = \frac{1}{4}$ **5. A** $\frac{2}{3}$ **B** $\frac{1}{2}$ **C** $\frac{5}{7}$ **D** $\frac{7}{2}$

B-51 USING THE IDEAS

1. A $\frac{4}{3}$ **B** $\frac{3}{8}$ **C** $\frac{2}{3}$ **D** $\frac{3}{7}$ **2. A** $\frac{3}{4} \div \frac{3}{2} = \frac{1}{2}$; $\frac{3}{4} \div \frac{1}{2} = \frac{3}{2}$

B $\frac{6}{18} \div \frac{1}{3} = \frac{6}{6}$; $\frac{6}{18} \div \frac{2}{6} = \frac{1}{3}$ **C** $\frac{15}{8} \div \frac{5}{2} = \frac{3}{4}$; $\frac{15}{8} \div \frac{3}{4} = \frac{5}{2}$
D $\frac{15}{64} \div \frac{3}{8} = \frac{5}{8}$; $\frac{15}{64} \div \frac{5}{8} = \frac{3}{8}$ **3. A** $\frac{3}{4}$ **B** $\frac{2}{5}$ **C** $\frac{3}{2}$ **D** $1\frac{1}{2}$ **E** $\frac{7}{6}$
F $\frac{3}{5}$ **G** $3\frac{1}{3}$ **H** $1\frac{1}{2}$ **4. A** $\frac{1}{2}$; $\frac{3}{8} \div \frac{3}{4} = \frac{1}{2}$ **B** $\frac{2}{3}$; $\frac{10}{27} \div \frac{5}{9} = \frac{2}{3}$ **C** $\frac{3}{5}$;
$\frac{9}{20} \div \frac{3}{4} = \frac{3}{5}$ **D** $\frac{3}{2}$; $\frac{9}{16} \div \frac{3}{8} = \frac{3}{2}$ **E** $\frac{3}{7}$; $\frac{12}{35} \div \frac{4}{5} = \frac{3}{7}$ **F** 8;
$4 \div \frac{1}{2} = 8$ **G** 5; $\frac{1}{2} \div \frac{1}{10} = 5$ **H** $\frac{1}{2}$; $\frac{3}{10} \div \frac{3}{5} = \frac{1}{2}$ **I** $\frac{5}{7}$;
$\frac{25}{49} \div \frac{5}{7} = \frac{5}{7}$ **J** $\frac{2}{4}$; $\frac{9}{7} \div \frac{4}{7} = \frac{9}{4}$ **K** $\frac{2}{2}$; $\frac{10}{24} \div \frac{5}{12} = \frac{2}{2}$ **L** 1; $\frac{3}{2} \div \frac{3}{2} = 1$
5. A $7\frac{1}{2}$ **B** $1\frac{2}{3}$ **C** $\frac{1}{2}$ **D** 1 **E** $\frac{1}{2}$ **F** $1\frac{1}{6}$ **G** $1\frac{1}{4}$ **H** $\frac{3}{10}$ **I** $\frac{1}{3}$
J $\frac{3}{10}$ **K** $\frac{8}{5}$ **L** $\frac{2}{3}$
Think 9

B-52 DISCUSSING THE IDEAS

1. step 1 — multiplying $\frac{3}{4}$ by $2 \cdot 5/2 \cdot 5$ or 1; step 2 — using the associative properly; step 3 — dividing numerator by 2 and denominator by 5; step 4 — rewriting fraction as the product of two fractions; step 5 — multiplying the two fractions **2. A** reciprocals **B** Invert the divisor and multiply; $\frac{5}{4}$ **3.** $\frac{2}{7} \div \frac{3}{4} = \frac{2}{7} \cdot \frac{4}{3} = \frac{8}{21}$ **4.** reciprocal **5. A** $\frac{10}{7}$
B $\frac{3}{8}$; $\frac{93}{40}$ **C** $\frac{17}{4} \div \frac{7}{3} = \frac{17}{4} \cdot \frac{3}{7} = \frac{51}{28}$ **D** $\frac{5}{1} \div \frac{10}{3} = \frac{5}{1} \cdot \frac{3}{10} \cdot \frac{3}{2}$

B-53 USING THE IDEAS

1. A $\frac{6}{5}$ **B** $\frac{5}{12}$ **C** $\frac{6}{7}$ **D** $\frac{1}{4}$ **E** 12 **F** $\frac{3}{5}$ **G** 10 **H** $\frac{7}{6}$ **I** $\frac{6}{7}$
J 12 **K** $\frac{80}{3}$ **L** $\frac{72}{5}$ **M** $\frac{25}{3}$ **N** 2 **O** $\frac{2}{3}$ **P** 3 **Q** $\frac{2}{3}$ **R** $\frac{4}{3}$
S $\frac{1}{2}$ **T** $\frac{7}{6}$ **2. A** $\frac{10}{6}$ **B** $\frac{7}{4}$ **C** $\frac{14}{15}$ **D** $\frac{32}{9}$ **E** $\frac{11}{6}$ **F** $\frac{8}{3}$ **G** $\frac{27}{10}$
H $\frac{15}{4}$ **I** $\frac{27}{50}$ **J** $\frac{127}{42}$ **K** 30 **L** 4 **M** $\frac{4}{11}$ **N** 5 **O** $\frac{4}{3}$ **P** 2
3. 10 **4.** 720 km **5.** $4\frac{14}{25}$ **6.** $\frac{16}{81}$

B-54 DISCUSSING THE IDEAS

1. A $\frac{4}{5}$ **B** $\frac{2}{5}$ **2. A** $\frac{2}{3}$ **B** $\frac{5}{8}$ **3. A** $\frac{1}{2}/\frac{2}{3}$ **B** $\frac{3}{4}/\frac{3}{8}$ **C** $\frac{5}{6}/\frac{1}{10}$
D $\frac{5}{4}/\frac{4}{3}$ **4. A** $\frac{2}{5} \div \frac{3}{4}$ **B** $\frac{1}{2} \div \frac{5}{6}$ **C** $\frac{9}{10} \div \frac{3}{5}$ **D** $\frac{17}{32} \div \frac{1}{4}$ **5. A** $\frac{6}{5}$
B $2\frac{1}{10}$ **C** $\frac{3}{4}$; $\frac{4}{3}$; 10 **D** $\frac{5}{8} \div \frac{7}{18} = \frac{5}{8} \cdot \frac{18}{7} = \frac{45}{28} = \frac{17}{28}$
6. Multiplicative property of one **A** $\frac{6}{5}$ **B** $2\frac{1}{10}$

B-55 USING THE IDEAS

1. A $\frac{4}{5}$ **B** $\frac{7}{16}$ **C** $\frac{8}{9}$ **D** $\frac{11}{3}$ **E** $\frac{19}{25}$ **F** $\frac{23}{4}$ **G** $\frac{17}{6}$ **H** $\frac{14}{3}$
I $\frac{1}{2}$ **J** $\frac{10}{3}$ **K** $\frac{9}{49}$ **L** $\frac{18}{18}$ **2. A** $\frac{5}{6}/\frac{1}{2}$ **B** $\frac{7}{6}/\frac{3}{4}$ **C** $2/\frac{3}{4}$ **D** $\frac{1}{2}/4$
E $\frac{17}{3}/\frac{7}{3}$ **3. A** $\frac{3}{5} \div \frac{1}{2}$ **B** $\frac{1}{8} \div \frac{5}{2}$ **C** $\frac{1}{3} \div \frac{7}{3}$ **D** $\frac{3}{10} \div \frac{5}{4}$ **E** $\frac{3}{4} \div \frac{3}{2}$
4. A 2 **B** $1\frac{2}{3}$ **C** 10 **D** $\frac{1}{2}$ **E** $\frac{1}{3}$ **F** $\frac{1}{2}$ **G** $\frac{1}{2}$ **H** $\frac{1}{3}$ **I** 3
J $\frac{3}{8}$ **K** $\frac{1}{4}$ **L** $1\frac{1}{8}$ **M** $\frac{1}{4}$ **N** $\frac{3}{2}$ **O** 2 **P** $\frac{48}{35}$ **Q** $\frac{3}{4}$ **R** 10
Think 1. $1\frac{1}{3} \cdot 6 = 1\frac{1}{3} + 6$; $1\frac{1}{6} \cdot 7 = 1\frac{1}{6} + 7$ **2.** $4 \cdot \frac{4}{5} = 4 - \frac{4}{5}$;
$5 \cdot \frac{5}{6} = 5 - \frac{5}{6}$ **3.** $4\frac{1}{6} \div \frac{5}{6} = 4\frac{1}{6} + \frac{5}{6}$; $5\frac{1}{7} \div \frac{6}{7} = 5\frac{1}{7} + \frac{6}{7}$
4. $7\frac{1}{5} \div 6 = 7\frac{1}{5} - 6$; $8\frac{1}{6} \div 7 = 8\frac{1}{6} - 7$

B-56 FRACTIONAL NUMBER FUNCTIONS

1. A $\frac{4}{3}$ **B** 2 **C** $\frac{2}{3}$ **D** 0 **E** 1 **F** $\frac{4}{5}$ **G** $\frac{7}{3}$ **H** $\frac{3}{2}$ **I** $\frac{9}{4}$
J $\frac{33}{10}$ **K** $\frac{6}{5}$ **L** 150 **2. A** 1 **B** $\frac{1}{10}$ **C** $\frac{1}{4}$ **D** $\frac{5}{16}$ **E** $\frac{5}{12}$
3. A $\frac{2}{25}$ **B** $2\frac{2}{3}$ **C** $8\frac{3}{4}$ **D** $2\frac{1}{2}$ **E** $1\frac{1}{25}$ **4. A** 0 **B** $\frac{10}{21}$
C $1\frac{11}{25}$ **D** $1\frac{1}{3}$ **E** $6\frac{1}{4}$ **5. A** $6\frac{1}{2}$ **B** $2\frac{3}{5}$ **C** 0 **D** $\frac{2}{3}$ **E** $6\frac{5}{7}$
6. A $\frac{1}{2}$ **B** $1\frac{3}{10}$ **7. A** $1\frac{1}{3}$ **B** 6 **C** $6\frac{4}{7}$ **D** $8\frac{5}{8}$

B-57 PROBLEM SOLVING — DIVISION

1. 16 **2.** 8 **3.** $1\frac{1}{8}$ **4.** $193\frac{3}{4}$ cm **5.** $1053\frac{1}{2}$ km/h
6. $5\frac{2}{25}$ km/h **7.** 80 km/h **8.** 4 hours **9.** $2\frac{1}{2}$ cm **10.** 8
11. $\frac{1}{8}$ **12.** $\frac{8}{15}$ **13.** $\frac{15}{8}$

B-58 THE NORTHWEST TERRITORIES

1. 10 100 000 km² **2.** 104 000 **3.** 19 130 km²
4. 547 000 km² **5.** 17 563 **6.** 1 675 400 km² **7.** 6360 km

B-59 THE PROVINCES

1. A Quebec **B** Prince Edward Island **C** Ontario
D Prince Edward Island **2. A** Prince Edward Island
B Newfoundland **3.** Answers will vary. Example: make a graph or chart that ranks the provinces in order of their size? their population?

B-60 DISCUSSING THE IDEAS

1. $x = 28 \cdot 15 \div 12$ or 35 **2. A** $4 \cdot x = 8 \cdot 6$ **B** 12 **3.** X
4. A, C

B-61 USING THE IDEAS

1. A $50 \cdot x = 80 \cdot 10$; 16 **B** $8\frac{1}{2} \cdot 20 = 12 \cdot x$; $14\frac{1}{6}$
2. A $48 \cdot 4\frac{1}{2} = 9 \cdot m$; 24 **B** $7 \cdot n = 6 \cdot 21$; 18 **3.** 224 cm
4. $1\frac{2}{3}$ g **5.** 60 cm

Think A

$\frac{1}{2}$	$\frac{7}{12}$	$\frac{1}{6}$
$\frac{1}{12}$	$\frac{5}{12}$	$\frac{3}{4}$
$\frac{2}{3}$	$\frac{1}{4}$	$\frac{1}{3}$

B

$6\frac{1}{4}$	5	$11\frac{1}{4}$
$12\frac{1}{2}$	$7\frac{1}{2}$	$2\frac{1}{2}$
$3\frac{3}{4}$	10	$8\frac{3}{4}$

B-62 REVIEWING THE IDEAS

1. $7 \cdot \frac{1}{4} = \frac{7}{4}$ **2. A** $\frac{1}{12}$ **B** $\frac{1}{10}$ **C** $\frac{5}{6}$ **D** $\frac{3}{7}$ **3. A** 4 **B** 3
C 9 **D** 9 **E** 7 **F** 7 **4. A** $\frac{3}{8}$ **B** 1 **C** $\frac{5}{3}$ **D** $\frac{4}{5}$ **E** $\frac{1}{5}$
F $\frac{1}{10}$ **5. A** $\frac{5}{12}$ **B** $\frac{3}{28}$ **C** $\frac{1}{10}$ **D** $\frac{1}{2}$ **E** $\frac{1}{6}$ **F** $\frac{2}{3}$ **6. A** 9
B 42 **C** 8 **D** $4\frac{1}{3}$ **E** $8\frac{1}{6}$ **F** 10 **7. A** $\frac{13}{17}$ **B** $\frac{3}{17}$ **C** $\frac{1}{5}$
D 6 **E** $\frac{2}{7}$ **F** $\frac{7}{31}$ **8. A** $1\frac{3}{5}$ **B** $\frac{2}{3}$ **C** $\frac{63}{88}$ **D** $3\frac{7}{16}$ **E** $1\frac{11}{32}$
F $2\frac{2}{3}$ **9. A** 2 **B** 4 **C** $\frac{5}{14}$ **10.** $27\frac{1}{2}$¢ **11.** $2880.00
12. $157\frac{1}{2}$ km **13.** $100 **14.** 12 cm

UNIT B: MODULE 4

B-66 DISCUSSING THE IDEAS

1. A 1000 **B** 100 **2.** 0.001 **3. A** 237 **B** 0.237
4. A $\frac{8}{10}$ **B** 5/10 000 **C** $\frac{2}{100}$

B-67 USING THE IDEAS

1. A hundreds; $4 \cdot 100$ **B** hundredths; $3 \cdot \frac{1}{100}$
C thousands; $9 \cdot 1000$ **D** thousandths; $5 \cdot \frac{1}{1000}$ **E** ten thousandths; $8 \cdot \frac{1}{10000}$ **F** hundred thousandths; $4 \cdot \frac{1}{100000}$
G millionths; $7 \cdot \frac{1}{1000000}$ **2. A** 0.9 **B** 0.17 **C** 0.247
D 0.03 **E** 0.018 **F** 0.1637 **3. A** $7\frac{52}{100}$ **B** $\frac{364}{1000}$ **C** $9\frac{452}{1000}$
D $\frac{2763}{10000}$ **E** $27\frac{584}{1000}$ **F** $8\frac{73}{1000}$ **4. A** 8.4 **B** 21.23 **C** 0.69
D 8.472 **E** 15.407 **F** 29.2152 **G** 614.03 **H** 50.047
I 50.0471 **J** 0.37019 **K** 4.000612 **L** 25.005607
5. A $6\frac{4}{10}$ **B** $17\frac{31}{100}$ **C** $5\frac{3}{100}$ **D** $1\frac{5}{10}$ **E** $2\frac{3}{1000}$ **F** $81\frac{426}{1000}$
G $919\frac{7154}{10000}$ **H** $40\frac{352}{10000}$ **I** $2\frac{1746}{10000}$ **J** $286\frac{111111}{1000000}$ **K** $27\frac{5680}{10000}$
L $3\frac{123}{1000000}$ **6. A** 1 **B** 2 **C** 3 **7. A** $(3 \cdot 10^1) + (5 \cdot 10^0) + (4 \cdot 1/10^1) + (2 \cdot 1/10^2)$ **B** $(1 \cdot 10^0) + (2 \cdot 1/10^1) + (5 \cdot 1/10^2) + (4 \cdot 1/10^3)$ **C** $(5 \cdot 10^2) + (6 \cdot 10^1) + (2 \cdot 10^0) + (3 \cdot 1/10^1) + (4 \cdot 1/10^2)$ **D** $(5 \cdot 10^2) + (3 \cdot 10^0) + (2 \cdot 1/10^1) + (8 \cdot 1/10^3)$
E $(3 \cdot 1/10^2) + (7 \cdot 1/10^4)$ **F** $(2 \cdot 10^1) + (7 \cdot 10^0) + (6 \cdot 1/10^1) + (4 \cdot 1/10^2)$ **G** $(8 \cdot 10^0) + (3 \cdot 1/10^1) + (7 \cdot 1/10^2) + (4 \cdot 1/10^3)$
H $(9 \cdot 10^2) + (7 \cdot 10^1) + (3 \cdot 10^0) + (5 \cdot 1/10^1) + (6 \cdot 1/10^2)$
I $(8 \cdot 10^2) + (6 \cdot 10^1) + (7 \cdot 1/10^2) + (9 \cdot 1/10^3)$
J $(5 \cdot 1/10^1) + (6 \cdot 1/10^3)$

B-68 DISCUSSING THE IDEAS

1. D **2.** B **3. A** < **B** < **C** yes **4. A** yes **B** no **5. A** < **B** > **C** = **D** < **E** > **F** = **6.** B, A, C, E, D

B-69 USING THE IDEAS

1. B, C, D **2. A** < **B** > **C** > **D** < **E** > **F** > **3. A** < **B** = **C** > **D** > **E** > **F** = **G** > **H** < **I** = **J** > **K** = **L** < **M** > **N** > **O** < **P** > **Q** < **R** =

Think

$$\begin{array}{r} 9567 \\ +\ 1085 \\ \hline 10652 \end{array}$$

B-70 DISCUSSING THE IDEAS

1. Decimal points should be lined up **2. A** $\frac{5}{10} = \frac{500}{1000}$ **C** 1591 **3.** decimal in wrong position **4. A** $\frac{4}{10} = \frac{400}{1000}$ **C** 48.463 **D** 48.463 + 8.937 = 57.400

B-71 USING THE IDEAS

1. A 11.66 **B** 22.71 **C** 5.64 **D** 7.52 **E** 19.12 **F** 1.18 **G** 0.14 **H** 236.463 **2. A** 11.84 **B** 15.12 **C** 5.65 **D** 24.32 **E** 43.34 **F** 9.83 **3. A** 89.712 **B** 76.619 **C** 23.565 **D** 32.673 **E** 44.8 **F** 58.077 **G** 90.857 **H** 2.061 **4. A** 31.48 **B** 7.9 **C** 3.52 **D** 39.84 **E** 436.8108 **F** 23.3 **G** 44.778 **H** 4.442 **I** 206.203 **J** 124.738 **K** 1.0122 **L** 14.54 **M** 99.99 **N** 1.7558 **O** 0.163 **P** 0.009 **Q** 99.99 **R** 9.99989

Think **A** 2208; 2208 **B** 4416; 4416

B-72 DISCUSSING THE IDEAS

1. A decametre **B** centigram **C** millimetre **D** decilitre **2. A** 3 **B** 0.25 **C** 341 **D** 120 **E** 2 **F** 23

B-73 USING THE IDEAS

1. A 10 **B** 100 **C** 1000 **D** $\frac{1}{10}$ **E** $\frac{1}{100}$ **F** $\frac{1}{1000}$ **G** 10 **H** 10 **I** $\frac{1}{10}$ **J** $\frac{1}{10}$ **K** $\frac{8}{10}$ **L** $\frac{1}{4}$ **M** $\frac{59}{100}$ **N** $\frac{4}{1000}$ **O** 2.9 **2. A** 1000 **B** $\frac{1}{1000}$ **C** 100 **D** $\frac{1}{10}$ **E** 1000 **F** $\frac{1}{100}$ **3.** 2.4 **4.** 9.51 **5. A** 0.13 **B** 13 **6.** 1.05

B-74 DISCUSSING THE IDEAS

1. A 0.24 **B** 0.1134 **2.** 29.202

B-75 USING THE IDEAS

1. A 0.12 **B** 0.025 **C** 0.48 **D** 0.035 **E** 0.072 **F** 0.0048 **G** 0.21 **H** 0.0054 **I** 0.000028 **J** 0.000015 **2. A** 112.6 **B** 8.7 **C** 16.2 **D** 44.1 **E** 9.9 **F** 1.224 **G** 4.543 **H** 0.1066 **I** 2.4843 **J** 18.0404 **K** 0.68341 **L** 6.43112 **M** 0.197566 **N** 0.37944 **O** 1.83998

B-75 SHORT STORIES — THE HUMAN BODY

1. 40.136 **2.** 1.28 **3.** 24.52 **4.** 0.6 **5.** 12.222 **6.** 5.625 **7.** 1217.7

B-76 DISCUSSING THE IDEAS

1. A 1.2 **B** 1.2 **2. A** 0.12 **B** 0.12 **3. B** 42193; 4219.3; 421.93; 4.2193; 0.42193; 0.042193 **C** 40; 4; 0.4; 0.004; 0.0004; 0.00004 **D** 264380; 26438; 2643.8; 26.438; 2.6438; 0.26438 **E** 6394712.5; 639471.25; 63947.125; 639.47125; 63.947125; 6.3947125

B-77 USING THE IDEAS

1. A 85 **B** 6.4 **C** 90.37 **D** 503 **E** 270 **F** 2910 **G** 5468 **H** 3140 **I** 820 **J** 93.58 **K** 9358 **L** 9358 **2. A** 456.2 **B** 49.6 **C** 8437 **D** 9.37 **E** 846.2 **F** 76.2 **G** 376 **H** 840 **I** 53762 **J** 563 **K** 88.91 **L** 5.7 **3. A** 0.83 **B** 2.517 **C** 43.208 **D** 3.27 **E** 0.426 **F** 0.0579 **G** 0.9643 **H** 0.08756 **I** 6.6842 **J** 0.105 **K** 0.039 **L** 0.0082 **4. A** 156.2 **B** 0.0726 **C** 84.274 **D** 0.8397 **E** 0.83 **F** 0.677 **G** 0.8464 **H** 0.0074

Think $1.19

B-78 DISCUSSING THE IDEAS

2. A 7 **B** 12 **C** 7; 12 **3. A** 23.8 **B** 23.8 **4.** 0.96 **5.** 0.24

B-79 USING THE IDEAS

1. A 63 **B** 342 **C** 10 **D** 5 **E** 60 **2. A** 30 **B** 56 **C** 60 **D** 217 **E** 240 **F** 261 **G** 5841 **H** 203 **3. A** 2.4 **B** 1.4 **C** 4 **D** 24 **E** 2.5 **F** 3.6 **4. A** 0.21 **B** 0.48 **C** 1.8 **D** 0.24 **E** 0.27 **F** 0.56 **5. A** 544 **B** 1000 **C** 1540 **D** 5 **E** 50 **F** 3 **G** 4500 **H** 5000 **I** 945 **J** 40000 **K** 6 **L** 12 **M** 0.96 **N** 9.7 **O** 4.62

B-80 ESTIMATING BODY HEIGHTS

1. 166 cm **2.** 160 cm **3.** 169 cm

B-81 PLANET MASSES

1. A 17 **B** 1.02×10^{26} **2.** 5.7×10^{26} **3.** 4.8×10^{24} **4.** 6.6×10^{23} **5.** 3×10^{23} **6.** 7 **7.** 2 **8.** 83

B-82 DISCUSSING THE IDEAS

1. multiply **2. A** 7.31

B-83 USING THE IDEAS

1. A 33.8 **B** 3.37 **C** 3.79 **D** 6.5 **E** 0.008 **F** 0.6 **G** 0.37 **H** 0.25 **2. A** 74 **B** 0.07 **C** 1.35 **D** 437 **E** 2690 **F** 0.936 **G** 530 **H** 2.5 **I** 446 **J** 7 **K** 17 **L** 0.09 **M** 61.8 **N** 9460 **O** 2.77 **P** 217.4 **Q** 3.7 **R** 610 **S** 8.4 **T** 990 **U** 0.51 **V** 3.9 **W** 0.64 **X** 410 **3. A** 0.38 **B** 4.65 **C** 47.05 **D** 49.46 **E** 5.21 **F** 6805.13 **G** 0.81 **H** 5148.25 **I** 0.44 **J** 0.84 **K** 0.58 **L** 0.01

Think

B-84 SHORT STORIES — SPACE TRAVEL

1. 7.8 **2.** 11.6 **3.** 6800 **4.** 5 **5.** 7.4 **6.** 72.5 **7.** 7.5 **8.** 9.5 **9. A** 8640 **B** 127008 **10.** 62.4 **11.** 5.02 **12.** 78 **13.** 3.4593

B-86 DISCUSSING THE IDEAS

1. 0.8 **2.** 0.875 **4.** 0.12 **5.** no **6.** 0.3333 . . .

B-87 USING THE IDEAS

1. A 46; 0.46 **B** 8; 0.8 **C** 28; 0.28 **D** 115; 0.115
E 18; 0.018 **F** 256; 0.256 **G** 75; 0.75 **H** 136; 0.136
I 2875; 0.2875 **2. A** 0.625 **B** 0.24 **C** 13.5 **D** 0.0625
E 0.225 **F** 0.0875 **3. A** 0.6 **B** 0.25 **C** 0.875
D 0.6875 **E** 0.35 **F** 0.76 **G** 0.54 **H** 1.5 **I** 0.725
J 0.5375 **K** 0.66 **L** 0.91875 **M** 0.035 **N** 0.0225
O 1.138 **P** 0.16 **Q** 0.744 **R** 0.49 **S** 0.8 **T** 0.7712
4. A 0.34 **B** $\approx$ 0.07 **C** 0.06 **D** $\approx$ 0.04 **E** $\approx$ 0.14
F $\approx$ 0.073 **G** $\approx$ 0.126 **H** $\approx$ 0.14 **I** $\approx$ 0.06 **J** $\approx$ 0.36
K $\approx$ 0.18 **L** 0.206 **M** $\approx$ 0.614 **N** 0.184 **O** $\approx$ 0.072
P $\approx$ 0.779 **Q** 5.54 **R** 0.08 **S** 0.36 **T** 0.38
U $\approx$ 0.79 **V** $\approx$ 0.076 **W** 0.168 **X** $\approx$ 0.63 **Y** $\approx$ 0.57
5. A $n = 2.99999$ ($x \approx \frac{3}{7}$) **B** $n = 15.9999$ ($x \approx \frac{16}{19}$)
C $n = 1.99998$ ($x \approx \frac{2}{5}$)

B-88 DISCUSSING THE IDEAS

1. 0 **2.** repeats **3. A** no **B** repeating
4. B = 0.833 . . . ; $0.8\overline{3}$ D = 0.571428571428 . . . ; $0.\overline{571428}$

B-89 USING THE IDEAS

1. A 0.666 . . . **B** $0.1\overline{66}$ . . . **C** 0.6363 . . .
D $0.111\ldots$ **E** $0.\overline{185}$ **F** 0.8333 . . . **G** $0.36\overline{36}$ . . .
H $0.054\overline{054}$ **I** 0.4666 . . . **J** $0.\overline{384615}$ **K** $0.\overline{142857}$
L 0.555 . . . **M** 0.41666 . . . **N** $0.\overline{2941176470588235}$
2. A 0.67 **B** 0.17 **C** 0.63 **D** 0.11 **E** 0.19 **F** 0.83
G 0.36 **H** 0.05 **I** 0.47 **J** 0.38 **K** 0.14 **L** 0.56
M 0.42 **N** 0.29 **3. A** $0.2\overline{3}$ **B** $0.41\overline{2}$ **C** $0.6\overline{17}$
D $0.\overline{2067}$ **E** $63.\overline{7}$ **F** $5.61\overline{2}$ **G.** $0.0\overline{1}$ **H** $0.\overline{43}$
I $0.7\overline{051}$ **4. A** 0.777 . . . **B** 0.2121 . . . **C** 7.4888 . . .
D 0.213213 . . . **E** 0.8333 . . . **F** 56.24999 . . .
G 0.78927892 . . . **H** 3.000 . . . **I** 0.4646 . . .
J 8.6333 . . . **5. A** $0.\overline{1}$ **B** $0.\overline{01}$ **C** $0.\overline{001}$ **D** $0.\overline{0001}$
E $0.\overline{00001}$ **6. A** $0.\overline{7}$ **B** $0.0\overline{8}$ **C** $0.0\overline{05}$ **D** $0.\overline{0002}$
E $0.0100\overline{001000}$ **7.** $\frac{347}{999}$ **8.** $0.\overline{123456789}$

Think

1.

1	8	10	15
12	13	3	6
7	2	16	9
14	11	5	4

2. sum = 34

B-90 DISCUSSING THE IDEAS

1. $0.6\frac{2}{13}$ **2.** $0.61\frac{7}{13}$ **3.** $\frac{1}{8}$ **4. A** $= 0.66\frac{1}{3}$ **B** $= 0.66\frac{1}{2}$
C $= 0.66\frac{2}{3}$

B-91 USING THE IDEAS

1. A $0.7\frac{4}{5}$ **B** $0.16\frac{1}{4}$ **C** $0.037\frac{1}{2}$ **D** $0.66\frac{2}{3}$ **E** $0.785\frac{1}{5}$
F $0.83\frac{5}{6}$ **2. A** $0.8\frac{1}{3}$; $0.83\frac{1}{3}$; $0.833\frac{1}{3}$ **B** $0.1\frac{1}{6}$; $0.18\frac{2}{3}$; $0.187\frac{1}{2}$
C $0.4\frac{4}{9}$; $0.44\frac{4}{9}$; $0.444\frac{4}{9}$ **D** $2.1\frac{2}{3}$; $2.16\frac{2}{3}$; $2.166\frac{2}{3}$ **E** $0.4\frac{2}{9}$; $0.42\frac{2}{9}$;
$0.428\frac{4}{7}$ **F** $0.2\frac{7}{9}$; $0.27\frac{7}{9}$; $0.277\frac{7}{9}$ **3. A** $\frac{1}{8}$ **B** $\frac{13}{200}$ **C** $\frac{5}{6}$ **D** $\frac{2}{3}$
E $\frac{127}{4000}$ **F** $\frac{3}{8}$ **4.** $x = 0.2\frac{1}{2}$; $y = 0.3\frac{1}{3}$; $z = 0.4\frac{2}{3}$ **5.** $x = \frac{1}{4}$;
$y = \frac{1}{3}$; $z = \frac{19}{40}$

Think **1.** 104 cm² **2.** 105 cm² **3.** The 8 × 13 rectangle does not fit the 5 × 21 rectangle exactly.

B-92 SCIENTIFIC NOTATION — DECIMALS

1. A 3.7 **B** 10^2 **C** 6.843 **D** 4756 **E** 4.8 **F** 10^4
G 583.726 **H** 9.46321 **I** 10^3 **J** 3 700 000
2. A 7×10^2 **B** 9×10^2 **C** 8×10^3 **D** 7×10^4
E 6×10^5 **F** 1×10^5 **3. A** 3.6×10^3 **B** 8.64×10^4
C 2.55×10^6 **D** 3.1556×10^7 **E** 2×10^9
F 7.82×10^9 **G** 5.24×10^{10} **H** 6×10^{15} **4. A** 4320
B 54 000 **C** 616 **D** 700 000 **E** 82 000 000 000
F 153 000 000 **G** 59 000 000 **H** 1.02

B-93 COMPARING UNITS WITHIN THE METRIC SYSTEM

1. 1800 **2. A** 1000 **B** 1000 **C** $\frac{1}{1000}$ **3.** 25 g
4. 1030 g **5. A** 680 g **B** 1.47 **6.** 33.12 kg
7. 51.8 ml

B-94 REVIEWING THE IDEAS

1. A 4 **B** 3 **C** 6 **D** 5 **E** 7 **F** 1 **2.** 5.376
3. $72\frac{936}{10\,000}$ **4.** 753.28 **5.** $(2 \times 10^1) + (6 \times 10^0) +$
$(3 \times 1/10^1) + (7 \times 1/10^2) + (4 \times 1/10^3) + (5 \times 1/10^4)$
6. A = **B** < **C** > **D** < **7. A** 17.09 **B** 527.385
8. A 60.6 **B** 14.359 **9. A** 3.43 **B** 0.000171
C 0.2568 **D** 15.437 **10. A** 57.36 **B** 85.6 **C** 5428
D 8.754 **E** 9.56 **F** 0.0876 **11. A** 58 **B** 57.6
C 57.56 **D** 57.565 **12. A** 400 **B** 396.75028
13. A 0.347 **B** $0.75\overline{14857}$ **C** 47.5 **14. A** 734 **B** 83.6
C \$9.95 **D** 9 **15. A** 0.8 **B** 0.85 **C** 0.28125
16. A $0.8\overline{1}$ **B** 0.82 **17.** $0.57\frac{1}{7}$ **18.** 1.49×10^8
19. A 1000 **B** 1 **C** 1 **D** 1

UNIT B: MODULE 5

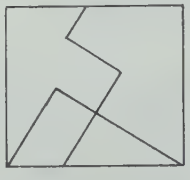
B-98 DISCUSSING THE IDEAS

2. A 32 **B** $\frac{8}{15}$ **C** $\frac{18}{60}$; $\frac{3}{20}$

B-99 USING THE IDEAS

6. $\frac{1}{5}$ **7.** $\frac{1}{10}$ **8.** $\frac{1}{100}$ **9.** $\frac{5}{26}$ **10.** $\frac{3}{5}$ **11.** $\frac{20}{50}$

Think



B-100 DISCUSSING THE IDEAS

2. A $\frac{3}{4}$ **B** yes **3.** $\frac{3}{4}$, $\frac{5}{8}$ **4. A** yes **B** same record
C no **5. A** means 6, 4; extremes 3, 8 **B** extremes

B-101 USING THE IDEAS

1. A $\frac{1}{3}$ **B** 4 to 5 **C** 5:3 **D** $\frac{3}{5}$ **E** 9 to 5 **2. A** means 5, 6; extremes 2, 15 **B** means 4, 7; extremes 1, 28
C means 3, 20; extremes 5, 12 **D** means 8, 200; extremes

5, 320 **3. A** $3 \cdot 4 = 12$; $2 \cdot 6 = 12$ **B** $2 \cdot 12 = 24$; $1 \cdot 24 = 24$ **C** $3 \cdot 15 = 45$; $5 \cdot 9 = 45$ **D** $8 \cdot 5 = 40$; $10 \cdot 4 = 40$ **4.** $9 : 3 = 12 : 4$; $9 : 12 = 3 : 4$; $4 : 3 = 12 : 9$; $4 : 12 = 3 : 9$ **5. A** $4 : 14$; $\frac{4}{14}$ **B** $\frac{4}{14} = \frac{4}{14}$ **6. A** $\frac{15}{42}$ **B** $\frac{5}{14}$ **C** $\frac{15}{42} = \frac{5}{14}$ **7. A** $\frac{4}{5}$; $\frac{4}{5}$ **B** yes

Think **1.** 4/20/80; 10/8/80; 8/10/80; 5/16/80

B-102 DISCUSSING THE IDEAS

2. 48 **3.** 128 **4.** 5 **5. A** $24/N$ **B** B **C** 40

B-103 USING THE IDEAS

1. A 6 **B** 15 **C** 10 **D** 6 **E** 20 **F** 4 **G** 3 **H** 141
2. $\frac{10}{3} = N/24$; $N = 80$ **3.** $2 : 5 = N : 80$; $N = 32$
4. $\frac{7}{1} = 266/N$; $N = 38$ **5.** $\frac{27}{4} = N/72$; $N = 486$
6. $\frac{3}{50} = 9/N$; $N = 450$ **7.** $\frac{2}{7} = N/210$; $N = 60$
8. A $\frac{7}{5} = 14/x$; $x = 10$ **B** $\frac{7}{5} = 35/x$; $x = 25$ **C** $\frac{7}{5} = x/15$; $x = 21$ **D** $\frac{7}{5} = x/50$; $x = 70$ **E** $\frac{7}{5} = 42/x$; $x = 30$
F $\frac{7}{5} = 119/x$; $x = 85$ **9. A** 5000 **B** 3000

B-104 DISCUSSING THE IDEAS

2. B $\frac{1000}{1}$ **C** $\frac{1}{10}$ **D** $\frac{1}{60}$ **E** $\frac{60}{1}$ **F** $\frac{1}{24}$ **3. A** $\frac{5}{2}$ **B** $\frac{3}{5}$ **C** $\frac{7}{12}$ **D** $\frac{2}{5}$ **E** $\frac{1}{4}$ **F** $\frac{6}{7}$

B-105 USING THE IDEAS

1. A 100; $\frac{6}{1}$ **B** 10; $\frac{30}{7}$ **C** 1000; $\frac{140}{17}$ **D** 1000; $\frac{1}{20}$
E 10; $\frac{12}{30}$ **F** 100; $\frac{21}{40}$ **2. A** $\frac{1}{20}$ **B** $\frac{20}{1}$ **C** $\frac{1}{10}$ **D** $\frac{1}{10}$ **E** $\frac{3}{200}$
F $\frac{1}{20}$ **G** $\frac{4}{1}$ **H** $\frac{49}{10}$ **I** $\frac{71}{10}$ **3. A** $\frac{9}{1}$ **B** $\frac{540}{1}$ **C** $\frac{288}{5}$ **D** $\frac{4}{5}$ **E** $\frac{1}{5}$
F $\frac{1}{8}$ **G** $\frac{1}{1000}$ **H** $\frac{1}{10}$ **I** $\frac{1}{2}$ **J** $\frac{3}{56}$ **K** $\frac{20}{11}$ **L** $\frac{11}{45}$ **4. A** $\frac{2}{3}$
B $\frac{4}{11}$ **C** $\frac{16}{25}$ **D** $\frac{1}{3}$ **E** $\frac{1234}{4321}$ **F** $\frac{12345697}{76543321}$

Think $N = 25$ **1.** 4225 **2.** 5625 **3.** 7225 **4.** 9025

B-106 DISCUSSING THE IDEAS

2. A $\frac{1}{10}$ **B** 10 **3.** 19 **4. A** 100 **B** 0.25 mm

B-107 USING THE IDEAS

1. A Venus **B** Earth **C** Mars **D** Mercury **2.** 7.77
3. A 320 km **B** 640 km **C** 640 km **4. A** 28 **B** 28
5. A 55 **B** 0.2

B-108 DISCUSSING THE IDEAS

2. B 16% **C** 8% **3. A** "41 out of 100" or $\frac{41}{100}$ or 0.41
B $\frac{41}{100}$ **4. A** $\frac{8}{100} = \frac{2}{25}$ **B** $\frac{27}{100}$ **C** $\frac{2}{100} = \frac{1}{50}$ **D** $\frac{90}{100} = \frac{9}{10}$
E $\frac{50}{100} = \frac{1}{2}$ **F** $\frac{40}{100} = \frac{2}{5}$ **5.** 23%

B-109 USING THE IDEAS

1. A 17% **B** 29% **C** 14% **D** 11% **E** 10% **F** 9%
G 1% **H** 97% **I** 66% **J** 75% **K** 100% **L** 125%
2. A 0.23 **B** 0.94 **C** 0.25 **D** 0.2 **E** 0.12 **F** 0.1
G 0.09 **H** 0.03 **I** 0.02 **J** 0.01 **K** 0.95 **L** 1.02
3. A 27% **B** 35% **C** 66% **D** 60% **E** 11% **F** 8%
G 4% **H** 1% **I** 40% **J** 30% **K** 100% **L** 116%
4. A $\frac{58}{100}$ **B** $\frac{10}{100}$ **C** $\frac{20}{100}$ **D** $\frac{5}{100}$ **E** $\frac{4}{100}$ **F** $\frac{75}{100}$ **G** $\frac{30}{100}$
H $\frac{16}{100}$ **I** $\frac{35}{100}$ **J** $\frac{40}{100}$ **K** $\frac{90}{100}$ **L** $\frac{22}{100}$ **5. B** 60% **C** 35; 0.35; 35% **D** 100; 1.00; 100%

Think 36 boys; 60 girls

B-110 DISCUSSING THE IDEAS

2. 6 **3.** no **4. A** $1.20 **B** $1.80 **5. A** 25% less than original price **B** $45.00

B-111 USING THE IDEAS

1. A 20 **B** 35 **C** 10 **D** 62 **E** 85 **F** 46 **G** 1
H 3 **I** 4 **2. A** 4.5 **B** 144.56 **C** 12.8 **D** 85.4
E 428.22 **F** 15.69 **G** 9.876 **H** 1.062 **I** 2.78
3. A $740 **B** $24.75 **C** $35.98 **D** $9.65 **E** $2.34
F $300 **G** $9.21 **H** $0.21 **I** $25.64 **J** $67.27
4. A $22.50 **B** $52.50 **5.** $172.70 **6.** 72 000
7. $272.00 **8.** $56.00 **9.** $20.00

B-112 DISCUSSING THE IDEAS

2. A 2.00 **B** c **3.** 0.005 **4.** 0.96

B-113 USING THE IDEAS

1. A 0.11 **B** 0.24 **C** 0.81 **D** 1.01 **E** 1.23 **F** 0.01
G 0.07 **H** 0.04 **I** 1.5 **J** 4.0 **K** 2.5 **L** 10.0
M 0.0003 **N** 0.1235 **O** 0.426 **P** 0.125 **Q** 0.342
R 0.0342 **S** 0.005 **T** 0.001 **U** 0.875 **V** 0.025
W 0.125 **X** 0.0325 **2. A** 29.4 **B** $16.\dot{5}$ **C** 60 **D** 250
E $15.\dot{6}$ **F** 2.718 **G** 150 **H** 4.25 **I** 52 **3.** 80
4. $\$7.5\dot{0}$ **5. A** $0.81 **B** $54.81 **6.** $30 **7.** 3 hours 19 minutes

Think **1.** yes **2.** no **3.** $26 \div 2 = 2$ R2

B-114 DISCUSSING THE IDEAS

1. A 50% **B** 25% **2. A** 50% **B** 12.5% **3.** 6.25%
5. 12.5%
6.

B-115 USING THE IDEAS

2. 25% **3.** 50% **4.** $62\frac{1}{2}\%$ **5.** $62\frac{1}{2}\%$ **6.** $37\frac{1}{2}\%$
7. $37\frac{1}{2}\%$ **8.** $18\frac{3}{4}\%$

B-116 DISCUSSING THE IDEAS

2. $6.30 **3.** $\$18.75 + (\$18.75 \times 0.07) = \$20.06$

B-117 USING THE IDEAS

1. $294 **2. A** $4.00 **B** $84.00 **3.** $19.80
4. $34.45 **5. A** $0.24 **B** $23.64 **6.** $5.50; $27.50
7. $25.44 **8.** $110.16

Think Lost $60.00

B-118 DISCUSSING THE IDEAS

3. $40.00 **4.** $120.00 **5.** $30.00

B-119 USING THE IDEAS

1. A \$12.00 **B** \$42.00 **C** \$25.00 **D** \$32.00
E \$12.00 **F** \$16.00 **G** \$14.00 **H** \$54.00 **I** \$25.00
J \$16.50 **K** \$87.50 **L** \$9.00 **2. A** \$35.50 **B** \$224.00
C \$1008.00 **D** \$8600.00 **E** \$40.50 **F** \$16.53
3. A 12% **B** 18% **C** 36% **D** 30% **E** 15% **F** 9%
4. A 1% **B** 1.5% **C** $\frac{1}{2}$% **D** $\frac{2}{3}$% **E** $\frac{3}{4}$% **F** 2.5%
5. \$36.00 **6. A** bank B **B** \$10.00 **7. A** \$6.75
B \$156.75 **8.** \$1631.25

B-120 DISCUSSING THE IDEAS

2. A 0.45 **B** 0.4 **C** 0.4375 **3. A** 45% **B** 40%
C 43.75% **4. A** 45% **B** 40% **C** 43.75%

B-121 USING THE IDEAS

1. A 20% **B** 30% **C** 50% **D** 80% **E** 1% **F** 27%
G 9% **H** 25% **I** 7.5% **2.** 50% **3.** 40% **4.** 20%
5. 60% **6.** 7% **7.** 80% **8. B** 50.9% **C** 19.4%
D 32.7% **E** 133.3% **F** 25.8% **G** 25.1% **H** 9%
9. A yes **B** 33.8% **10. A** 0.289 **B** 5 **C** 0.283

B-122 DISCUSSING THE IDEAS

2. 52% **3.** less **5. A** 40 **B** 420 **C** 140
6. 8 700 000 **7.** 60%

B-123 USING THE IDEAS

1. A 75% **B** 30% **C** 90% **D** 50% **2. A** 25%
B 55% **C** 75% **D** 35% **3. A** North America
B Europe **C** 30% **4.** B **5. A** \$125 **B** \$150
C \$175 **D** \$200 **E** \$225 **6. A** 10 **B** 12 **C** 3.7
D 45 **E** 34 **F** 66 **G** 16.7
Think 4 kg

B-124 REVIEWING THE IDEAS

1. A 2 : 3 **B** 2 : 5 **C** 7 : 5 **D** 4 : 1 **2.** means 8, 15; extremes 3, 40 **3. A** 8 **B** 45 **C** 9 **D** 15 **4. A** $\frac{3}{5}$
B $\frac{4}{3}$ **C** $\frac{2}{1}$ **D** $\frac{3}{5}$ **5.** 620 by 950 cm **6. A** 0.23 **B** 0.06
C 2.5 **D** 0.015 **E** 0.0001 **F** 0.00003 **7. A** 35%
B 4% **C** 125% **D** 500% **E** 70% **F** 0.9% **8. A** $\frac{1}{2}$
B $\frac{3}{10}$ **C** $\frac{3}{4}$ **D** $\frac{4}{5}$ **E** $\frac{1}{50}$ **F** $\frac{1}{200}$ **9. A** 25% **B** 60%
C 70% **D** 30% **E** 37.5% **F** 0.2% **10. A** 36.54
B 25.47% **C** 43.77 **D** 300 **11. A** \$0.98 **B** \$14.98
12. A \$14 **B** \$81 **C** \$27 **D** \$10 **13.** 15%
14. 42.9% **15.** 8% **16.** schools 60%; police and streets 17%; misc. 6% **17.** C **18. A** \$100 **B** \$250

B-126 CUMULATIVE REVIEW

1. A $\frac{5}{6}$ **B** $\frac{5}{8}$ **2.** D **3. A** $\frac{2}{6}$; $\frac{3}{9}$; $\frac{4}{12}$ **B** $\frac{6}{20}$; $\frac{9}{30}$; $\frac{12}{40}$ **C** $\frac{4}{18}$; $\frac{6}{27}$; $\frac{8}{36}$ **4. A** $\frac{7}{12}$ **B** $\frac{9}{14}$ **5. A** $\frac{3}{5}$ **B.** $\frac{2}{5}$ **C** $\frac{2}{5}$ **D** $\frac{1}{6}$ **6. A** $\frac{3}{4}$
C $1\frac{2}{3}$ **7. A** > **B** > **C** < **D** > **8. A** $\frac{13}{8}$ **B** $\frac{17}{5}$ **C** $\frac{49}{10}$
9. A $1\frac{4}{7}$ **B** $4\frac{3}{10}$ **C** $5\frac{1}{4}$ **10.** 20 **11. A** $1\frac{1}{6}$ **B** $2\frac{1}{2}$ **C** $1\frac{1}{12}$
D $7\frac{7}{20}$ **12. A** $\frac{1}{6}$ **B** $4\frac{3}{8}$ **C** $3\frac{3}{5}$ **D** $1\frac{5}{12}$ **13. A** $\frac{2}{3}$ **B** $\frac{1}{6}$
C $\frac{17}{30}$ **D** $\frac{5}{6}$ **E** $\frac{3}{4}$ **F** $3\frac{3}{8}$ **G** $3\frac{1}{12}$ **H** $9\frac{1}{4}$ **14. A** 4 **B** 6
C 7 **15.** $90\frac{2}{3}$ **16. A** $\frac{1}{6}$ **B** $\frac{1}{24}$ **C** $\frac{3}{4}$ **D** $\frac{5}{8}$ **17. A** 3 **B** 9
C 12 **D** 25 **18. A** $\frac{1}{5}$ **B** $\frac{3}{5}$ **C** $1\frac{1}{4}$ **D** 19 **19. A** $3\frac{1}{2}$
B $\frac{1}{5}$ **C** $\frac{3}{7}$ **20. A** $1\frac{1}{2}$ **B** $\frac{9}{10}$ **C** $1\frac{1}{3}$ **D** 2 **21. A** $\frac{9}{10}$ **B** $1\frac{1}{3}$
22. 2 **23. A** $1\frac{7}{10}$ **B** $\frac{152}{500}$ **C** $\frac{4}{100}$ **24. A** 0.37 **B** 1.356
C 6.09 **25. A** 25.338 **B** 324.206 **26. A** 2.87
B 65.626 **27. A** 0.744 **B** 327.484 **28. A** 3.8 **B** 3900
29. $0.\overline{4}$ **30.** $0.83\frac{1}{3}$ **31.** 3 **32.** 3 to 5 **33. A** 0.27
B 1.15 **C** 0.08 **34. A** 89% **B** 90% **C** 3% **35. A** $\frac{1}{5}$
B $\frac{7}{20}$ **C** $\frac{1}{25}$ **36. A** 40% **B** 75% **C** 62.5% **37.** 96.3
38. \$19.80 **39.** \$49.50 **40.** 30%

UNIT C: MODULE 1

C-2 DISCUSSING THE IDEAS

1. 13 **4.** uncountable **5.** $\overline{RT}$; $\overline{ST}$; $\overline{SR}$; $\overline{TS}$ **6.** no

C-3 USING THE IDEAS

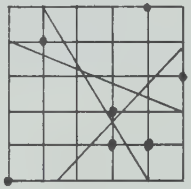
2. 3 **4. A** 4 **B** $\overline{AD}$; $\overline{ED}$ **C** $\overline{AB}$; $\overline{BC}$ **D** *C* **E** *A*
5. 6 **6.** 10 **7.** 15
Think Three colors are not enough.

C-4 DISCUSSING THE IDEAS

1. no; $\overline{AC}$ and $\overline{BD}$ intersect **4. A** yes **B** no

C-5 USING THE IDEAS

1. 10 **2.** *C*; 3 **3.** $\overline{AE} \| \overline{BF}$; $\overline{AE} \| \overline{DH}$; $\overline{AE} \| \overline{CG}$; $\overline{BF} \| \overline{DH}$; $\overline{BF} \| \overline{CG}$; $\overline{DH} \| \overline{CG}$; $\overline{AB} \| \overline{EF}$; $\overline{AB} \| \overline{GH}$; $\overline{AB} \| \overline{CD}$; $\overline{EF} \| \overline{GH}$; $\overline{EF} \| \overline{CD}$; $\overline{GH} \| \overline{CD}$ **4.** Answers may vary. Example: $\overline{AE}$ and $\overline{BD}$
5. yes
Think

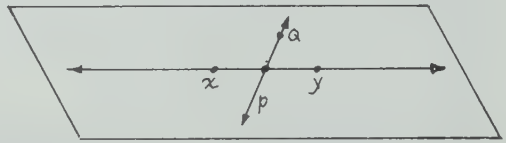


C-6 DISCUSSING THE IDEAS

1. line **2.** 3 **3.** 7 **4.** yes

C-7 USING THE IDEAS

3. A *D* **B** *F* **C** *A* **D** *E* **E** *G* **F** *H* **G** *C* **H** *B*
4.



5. *DCB*; *ABC*; *ADB* **6.** 10

C-8 DISCUSSING THE IDEAS

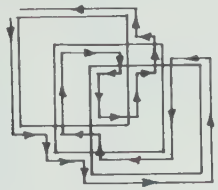
1. A segment has end points. **3.** Answers may vary.
4. $\overline{IJ} \cong \overline{UV}$; $\overline{MN} \cong \overline{OP}$; $\overline{KL} \cong \overline{ST}$; $\overline{QR} \cong \overline{WX}$

C-9 USING THE IDEAS

1. $\overline{AC} \cong \overline{DE}$; $\overline{BC} \cong \overline{DF}$; $\overline{AB} \cong \overline{EF}$ **2. A** $\overline{BC}$; $\overline{CD}$; $\overline{DE}$; $\overline{EF}$; $\overline{FA}$ **B** $\overline{BD}$; $\overline{CE}$; $\overline{DF}$; $\overline{EA}$; $\overline{FB}$ **C** $\overline{BE}$; $\overline{CF}$ **3.** 15 **4.** $\overline{HG}$

6. A 1 **B** 3 **C** 6 **D** 10 **7.** $\overline{BM_1} \cong \overline{AM_1}$; $\overline{BM_2} \cong \overline{CM_2}$; $\overline{CM_3} \cong \overline{DM_3}$; $\overline{AM_4} \cong \overline{DM_4}$

Think



C-10 DISCUSSING THE IDEAS

1. yes **2. A** $\overline{RS} < \overline{TV}$ **B** $\overline{EF} < \overline{GH}$ **C** $\overline{MN} > \overline{PQ}$ **3.** $\overline{AC}$; $\overline{AE}$; $\overline{AK}$; $\overline{CE}$; $\overline{CK}$; $\overline{EK}$ **4.** Answers may vary. Example: $\overline{AC} < \overline{AK}$; $\overline{CE} < \overline{AK}$; $\overline{EK} < \overline{AK}$ **5. A** > **B** = **C** <

C-11 USING THE IDEAS

1. A *C* **B** *A* **C** *C* **D** *B* **E** *B* **2. A** *F* **B** *E* **C** *G* **D** *F* **E** *D* **F** *E* **3. A** $\overline{AB} \cong \overline{EF}$ **B** $\overline{AB} > \overline{EF}$ **C** $\overline{AB} < \overline{EF}$ **4.** $\overline{ST} > \overline{UV}$; $\overline{ST} < \overline{YZ}$; $\overline{ST} \cong \overline{WX}$; $\overline{UV} < \overline{YZ}$; $\overline{UV} < \overline{WX}$; $\overline{YZ} > \overline{WX}$ **5. A** 40 **B** 30 **C** 20 **D** 10

C-12 DISCUSSING THE IDEAS

1. Answers may vary. Example: mid point on football field. **2.** no **3.** Answers may vary. Example: compass needle **4.** no

C-13 USING THE IDEAS

1. B yes **C** no; yes **2. D** 2 **3. A** *Y* **B** *V* **C** *X* **D** *V* **E** *W* **4. D** inside the triangle

C-14 DISCUSSING THE IDEAS

1. *A* **2. A** $\angle X < \angle Y$ **B** $\angle R \cong \angle S$ **C** $\angle M > \angle N$ **3. A** set of points on *C*-side of $\overline{AB}$ ∩ set of points on *B*-side of $\overline{AC}$

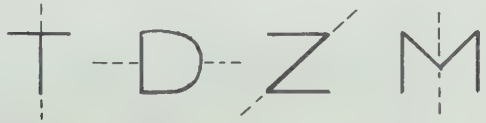
C-15 USING THE IDEAS

1. A $\overline{SR}$ and $\overline{ST}$ **B** *S* **C** $\angle S$; $\angle 2$; $\angle RST$ **2.** $\angle KML$; $\angle JML$ **3. A** < **B** > **C** ≅ **4. A** ≅ **B** < **C** > **5. A** $\angle AQB$ **B** $\angle AQB$ **C** $\angle AQC$ **D** $\angle BQC$ **6. A** interior of $\angle 2$ **B** interiors of $\angle 1$, $\angle 2$ and $\angle 3$ and rays of $\angle 2$

C-16 DISCUSSING THE IDEAS

1. yes **3.** yes

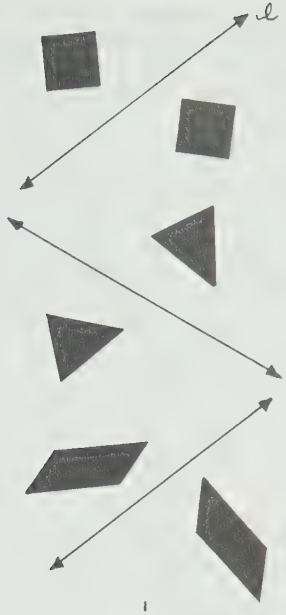
4.



C-17 USING THE IDEAS

1. A *S* **B** *T* **C** *R* **D** *D* **E** *U* **2. A** yes **B** yes **C** no **D** yes **E** no **F** yes

3. A



B

C

4. A

B

C no **D** no

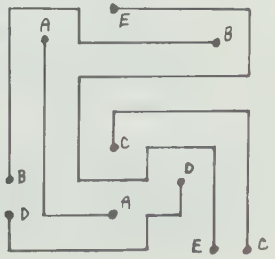
C-18 DISCUSSING THE IDEAS

1. yes **4.** $\angle USV$; $\angle RSV$; $\angle UST$; $\angle RST$

C-19 USING THE IDEAS

1. *B*; *D* **2. C** parallel

Think



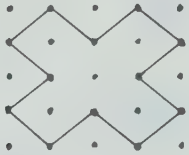
C-20 DISCUSSING THE IDEAS

1. A not same vertex **B** interiors overlap **2.** $\angle BAE$ and $\angle EAC$; $\angle BAD$ and $\angle DAC$ **3. A** yes **4.** yes **5. A** acute **B** adjacent or supplementary

C-21 USING THE IDEAS

1. A $\angle 4$ and $\angle 5$; $\angle 5$ and $\angle 1$; $\angle 1$ and $\angle 2$; $\angle 2$ and $\angle 3$; $\angle 3$ and $\angle 4$ **B.** $\angle 4$ and $\angle 1$; $\angle 4$ and $\angle 2$; $\angle 5$ and $\angle 2$; $\angle 5$ and $\angle 3$; $\angle 1$ and $\angle 3$ **2. A** supplementary **B** vertical **C** supplementary **D** vertical **E** supplementary **3. A** $\angle 1$ and $\angle 3$ **B** $\angle 2$ and $\angle 3$; $\angle 1$ and $\angle 2$ **4. A** $\angle 1$ **B** $\angle 5$; $\angle 4$; $\angle 2$ **C** $\angle 3$ **D** none **5. A** obtuse **B** obtuse **C** acute **D** acute **E** acute **F** obtuse

Think



C-22 DISCUSSING THE IDEAS

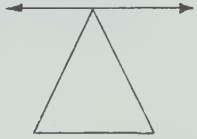
1. 9 cm, 5 cm, 3 cm **2.** no **4. A** 3 **B** 3 **5.** no

C-23 USING THE IDEAS

1. C $\triangle RTS$; $\triangle TRS$; $\triangle TSR$; $\triangle SRT$ **2.** $\triangle ABC$; $\triangle ACE$; $\triangle CDE$; $\triangle ADE$; $\triangle ABE$

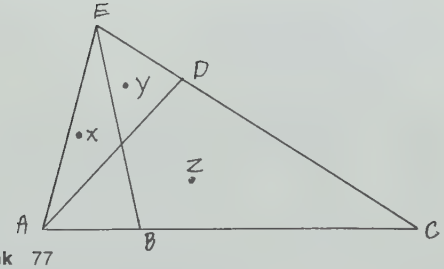
3. A

B



C

4.



Think 77

C-24 DISCUSSING THE IDEAS

1. *C* **2.** no **3.** 1 **4.** 3 **5. B** opposite the right angle **6.** no

C-25 USING THE IDEAS

1. A isosceles **B** right **C** scalene **D** isosceles right **2.** right angle with adjacent sides congruent

Think

$\triangle ABC$	$\triangle IJE$	$\triangle ABE$	$\triangle CJE$	$\triangle CDF$
$\triangle CFI$	$\triangle JEH$	$\triangle ABF$	$\triangle CFE$	$\triangle CDE$
$\triangle CIE$	$\triangle JHD$	$\triangle FBC$	$\triangle FJE$	$\triangle AEF$
$\triangle FIJ$	$\triangle EHB$	$\triangle AJF$	$\triangle EDB$	$\triangle ADE$
$\triangle FGJ$	$\triangle HDB$	$\triangle ADJ$	$\triangle EDJ$	$\triangle FDB$
$\triangle AFG$	$\triangle ADC$	$\triangle ADF$	$\triangle EJB$	$\triangle FBE$
$\triangle AGD$	$\triangle CDB$	$\triangle DJF$	$\triangle BJD$	$\triangle IDE$
$\triangle JGD$	$\triangle AEC$	$\triangle CFJ$	$\triangle DEF$	$\triangle IFD$
$\triangle FEG$	$\triangle EGD$	$\triangle FEH$	$\triangle FDH$	$\triangle AJC$
$\triangle ABJ$	$\triangle CJB$			

C-26 DISCUSSING THE IDEAS

2. no **3.** false

4. A

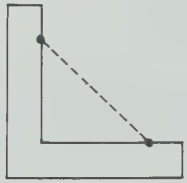
B

C-27 USING THE IDEAS

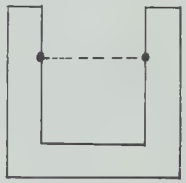
1. A convex **B** convex **C** concave **D** convex **E** concave **F** convex **G** concave **H** convex

2.

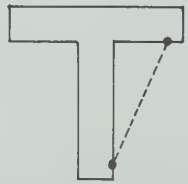
A



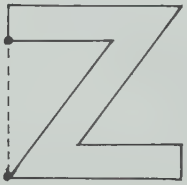
B



C

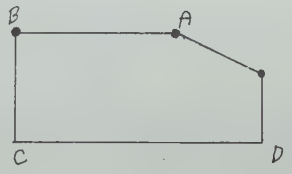


D

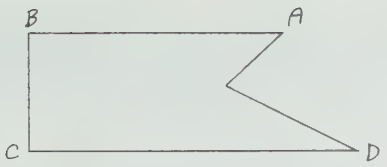


3. A convex

B



C



Think **A** 10°C **B** 20°C **C** 30°C **D** 5°C **E** 40°C

C-28 DISCUSSING THE IDEAS

1. convex **3.** yes

4. no

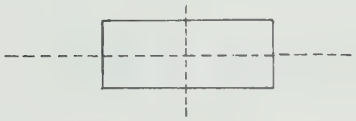
C-29 USING THE IDEAS

1. A trapezoid; parallelogram; rectangle; square; rhombus **B** parallelogram; rectangle; square; rhombus **C** rectangle; square **D** square; rhombus

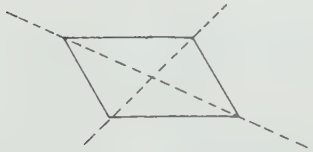
2. A rectangle **B** trapezoid **C** rhombus **D** square

3.

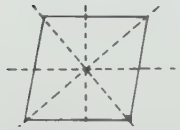
4. 2;



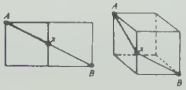
5.



6.



Think When the left side and front of the cube are spread flat (see figure), they form a rectangle. Thus, a straight line between the vertices will be the shortest distance.



C-30 DISCUSSING THE IDEAS

1. yes **2. A** equilateral triangle or equiangular triangle **B** square

3. A

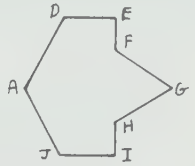
B

4. A no **B** 2

C-31 USING THE IDEAS

1. 5; 5; 6; 9; 7; 14; 8; 20; 9; 27; 10; 35 **2. A** square **B** triangle **C** rectangle **D** pentagon **E** trapezoid **F** hexagon

3.



4. 3; *ADGJ*; *CMOE*; *KMOI*

Think

C-32 REVIEWING THE IDEAS

1. *M*; $\overline{AB}$; $\overline{AC}$; $\overline{BC}$ **2.** *C* **3.** $\overline{AB} \cong \overline{CE}$; $\overline{BC} \cong \overline{AE}$; $\overline{CE} \cong \overline{CD}$; $\overline{AB} \cong \overline{CD}$ **4. A** $\overline{MN}$ **B** $\overline{CD} < \overline{PQ}$ **5. A** half-line **B** half-plane **6.** $\overline{AB}$; $\overline{BC}$; $\overline{CB}$; $\overline{BA}$ **7.** $\angle JKM$; $\angle JKL$; $\angle MKL$ **8.** yes **9.** *A*; *C* **10. A** parallel **B** skew **C** perpendicular **11. A** $\angle AQD$ **B** $\angle AQB$; $\angle AQC$ **C** $\angle DQC$; $\angle CQB$ **D** $\angle DQB$ **12. A** $\angle 1$ and $\angle 3$ **B** $\angle 1$ and $\angle 2$; $\angle 2$ and $\angle 3$ **13. A** right **B** isosceles **C** equilateral **D** scalene **14.** *A* **15. A** *ABFD*; *ABFD* **B** *BCGF* **C** *EFJIH* **D** *EFKJIH*

UNIT C: MODULE 2

C-36 DISCUSSING THE IDEAS

3. approximately $9\frac{1}{2} \times 11\frac{1}{2}$ (or) about 108 units **4.** 8 cubic units

C-37 USING THE IDEAS

1. A length **B** volume **C** area **D** length **E** area **F** volume **2. A** area **B** volume **C** length **D** volume **E** length **F** area **3. A** 4 units **B** $7\frac{1}{2}$ units

C $8\frac{1}{2}$ units **D** 3 units **4. A** 6 units **B** 4 units **C** 7 units **5. A** 12 units **B** 18 units **C** 36 units **6.** 96

C-38 DISCUSSING THE IDEAS

3. A 1000 **B** 10 **C** $\frac{1}{100}$ **D** $\frac{1}{1000}$ **E** $\frac{1}{10}$ **F** 100
4. A 4 **B** 43 **C** 0.43 **D** 0.043

C-39 USING THE IDEAS

1. A 100 **B** 10 **C** 10 **D** 100 **E** 1000 **F** 20 **G** 3 **H** 5 **I** 8 **J** 230 **K** 700 **L** 10 **2. A** 53 mm; 5.3 cm **B** 60 mm; 6 cm **C** 37 mm; 3.7 cm **D** 25 mm; 2.5 cm **4. A** 5.62 m **b.** 1.34 m **C** 0.85 m **D** 0.67 m **E** 0.238 m **F** 2.222 m **G** 18.7 m **H** 1750 m **I** 1000.053 m **5. A** 10 **B** 100 **C** 1000 **D** 10 000 **E** 100 000 **F** 1 000 000

C-40 DISCUSSING THE IDEAS

1. 12 cm **2.** Find the sum of the lengths of the four sides. **3. A** 28 cm **B** 8.4 m **4.** $P = 3 \cdot s$; $P = 5 \cdot s$; $P = 6 \cdot s$

C-41 USING THE IDEAS

1. 14 cm **2. A** 28 **B** 40 **C** 12 **D** 38 **3. A** 49 **B** $23\frac{1}{4}$ **C** 23.4 **4.** 2; 2; 2 **5. A** 38 **B** 14.4 **C** 69 **6.** 11 and 1 or 10 and 2 or 9 and 3 or 8 and 4 or 7 and 5 or 6 and 6

Think large region: 10 units; star: 10 units

C-42 DISCUSSING THE IDEAS

2. A no **B** yes **C** no

C-44 DISCUSSING THE IDEAS

2. A 100 cm² **B** 10 000 cm² **3. A** 15 sq. units **B** 15 sq. units **C** 32 sq. units

C-43 USING THE IDEAS

1. A 4.5 km, 3.5 km **B** 19.5 m, 18.5 m **C** 37.5 cm, 36.5 cm **D** 19.5 mm, 18.5 mm **E** 12.5 cm, 11.5 cm **F** 154.5 dm, 153.5 dm **2. A** $3 + 2 + 2 + 5 = 12$ cm **B** $34 + 24 + 16 + 52 = 126$ **C** 6 mm **3.** No. Because it is greater than 500 000 km more. **5.** 0.5 trillion km **6. A** 0.125 **B** 0.03125 **C** 0.02 **D** 0.02778

C-45 USING THE IDEAS

1. A 6 cm² **B** 3 cm² **C** 6 cm² **2. A** 19 **B** 19 **C** 31 **D** 15 **E** 10 **F** 24 **G** 24 **H** 12 **I** 12 **J** 12 **3.** 100 sq. m **4.** 100 ares **5.** 100 hectares **6.** 9 975 000 km²

C-46 DISCUSSING THE IDEAS

1. yes **2. A** 8 **B** 4 **C** 32 **D** $A = l \cdot w$ **3. A** 28 sq. units **B** 28 sq. units **C** 28 sq. units **4.** $A = b \cdot h$

C-47 USING THE IDEAS

1. A 36 sq. units **B** $39\frac{3}{8}$ sq. units **C** 16.38 sq. units
2. A 25 sq. units **B** $24\frac{7}{8}$ sq. units **C** 22.5 sq. units
3. A 35 sq. units **B** 13.5 sq. units **C** 47.7 sq. units
4. 15.3 m² **5.** $228.74 **6.** $87.50 **7.** $48

C-48 DISCUSSING THE IDEAS

1. the triangle area = $\frac{1}{2}$ rectangle area **2. A** 12 **B** 6 **C** 32 **D** 16 **E** 22 **3. A** 4 **B** $A = \frac{1}{2} \cdot 6 \cdot 4$ **C** 12 **4.** $7\frac{13}{16}$

C-49 USING THE IDEAS

1. A 20 **B** $17\frac{1}{20}$ **C** 17.05 **2. A** 6 **B** $9\frac{1}{24}$ **C** 10.85
3. 12.5 sq. units **4.** $18.14 **5. A** 37 800 mm² **B** $6\frac{3}{4}$ m²
C 4 m² **D** 8.82 cm²
6. $\frac{1}{2} \rightarrow \cdot b \rightarrow \cdot h \rightarrow A$

Think [A] 8 [B] 8 [C] $\frac{1}{2} \cdot (a + \ ) \cdot h$

C-50 DISCUSSING THE IDEAS

1. yes **2.** $a = 3, b = 4, c = 5$ **3.** 16 sq. units; 4 units

C-51 USING THE IDEAS

1.

A	16, 16		32	32
B	9, 25		34	34
C	4, 36		40	40
D	16, 9		25	25

2. A 100, 10 **B** 9, 3 **C** 169, 13

3.

	a	b	a^2	b^2	c^2	c
A	3	4	9	16	25	5
B	8	6	64	36	100	10
C	12	5	144	25	169	13
D	8	6	64	36	100	10
E	12	9	144	81	225	15

C-52 DISCUSSING THE IDEAS

1. 27 **2. A** multiply $h \cdot w \cdot l$ **B.** 28 **C** 28 **D** 4 **E** 112 **3. A** 72 **B** $\frac{1}{2}$ the volume of the box

C-53 USING THE IDEAS

1. 72 **2.** 45 **3.** 68.25 **4.** 51.7 **5.** 59.5 **6.** $12\frac{3}{16}$ **7.** $115\frac{1}{2}$ **8.** 130.83 **9.** 77.26 **10.** 90 **11.** 45 **12.** 45
13. A 60 cubic centimeters **B** 4 times
Think 32 000 km

C-54 DISCUSSING THE IDEAS

1. A 15 **B** 10 **C** 6 **D** 62 **2. A** 4 **B** 18 **3.** length height and width of room **4.** volume = 15, surface area = 46

C-55 USING THE IDEAS

1. A 1, 6 **B** 2, 10 **C** 3, 14 **D** 4, 18 **2. A** 8, 24 **B** 16, 40 **C** 45, 78 **D** 64, 96 **3. A** 5040 cm³

B 1984 cm^2 **4. A** 4.743 cm^3 **B** 19.18 dm^2
5. A 60, 100 **B** 16.2, 46.44 **C** $36\frac{1}{4}, 87\frac{4}{5}$ **6. A** the same number of units are still on the surface **B** any one of the corner cubes can be removed

Think $l = 6$ units, $w = 6$ units, $h = 6$ units

C-56 DISCUSSING THE IDEAS

1. 1000 cm^3 **2. A** $\frac{1}{1000}$ **B** 1 cm^3 **3. A** 1000 **B** $\frac{1}{10}$ **C** $\frac{1}{1000}$ **D** 100 **E** $\frac{1}{100}$ **F** 10 **4. A** 0.5 **B** 2.749 **C** 72 **D** 3.8 **5.** 30 ml

C-57 USING THE IDEAS

1. A 96 ml **B** 30.6 ml **C** 56.96 ml **2. A** 0.096 **C** 0.0306 **C** 0.05696 **3.** 1 kl **4. A** 25 g **B** 1000 g **C** 1 000 000 g **5.** 1.416 *l* **6.** $6.09, $316.68 **7.** $7.14

Think **(1)** Fill the 9-*l* pail and then pour 4 litres into the 4-*l* pail. **(2)** Dump out the 4-*l* pail and pour 4 litres more from the 9-*l* pail into the 4-*l* pail. **(3)** Again, dump out the 4-*l* pail and pour the 1 litre pail from the 9-*l* pail to the 4-*l* pail. **(4)** Refill the 9-*l* pail and then pour into the 4-*l* pail until full. Thus the 9-*l* pail will have 6 *l* remaining.

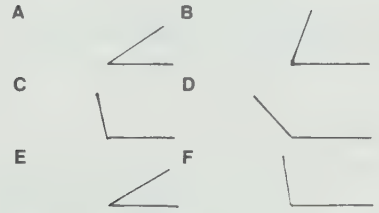
C-58 DISCUSSING THE IDEAS

2. $\frac{1}{180}$ **3.** $\angle AOB = 50°$, $\angle BOC = 30°$, $\angle AOC = 80°$; yes **4.** Their sum is 180. **5.** Their sum is 180° or a straight angle.

C-59 USING THE IDEAS

1. A 25° **B** 62° **C** 90° **D** 135° **E** 155° **F** 180° **G** 37° **H** 65° **I** 110° **J** 130° **K** 155° **L** 28° **M** 73° **N** 93° **O** 118° **P** 45° **Q** 65° **R** 90° **S** 20° **T** 45° **U** 25° **V** 118°

2.



3. A 55° **B** 68° **C** 60° **D** 74° **E** 55° **F** 125°

C-60 DISCUSSING THE IDEAS

1. A $\frac{1}{2}$ **B** $\frac{1}{4}$ **C** $\frac{1}{6}$ **D** $\frac{1}{10}$ **E** $\frac{3}{4}$ **F** $\frac{1}{60}$ **G** $\frac{1}{120}$ **H** $\frac{1}{240}$
2. 65′ = 1°5′ **3.** 118°72′; 63°27′ **4.** 48°30′
5. 116°20′17″

C-61 USING THE IDEAS

1. A 81°43′ **B** 73°32′ **C** 27°2′ **2. A** 58°43′ **B** 122°28′ **C** 96°52′ **3. A** 75°51′53″ **B** 123°50′25″ **C** 155°24′25″ **D** 135° **4. A** 105°43′15″ **B** 28°13′44″ **C** 65°58′29″ **D** 99°58′11″ **5. A** 52° **B** 62°11′ **C** 47°43′22″ **D** 31°59′11″ **E** 46°59′11″ **6. A** 124° **B** 74°40′ **C** 95°33′21″ **D** 54°1′17″ **E** 175°59′2″

C-62 DISCUSSING THE IDEAS

1. It is equal to a straight angle. **2.** 180° − (70° + 50°) = 60° **3.** 360°

C-63 USING THE IDEAS

1. A 20° **B** 121° **C** 90° **D** 47°43′ **E** 57°40′ **F** 52°55′ **2.** 60° **3.** $\angle B = 120°50'$, $\angle C = 59°10'$, $\angle D = 120°50'$ **4. A** 180° **B** 180° **C** 360° **5. A** 180° **B** 180° **C** 180° **D** 540° **6.** 720° **7.** $180°(n - 2)$

C-64 REVIEWING THE IDEAS

1. $\overline{AB}$ = 25 mm, $\overline{BC}$ = 17 mm, $\overline{AC}$ = 30 mm **2.** 72 mm **3. A** 100 **B** 10 **C** 0.1 **D** 2.3 **E** 0.47 **F** 5000 **4.** 8.4 cm, 7.5 cm **5. A** 7 **B** 2.7 **C** $18\frac{3}{4}$ **D** 18.6 **6.** 10 **7.** 280 m^3 **8.** 286 sq m^2 **9.** 1000 **10.** 64 **11.** $\angle X = 55°$, $\angle Y = 115°$ **12. A** 75°51′19″ **B** 124°18′1″ **13. A** 45°42′35″ **B** 68°32′44″ **14. A** 90° **B** 129°29′ **C** 39°29′ **15.** 76°18′ **16.** 20°47′

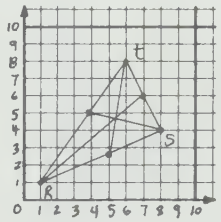
C-68 DISCUSSING THE IDEAS

1. (4,6) **2.** (9,9) **3.** yes; example (7,7), (3,7) **4. A** 5 **B** 5 **C** (5,0), (5,1), (0,1) **5. A** (3,9), (5,3), (7,7) **B** four times as large

C-69 USING THE IDEAS

1. (0,4), (2,8), (8,6), (10,2) **2.** (1,6), (5,7), (9,4), (5,3) **3.** (4,5), (6,5) **4.** $5\frac{1}{2}$, $5\frac{1}{2}$ **5.** (1,1), (9,4), (7,8) **6.** $(5,2\frac{1}{2})$, $(8,6)$, $(4,4\frac{1}{2})$

7.



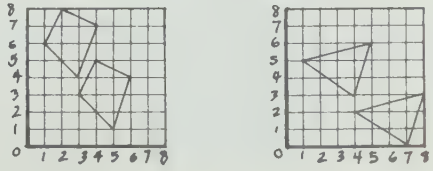
8. $5\frac{1}{2}$, $4\frac{1}{2}$

C-70 DISCUSSING THE IDEAS

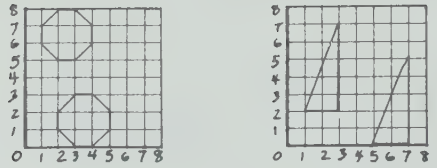
1. The position of the hexagon is moved but the size and shape are the same. **2.** Answers will vary. Example: subtract 2 from the first number and add 3 to the second number in the pair. **3.** yes **5. A** yes **B** 3,4 **C** left 3, down 4

C-71 USING THE IDEAS

1. A **B**



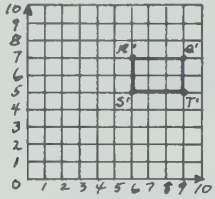
C D



2. A right 4, down 2 **B** right 2, up 1 **C** left 3, down 3
3. parallel

C-72 DISCUSSING THE IDEAS

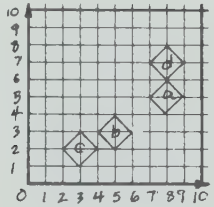
1. right 2, down 2; up 3; left 3, down 3
2. A



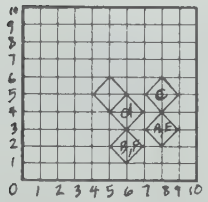
B right 5 **C** no

C-73 USING THE IDEAS

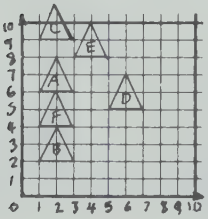
1.



2.



3.



Think

C-74 DISCUSSING THE IDEAS

1. (5,5) **2. A** R(1,2); Q(9,5) **B.** yes **3.** yes
4. A P(8,7); Q(5,1); R(2,9) **B** P(7,2); Q(1,5); R(9,8)
C P(2,3); Q(5,9); R(8,1)

C-75 USING THE IDEAS

1. A (1,1) **B** (2,7) **C** (6,8) **D** (8,6) **E** (7,2)
F (5,5) **2.** P′(1,9); Q′(7,8); R′(8,4); S′(6,2); T′(2,3)
3. P′(9,9); Q′(8,3); R′(4,2); S′(2,4); T′(3,8) **4.** A′(1,7); B′(0,4); C′(3,5); D′(8,2); E′(4,0); F′(5,2); G′(6,6); H′(2,1); I′(3,8); O′(4,4) **5.** A′(7,7); B′(4,8); C′(5,5); D′(2,0); E′(0,4); F′(2,3); G′(6,2); H′(1,6); I′(8,5) **6.** A′(7,1); B′(8,4); C′(5,3); D′(0,6); E′(4,8); F′(3,6); G′(2,2); H′(6,7); I′(5,0); O′(4,4)
7. A A(1,5); B(3,7); C(6,6) **B** A(7,5); B(5,7); C(2,6) **C** no

C-76 DISCUSSING THE IDEAS

1. square, equilateral triangle, rhombus, rectangle
2. 1 **3.** 2

C-77 USING THE IDEAS

1. A 2 **B** 4 **C** 2 **D** 4 **2. A** 4 **B** 2 **C** 4 **D** 4
3.

	A	B	C	D
$\frac{1}{4}$	(4,7)	(7,4)	(4,1)	(1,4)
$\frac{1}{2}$	(7,4)	(4,1)	(1,4)	(4,7)
$\frac{3}{4}$	(4,1)	(1,4)	(4,7)	(7,4)
1	(1,4)	(4,7)	(7,4)	(4,1)

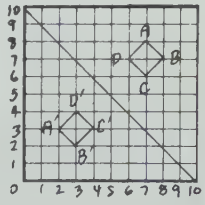
Think 96, 98

C-78 DISCUSSING THE IDEAS

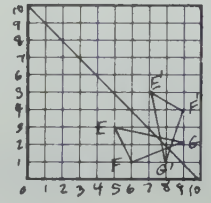
1. A same size and shape but different position on grid
B yes **2.** no **3.** The $\frac{1}{2}$ turn is not the same as the reflection.

C-79 USING THE IDEAS

1. A

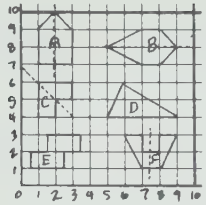


B A′(2,3); B′(3,2); C′(4,3); D′(3,4
2.



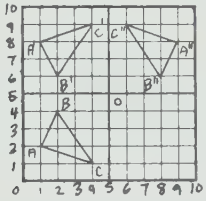
B E′(7,5); F′(9,4); G′(8,1)

3. A & C



B A, B, C, F

4. A, B, & C



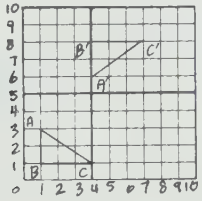
D yes if O is used as point of reflection **E** Yes, rotation of $\frac{1}{2}$ turn clockwise.

C-80 DISCUSSING THE IDEAS

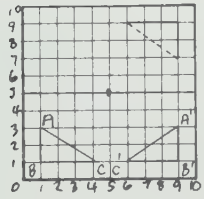
1. no **2. A** (9,4) **B** R″(4,9); Q″(1,3) **3.** yes

C-81 USING THE IDEAS

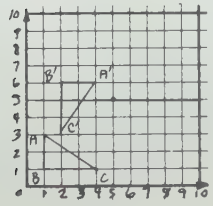
1.



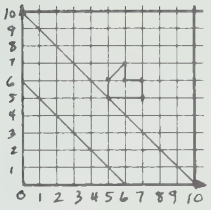
2.



3.



4. A



B reflection through line halfway between r and s

Think B ↙, 10 revolutions; C ↗, 40 revolutions

C-82 DISCUSSING THE IDEAS

1. yes **4.** C

C-83 USING THE IDEAS

Think yes

C-84 DISCUSSING THE IDEAS

1. no

2.

3. A yes, 180° rotation **B** B **4. A** translational **B** translational, rotational, reflectional **C** reflectional, translational, rotational

C-85 USING THE IDEAS

1. rotational, translational, reflectional **2.** translational, reflectional **3.** translational, reflectional **4.** translational **5.** rotational, translational, reflectional **6.** rotational, translational, reflectional

C-86 DISCUSSING THE IDEAS

1. yes **2. A** reflection in line **B** $A \rightarrow D, B \rightarrow E, C \rightarrow F$ **3.** $R \rightarrow Q, T \rightarrow E, S \rightarrow K$ **4.** $\overline{BC} \cong \overline{EF}, \overline{AC} \cong \overline{DF}, \angle B \cong \angle E, \angle C \cong \angle F$

C-87 USING THE IDEAS

1. A 3,6 **B** $\angle A \cong \angle Y, \angle B \cong \angle X, \angle C \cong \angle Z$ **C** $\overline{AB} \cong \overline{YX}, \overline{BC} \cong \overline{XZ}, \overline{AC} \cong \overline{YZ}$ **2. A** $\overline{AB} \cong \overline{MK}, \overline{BC} \cong \overline{HK}, \overline{CA} \cong \overline{HM}$ **B** $\angle A \cong \angle M, \angle B \cong \angle K, \angle C \cong \angle H$ **3. A** $\overline{AB} \cong \overline{PQ}, \overline{BC} \cong \overline{QR}, \overline{CA} \cong \overline{PR}$ **B** $\angle A \approx \angle P, \angle B \approx \angle Q, \angle C \approx \angle R$ **4. A** reflection image in line *n* **B** $\frac{1}{4}$ turn about point *O* **C** translation of right 6 down 5 **5.** $\triangle RST \cong \triangle JKL, \triangle RST \cong \triangle LNM, \triangle RST \cong \triangle EDF$

C-88 DISCUSSING THE IDEAS

1. yes **2.** 2 small triangles **3.** 2 small triangles **4.** yes; the two small triangles **5.** 2 small triangles

6.

7. yes, use square and 2 small triangles

C-89 USING THE IDEAS

1. A B

C D

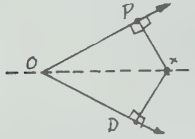
E F

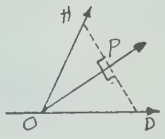
2.

3.

Think 4

C-90 INVESTIGATING THE IDEAS

1. 

2. 

C-91 USING THE IDEAS

Reflection constructions are left to the student

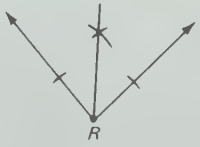
Think DCXXII; 51

C-92 DISCUSSING THE IDEAS

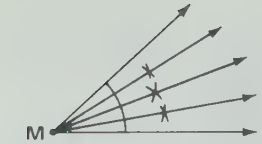
1. a. Label a point B' on your paper **b.** Draw a straight line through B' **c.** make an arc using B as vertex **d.** keeping same opening of compass make a similar arc using B' as vertex **e.** measure opening of AC and measure same opening using C' as vertex of arc making $A'C'$ **f.** draw $\overline{B'A'}$

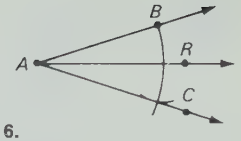
C-93 USING THE IDEAS

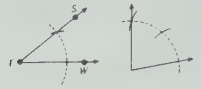
1. B, C

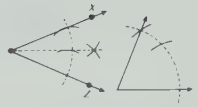


2.

3. 



4. 

6. 

Think

C-94 DISCUSSING THE IDEAS

1. Construction left to the student. **2.** equilateral triangle **3. & 4.** construction left to the student

C-95 USING THE IDEAS

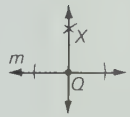
1 to 4. Constructions left to the students. **5.** Yes; for example segments with lengths 6 cm, 2 cm, 3 cm. **6.** Answers will vary.

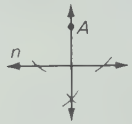
C-96 DISCUSSING THE IDEAS

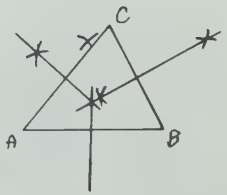
1. B right angles **C** $\overline{AM} \cong \overline{MB}$ **2, 3.** Construction is left to the student.

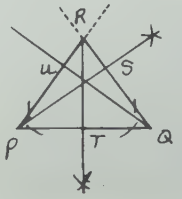
C-97 USING THE IDEAS

1.

2. 

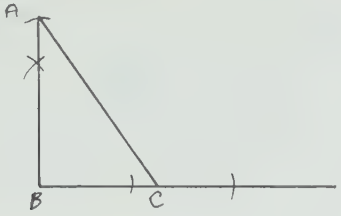
3. 

4. 

5. 

altitudes $\overline{PS}$, $\overline{RT}$, $\overline{QU}$

6.



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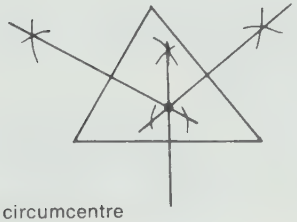
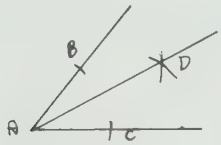


O is the circumcentre

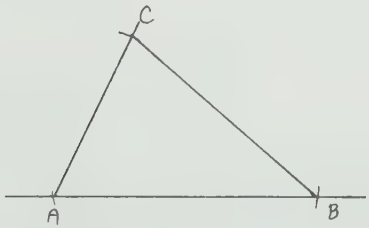
C-98 REVIEWING THE IDEAS

1. $A(1,2)$; $B(3,4)$; $C(3,1)$ **2.** $A'(3,3)$; $B'(5,5)$; $C'(5,2)$ **3.** $A(2,7)$; $B(4,5)$; $C(1,5)$ **4.** $A'(6,7)$; $B'(4,5)$; $C'(7,5)$ **5.** *A* **6.** *B* **7.** congruent **8.** $\triangle RST \cong \triangle CBA$ **9. A** $\overline{JK} \cong \overline{GE}$; $\overline{KL} \cong \overline{EF}$; $\overline{JL} \cong \overline{GF}$ **B** $\angle J \cong \angle G$; $\angle K \cong \angle E$; $\angle L \cong \angle F$ **10. B**



AD bisects $\angle BAC$

11.



12. B

UNIT C: MODULE 4

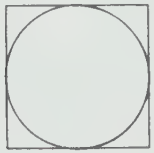
C-102 DISCUSSING THE IDEAS

1. answers will vary **2. A** infinite **B** yes **3.** twice the radii **4.** yes; endpoints are on the circle **5. A** $\angle 4$ cm **B** exterior

C-103 USING THE IDEAS

1. B 6 cm **D** yes **2.** rectangle **3.** square

5.



C-104 DISCUSSING THE IDEAS

1. A $\angle AOC$, $\angle 2$ **B** $\angle ABC$, $\angle 1$ **2.** a part of a circle **3. A** inscribed **B** *BC* **C** *BC* **4.** diameter **5.** twice

C-105 USING THE IDEAS

1. A $\angle AOB$, $\angle COD$, $\angle BOC$, $\angle DOA$ **B** $\angle BAC$, $\angle ACD$ **C** $\angle BAC$, $\angle BOC$ **D** $\angle COD$ **2. A** $\angle AOE$ **B** $\angle BOC$ **B** $\angle BEC$ **D** $\angle COE$ **E** $\angle COE$ **F** *AB* **G** *BCD* **H** $\angle CED$ **3. A** 90° **B** 90° **C** right **4. A** 50° **B** 132° **C** 108° **D** 16°

Think arc, angle, centre, diameter, chord, circle, subtend, inscribed

C-106 DISCUSSING THE IDEAS

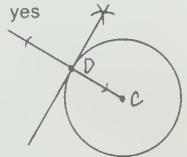
1. A one **B** perpendicular **2.** 4 **3.** Find which line is perpendicular to $\overline{PO}$. **4. B** yes

C-107 USING THE IDEAS

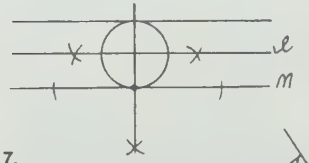
1. A perpendicular **B** perpendicular **2.** *t*, *s*, *n*; no

3.

4. yes



5.



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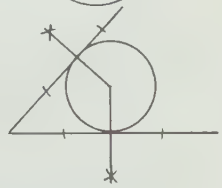
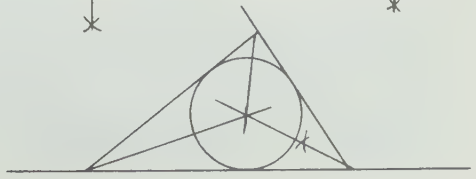


7.



C-108 DISCUSSING THE IDEAS

1. B 3.14 **2. A** 3.14 **B** yes **3.** $C = \pi \cdot d$ **4.** $d = c/\pi$

C-109 USING THE IDEAS

1. A 94.2 cm **B** 15.7 cm **C** 5.966 cm **D** 50.24 cm **E** 33.912 cm **2. A** 44 **B** 11 **C** 3.3 **D** $5\frac{1}{2}$ **E** $6\frac{3}{5}$ **F** $13\frac{1}{3}$ **3. A** 5 cm **B** 12.1 mm **C** 2.7 m **D** 0.4 m **E** 0.06 m **4.** 17.8 cm **5. A** 2.07 m **B** 20.7 m **C** 248.4 m **D** 483 revolutions **6. A** 6 cm **B** 8 cm **C** 7 cm **D** 62.8 mm **E** 7.2 cm

C-110 DISCUSSING THE IDEAS

1. A yes **B** only $\frac{1}{2}$ the sectors lie on the base **3. A** $A = \pi \cdot r^2 = 3.14 \cdot 2^2 = 12.56$ cm²

C-111 USING THE IDEAS

1. A 28.26 cm² **B** 254.34 m² **C** 3.14 m² **D** 11 304 mm² **E** 78.5 cm² **F** 16.61 m² **G** 1823.74 mm² **H** 1193.99 cm² **2.** $342\frac{2}{3}$ cm² **3.** 138.16 m² **4. A** 34¢ **B** $2.40 **5.** 78.5% **6.** 78.5%

Think square

C-112 CIRCLE PROBLEMS

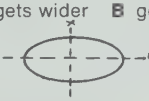
1. 6.28 cm **2.** 12.56 cm **3.** 3.14 cm² **4.** 9.42 cm² **5.** 400 **6.** 38 m **7.** 9.1 m/s **8.** 7.8 m/s **9.** 10 273 m² **10.** 5247 m²

C-113 THE SHOEMAKER'S KNIFE

2. 18.84 cm **3.** 18.84 cm **4.** 56.52 cm **5.** 6.28 cm² **6.** 25.12 cm² **7.** 25.12 cm² **8.** rectangle **9. A** no **B** quadrilateral

C-114 DISCUSSING THE IDEAS

2. A it gets wider **B** gets longer

3. yes,  lines x and y are the lines of symmetry

C-115 USING THE IDEAS

1. 50 mm **2.** 50 mm **3.** 50 mm **4.** 50 mm **5.** equal **6.** 40 mm **7.** 1570 mm² **8.** $\frac{3}{5}$

Think

C-116 DISCUSSING THE IDEAS

1. It has 8 faces. **2.** 8 faces, 12 edges, 6 vertices **3. A** The faces do not meet at right angles. **B** faces, edges are not equal to each other **C** The base is a pentagon. **4. A** pyramid with a square base **B** pyramid with a triangular base **C** pyramid with a hexagonal base **D** pyramid with an octagonal base **5.** Answers will vary.

C-117 USING THE IDEAS

Models are left for the students to do.

C-119 FACES, EDGES, AND VERTICES

1. 2 **2.** 7, 15, 17, 2 **3.** 6, 8, 12, 14, 2 **4.** 6, 5, 9, 11, 2 **5.** 4, 4, 6, 8, 2 **6.** 6, 5, 9, 11, 2 **7.** 8, 6, 12, 14, 2 **8.** 6, 6, 10, 12, 2 **9.** 10, 7, 15, 17, 2 **10.** 10, 7, 15, 17, 2 **11.** 12, 8, 15, 20, 2 **12.** 20, 12, 30, 32, 2 **13.** 2

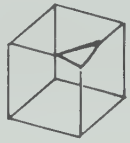
C-120 DISCUSSING THE IDEAS

Models are left to the students to make.

C-121 USING THE IDEAS

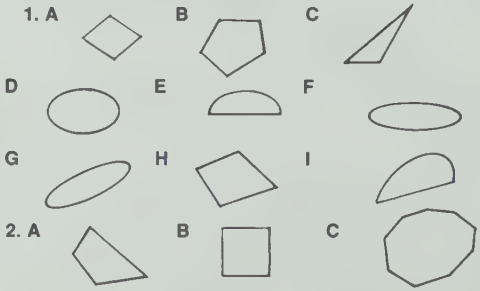
1. A 36 cm³ **B** 158.4 cm³ **C** 189.9 cm³ **2. A** 12 cm³ **B** 52.8 cm³ **C** 63.4 cm³ **3. A** 785 cm³ **B** 615.4 cm³ **C** 6.4 cm³ **D** 20.2 cm³ **E** 232.6 cm³ **F** 134 365 310 mm³ **4. A** 262 cm³ **B** 1.05 cm³ **C** 26.2 cm³ **D** 0.4 cm³ **E** 16.3 m³ **F** 14 243 mm³

C-122 DISCUSSING THE IDEAS

1. A  **B** yes

2. A rectangle **3. A** triangle **B** square **C** circle

C-123 USING THE IDEAS



C-124 REVIEWING THE IDEAS

1. A $\overline{BOC}$ **B** radius **C** $\overline{AB}$ **D** AC **E** $\angle AOC$ **F** $\angle ABC$ **G** 50° **2.** $\angle AOB = 90°$ **3.** 5.3 cm **4.** 3.14 **5.** 66 cm **6.** 38.5 cm² **7.** equal **8.** B **9. A** circle **B** square pyramid **C** cone **D** tetrahedron **E** cube **F** circular cylinder **G** triangular prism **H** hexagonal prism **10. A** 12 **B** 8 **C** 18 **D** 2 **11. A** 378 cm³ **B** 126 cm³ **12. A** 301.4 **B** 100.5 **13.** B

C-126 CUMULATIVE REVIEW

1. $\overline{XY}$, $\overline{YZ}$, $\overline{XZ}$ **2.** B **3. A** parallel **B** intersecting **C** skew **4.** C **5.** half plane **6.** $\overline{DE} \cong \overline{FG}$; $\overline{EG} \cong \overline{EF}$ **7.** $\angle KRP$, $\angle PRS$, $\angle KRS$ **8.** R **9. A** acute **B** vertical

C straight **D** obtuse **E** supplementary
F perpendicular **10. A** equilateral **B** scalene
C isoceles **11. A** parallelogram **B** square
C trapezoid **D** rectangle **E** hexagon **F** triangle
12. no **13.** 380, 0.38 **14. A** 54, 180 **B** 58, 165
C 36, 48 **15.** 10 **16. A** 120 **B** 164 **17. A** 125°
B $\angle DBC$ **18. A** 180°21′45″ **B** 38°27′48″ **19.** 42°32′
20. $A(1,2)$; $B(2,4)$; $C(3,1)$ **21.** $A(5,5)$; $B(6,7)$; $C(7,4)$
22. $A(7,6)$; $B(6,4)$; $C(5,7)$ **23.** $A'(7,2)$; $B'(6,4)$; $C'(5,1)$
24. A, C **25.** $\angle R \cong \angle X$; $\angle S \cong \angle Y$; $\angle T \cong \angle Z$; $\overline{RS} \cong \overline{XY}$; $\overline{ST} \cong \overline{YZ}$; $\overline{RT} \cong \overline{XZ}$ **26.** equal **27. A** $\angle ACB$ **B** $\angle AOB$
C 20° **28.** tangent **29.** 5 faces; 5 vertices; 8 edges
30. A 188.4 cm **B** 2826 cm^2 **31. A** 70 cm^3
B 141.3 cm^3 **C** 2.2 cm^3 **D** 62.8 cm^3 **32.** *B*

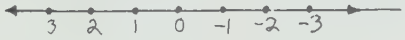
UNIT D: MODULE 1

D-2 DISCUSSING THE IDEAS

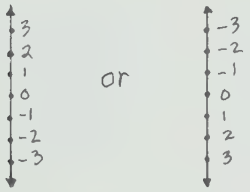
2. −5, −4, −3, −2 **3. A** −7 **B** −21 **C** 2 **D** 19
E −47 **F** 0

D-3 USING THE IDEAS

1. A −9 **B** −7 **C** −5 **D** 4 **E** 9 **F** 11 **G** −25
H −59 **I** 86 **J** −586 **K** 1079 **L** −1 687 943
M 5936 **N** −100 **O** 863 650 **2. A** −5 **B** 3 **C** −2
D 4 **E** −4 **F** −2 **G** 0 **H** 0 **I** 5 **J** −8 **K** 4 **L** 0
3. A −17 **B** 8 **C** 15 **D** −6
4. A



B



D-4 DISCUSSING THE IDEAS

1. 3 **2.** 0 **3.** 0 **4.** Opposites Principle **5.** Zero Principle **A** 2 **B** −5 **C** −7 **D** 8 **6. A** Commutative Principle **B** Opposites Principle **C** Associative Principle

D-5 USING THE IDEAS

1. A 0 **B** 0 **C** 0 **D** 0 **E** −3 **F** −9 **G** −18
H −76 **2. A** $a = 0; b = 0$ **B** $a = 0; b = 0$ **C** $a = 0; b = 0$ **D** $a = -8; b = -8$ **E** $a = -72; b = -72$
F $a = -963; b = -963$ **3. A** −8 **B** 17 **C** −58
D 867 **E** −1083 **4. A** $a = 2; b = 2$ **B** $a = 6; b = 6$
C $a = -3; b = -3$ **D** $a = -26; b = -26$ **5. A** 0
B −7 **C** −66 **D** 12 **E** −9 **F** 0 **G** 6 **H** −8
I −45 **6. A** $a = 0; b = 0$ **B** $a = -8; b = -8$ **C** $a = 17; b = 17$ **7. A** −7 **B** 0 **C** 4 **D** 15 **E** 6 **F** 0

D-6 DISCUSSING THE IDEAS

1. 5 **2.** 3° above 0° **3.** 2 **5.** 4 **6.** −7

D-7 USING THE IDEAS

1. A $6 richer **B** $9 poorer **C** $3 richer **2. A** 15° above 0° **B** 19° below 0° **C** 4° above 0° **3. A** 2 units right **B** 8 units left **C** 3 units right **4. A** 1 **B** 2
C 3 **D** 4 **E** 6 **F** 16 **5. A** 0 **B** 3 **C** 6 **D** 8 **E** 2
F −16 **G** 10 **H** −5 **I** −15 **J** −10 **K** 6 **L** −1
M −2 **N** 6 **O** −1 **P** −7 **Q** −20 **R** −3 **S** −5
T 0 **U** −9 **V** −19 **W** 48 **X** 7 **Y** 22 **Z** −1712

D-8 DISCUSSING THE IDEAS

1. $5.00 poorer **2.** −3 **3. A** −4 **B** −5 **C** −4
D −1 **4.** 4° below 0° **5. A** −7 **B** −1 **C** −6
6. positive **7.** negative **8.** by which numeral is greater
9. opposites **10.** The integer other than zero.

D-9 USING THE IDEAS

1. A $3 poorer **B** $4 poorer **C** $2 poorer **2. A** 3 units left **B** 5 units left **C** 13 units left **3. A** 2° below 0° **B** 3° below 0° **C** 5° below 0° **4. A** −4 **B** −1
C −4 **D** −14 **E** −3 **F** −2 **G** −5 **H** −4 **5. A** −1
B −1 **C** −7 **D** −4 **E** −4 **F** −6 **G** −3 **H** −1
I −3 **J** −7 **K** −5 **L** 0 **M** 7 **N** −5 **O** 5 **P** 5
Q 0 **R** 2 **S** −7 **T** −10 **U** 6 **V** −9 **W** −15 **X** 8
Y −7

D-10 DISCUSSING THE IDEAS

1. B $n + 4 = 3$ **C** $n + -2 = -9$ **D** $n + -5 = -3$
2. A 5 **B** $-3 + n = 2$ **3. A** $n + -7 = -1$ **B** 6
4. A $n + -2 = 8$ **B** 10

D-11 USING THE IDEAS

1. A 4 **B** −11 **C** 9 **D** −6 **2. A** $n + -5 = 4; n = 9$
B $n + -9 = -4; n = 5$ **C** $8 + n = -8; n = -16$
D $-6 + n = 1; n = 7$ **3. A** −4 **B** −1 **C** −7 **D** −2
E −1 **F** −3 **G** 7 **H** 6 **I** 8 **J** −5 **K** −2 **L** −10
M −4 **N** −4 **O** −7 **P** 15 **Q** 14 **R** −5 **S** −3
T −18 **4. A** $n + -9 = 3; n = 12$ **B** $n + 12° = 8°; n = 4°$ below 0° **C** $-19 + n = 7; n = 26$
Think **1.** 6 **2.** 0 **3.** 6 **4.** 12

D-12 DISCUSSING THE IDEAS

2. yes; $-7 - 2 = -9$ because $-9 + 2 = -7$; $1 - -3 = 4$ because $4 + -3 = 1$ **3. A** −5 **B** 4 **C** −1 **4. A** 2
B −11 **C** −7 **D** 3 **E** 5 **F** 13 **G** 1

D-13 USING THE IDEAS

1. A −2 **B** 3 **C** 9 **D** 15 **E** 1 **F** −1 **2. A** 6
B 5 **C** −3 **D** −1 **E** 0 **F** −9 **G** −2 **H** −3 **I** 1
J 4 **K** −1 **L** −1 **M** −4 **N** −1 **O** −1 **P** −5
Q 20 **R** 14 **S** −2 **T** −5 **U** −8 **V** −7 **W** 77
X −2 **3. A** −10 **B** −4 **C** −10 **D** −6 **E** −10
F −6 **G** 6 **H** 10 **I** −8 **J** 8 **K** 5 **L** 10 **M** −2
N −9 **O** −1 **P** −20 **Q** 18 **R** −15 **S** −15 **T** 1

4. A −2 **B** −8 **C** 8 **D** −2 **E** 13 **F** −7 **G** −13 **H** 13 **I** −5 **J** 0 **K** −14 **L** −8 **M** −10 **N** −4 **O** −4 **P** −5

D-14 DISCUSSING THE IDEAS

1. A 6 **B** −2 **C** −10 **D** −8 **E** 19 **F** −23
2. A $-3 \cdot -4 = -4 \cdot -3$ **B** $(13 \cdot 7) \cdot -5 = 13 \cdot (7 \cdot -5)$ **C** yes; Commutative and Associative Principles
3. $(-2 \cdot 3) + (-2 \cdot -4)$

D-15 USING THE IDEAS

1. A $a = -7; b = -7$ **B** $a = -12; b = -12$ **C** $a = 0; b = 0$ **D** $a = 0; b = 0$ **2. A** $a = -6; b = -6$ **B** $a = -8; b = -8$ **C** $a = -10; b = -10$ **D** $a = -14; b = -14$ **E** $a = -116; b = -116$ **F** $a = -12; b = -12$ **G** $a = -20; b = -20$ **H** $a = -32; b = -32$ **3. A** $a = -6; b = -6$ **B** $a = -10; b = -10$ **C** $a = -21; b = -21$ **D** $a = -24; b = -24$ **4. A** 0; 0; 0; 0; −6 **B** 0; 0; 0; 0; 6 **C** 0; 0; 0; 0; 20 **5. A** negative **B** positive

Think first number = 346; second number = 37

D-16 DISCUSSING THE IDEAS

2. $-12 \times 2 = n$ **3. A** −24 **B** 12 **C** 12 **D** −35

D-17 USING THE IDEAS

1. A $4 \times -3 = -12$ **B** $-2 \times -4 = 8$ **C** $-3 \times 3 = -9$ **D** $6 \times 1 = 6$ **2. A** −20 **B** −15 **C** −10 **D** −5 **E** 0 **F** 5 **G** 10 **H** 15 **I** 20 **J** 25 **K** 30 **L** 35 **3. A** −8 **B** −12 **C** 12 **D** −1 **E** 1 **F** 8 **G** 0 **H** −36 **I** −36 **J** 36 **K** −32 **L** 32 **M** 0 **N** −42 **O** 42 **P** −156 **Q** 156 **R** −168 **S** −169 **T** 169 **U** −270 **V** −3800 **W** 3800 **X** −468 000 **4. A** −12 **B** −32 **C** 3 **D** 20 **E** −8 **F** 27 **G** −30 **H** 36 **I** −24 **J** 30 **K** −1 **L** 1 **5. A** −56 **B** −30 **C** 63 **D** 4 **E** −3 **F** −8 **G** −52 **H** −10 **I** 7 **J** 9 **K** −5 **L** −9

D-18 DISCUSSING THE IDEAS

2. yes; $-12 \div 3 = -4$ because $-4 \cdot 3 = -12$; $-8 \div 4 = -2$ because $-2 \cdot 4 = -8$ **3. A** 4 **B** −3 **C** −8 **D** 8 **4. A** −4 **B** 3 **C** −5

D-19 USING THE IDEAS

1. A 4 **B** −2 **C** −1 **D** 4 **E** −3 **F** 0 **G** −1 **H** −3 **I** 1 **J** −8 **K** −6 **L** −6 **M** −5 **N** 4 **O** −4 **P** 4 **Q** −4 **R** −4 **S** −5 **T** −3 **2. A** −3 **B** 4 **C** −6 **D** 4 **E** −3 **F** 4 **G** −4 **H** −5 **I** −8 **J** −6 **K** −3 **L** −4 **M** −4 **N** −5 **O** −1 **P** −6 **Q** −1 **R** 0 **S** 1 **T** −2 **3. A** −5 **B** −3 **C** 1 **D** −4 **E** 16 **F** 3 **G** 3 **H** 3 **I** 0 **J** −4 **K** −16 **L** 153 **M** −3 **N** −1 **O** −4 **P** 4 **Q** −3 **R** 7 **4.** 4; −5

Think (Answers will vary.) **A** $0 = (4 - 4) \times 4$ **B** $3 = 4 - (4 \div 4)$ **C** $4 = (4 \times 4) \div 4$ **D** $5 = 4 + (4 \div 4)$ **E** $12 = (4 + 4) + 4$ **F** $20 = (4 \times 4) + 4$ **G** $32 = (4 + 4) \times 4$ **H** $64 = (4 \times 4) \times 4$ **2. A** $1 = \sqrt{4} - (4 \div 4)$ **B** $6 = (4 + 4) - \sqrt{4}$ **C** $8 = (4 \times 4) \div \sqrt{4}$ **D** $25 = 4! + (4 \div 4)$

D-20 DISCUSSING THE IDEAS

1. A 1 **B** 3 **C** −6 **2. A** greater than **B** less than **C** greater than **D** less than **E** greater than **F** less than **3. A** $-1 + 2 = 1$ **B** $-5 + 5 = 0$ **C** $-6 + 13 = 7$ **D** $-7 + 11 = 4$

D-21 USING THE IDEAS

1. A > **B** > **C** > **2. A** > **B** < **C** > **D** < **E** < **F** > **G** > **H** > **3. A** *E*; *F*; *G* **B** *A*; *B*; *C*; *D*; *E*; *F* **C** *A*; *B*; *L*; *M*; *N*; *O*

4. A

Think

D-22 DISCUSSING THE IDEAS

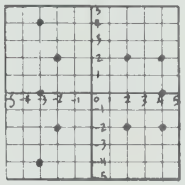
1. (0,0) **2. A** (3,−4) **B** (−5,6) **C** (−2,0) **D** (−4,−5) **E** (0,−3) **F** (6,6)

3.

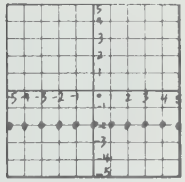
4. Answers may vary, examples: (1,2); (3,4); (−5,−4); (−8,−7)

D-23 USING THE IDEAS

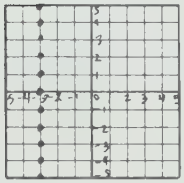
1. A



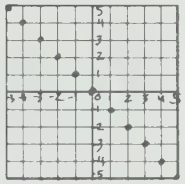
B



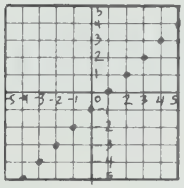
C



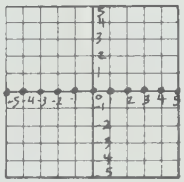
D



E

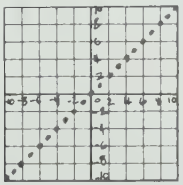


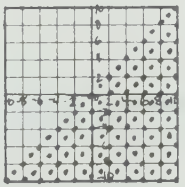
F



B

2. A





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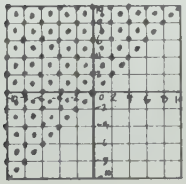

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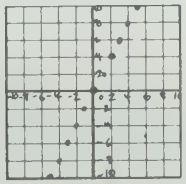

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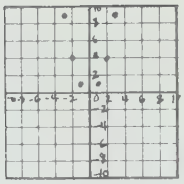

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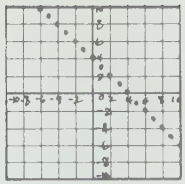

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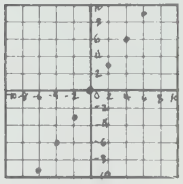
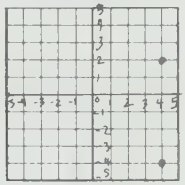
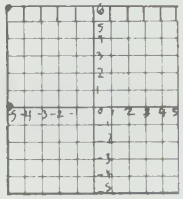


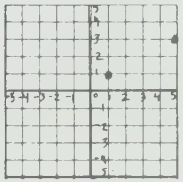
3. A



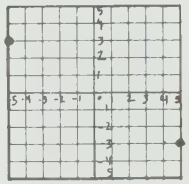
B



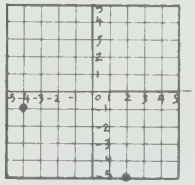
C



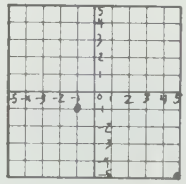
D



E



F



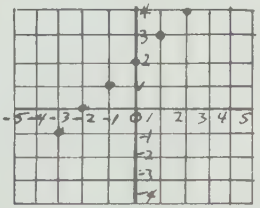
A (4,−1) **B** (−5,3) **C** (3,2) **D** (0,0) **E** (−1,−3) **F** (2,−3)

Think No! 1, 2, 4, 8, 16, 32, 64, 1.28, 2.56, 5.12, . . . Allowance for tenth week is $5.12. For the twentieth week it is $5242.88.

D-24 DISCUSSING THE IDEAS

1. yes; straight line **2. A** (−3,−1); 0, (−2,0); 1, (−1,1); 2, (0,2); 3, (1,3); 4, (2,4)

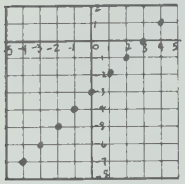
B



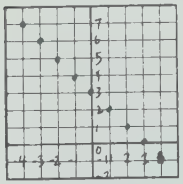
C Both are straight lines — the graph in part B is 1 less on the *y*-axis for all values of *x*.

D-25 USING THE IDEAS

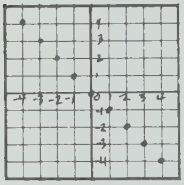
1. A −6 **B** −5 **C** −4 **D** −3 **E** −2 **F** −1 **G** 0 **H** 1



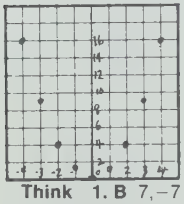
2. A 6 **B** 5 **C** 4 **D** 3 **E** 2 **F** 1 **G** 0 **H** 1



3. A 3 **B** 2 **C** 1 **D** 0 **E** −1 **F** −2 **G** −3 **H** −4



4. A 4 **B** 1 **C** 0 **D** 1 **E** 9 **F** 16



Think **1. B** 7,−7 **C** −4,−4 **D** 4,4 **E** −4,0 **F** 5,−3 **G** 5,−4 **H** −10,−10 **2. B** −8,−1 **C** −1,−1 **D** −8,4 **E** −10,−10 **F** 10,−2 **G** −35,−5 **H** 0,−8

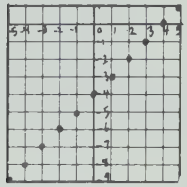
D-26 REVIEWING THE IDEAS

1. A −5 **B** 3 **C** −27 **D** 16 **E** 365 **F** −273 **G** 5806 **H** 0 **2. A** *E* **B** *F* **C** *A* **D** *H* **E** *G* **F** *B* **G** *C* **3. A** 0 **B** 2 **C** −12 **D** −11 **E** −3 **F** −6 **G** 3 **H** 0 **I** −27 **4. A** −3 **B** −5 **C** −4 **D** −3 **E** −5 **F** 1 **5. A** −3 **B** −4 **C** −5 **D** −7 **E** 1 **F** 2 **G** 6 **H** −2 **I** 4 **6. A** 72 **B** −36 **C** −84 **D** −25 **E** −121 **F** 56 **G** 860 **H** −140 **I** 13 **7. A** 3 **B** −5 **C** −9 **D** −1 **E** −5 **F** −1 **8. A** −4 **B** 4 **C** −4 **D** −6 **E** 3 **F** 0 **G** 8 **H** 5 **I** 8 **J** −9 **K** −4 **L** −1 **9. A** > **B** > **C** > **D** > **E** > **F** > **G** < **H** > **I** < **J** > **10.** (−3,3); (4,0); (1,−4)

11. Function Rule

$N + (-4)$

N	$f(N)$
−5	−9
−4	−8
−3	−7
−2	−6
−1	−5
0	−4
1	−3
2	−2
3	−1
4	0
5	1



UNIT D: MODULE 2

D-30 DISCUSSING THE IDEAS

1. A yes **B** yes **2.** false **4.** *B*

D-31 USING THE IDEAS

1. A 1, 2, 3, 4 **B** equally likely **2. A** *A*, *B* **B** not equally likely **C** *A* **3. A** 1, 2, 3 **B** equally likely **4. A** red, blue **B** not equally likely **C** red **5. A** 4H; 3H, 1T; 2H, 2T; 1H, 3T; 4T **B** not equally likely **C** 2H, 2T **6. A** 4 up; 3 up 1 side; 2 up, 2 side; 1 up 3 side; 4 side **B** not equally likely **C** 2 up, 2 side

Think $(7 \div 7 + 7) \times 7$

D-32 DISCUSSING THE IDEAS

1. A 6 **B** 6 **C** 6 **2.** Pair each letter with each number *A*1, *A*2, *A*3, *A*4, *A*5, *A*6, *B*1, *B*2, *B*3, etc. **3.** 36 **4.** $\{1, 2, 3, 4, 5, 6\}$ **5. A** 4 **B** penny heads, dime tails **C** dime heads, penny tails

D-33 USING THE IDEAS

1. A H, T **B** 1, 2, 3, 4, 5, 6 **C** 1, 2, 3, 4, 5

2.

Penny	Nickel	Dime
H	H	H
H	H	T
H	T	H
H	T	T
T	H	H
T	H	T

T	T	H
T	T	T

3. 1*A*, 1*B*, 1*C*, 2*A*, 2*B*, 2*C*, 3*A*, 3*B*, 3*C*, 4*A*, 4*B*, 4*C* **4.** {(H,1), (T,1), (H,2), (T,2), (H,3), (T,3), (H,4), (T,4), (H,5), (T,5), (H,6), (T,6)} **5. A** 49 **B** 13

D-34 DISCUSSING THE IDEAS

1. B 20 **C** 10 **2. A** $\frac{1}{3}$ **B** 1 chance in 6 **C** $\frac{1}{6}$ **3. A** 1 chance in 2 **B** $\frac{1}{2}$ **C** 1 chance in 2 **D** $\frac{1}{2}$ **4. A** $\frac{1}{3}$ **B** 1

D-35 USING THE IDEAS

1. A 2 **B** $\frac{2}{3}$ **C** 1 **D** $\frac{1}{3}$ **2. A** $\frac{4}{6}$ **B** $\frac{1}{3}$ **C** $\frac{2}{6}$ **3. A** $\frac{1}{3}$ **B** $\frac{2}{3}$ **C** 1 **D** 0 **4. A** $\frac{3}{5}$ **B** $\frac{2}{5}$ **C** $\frac{4}{5}$

D-36-37 PROBABILITY EXPERIMENTS

1. C 25 **2. B** $\frac{7}{10}$ **7. A** 4

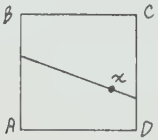
D-38 DISCUSSING THE IDEAS

1. 250 **2. A** 5 **B** multiply by 50 **3. A** 4 **B** 6 **C** average them **5.** 280; answers may vary

D-39 USING THE IDEAS

1. 1600 **2.** 1397 **3.** 1667 **4.** 18 300 **5.** 69

Think



D-40 DISCUSSING THE IDEAS

1. yes **2.** yes **3.** Divide the number of students in school by 10. **4. B** 18 **C** 1.8

D-41 USING THE IDEAS

1. A Mary Jacobs **B** Mary 70; Bob 30 **2. A** 600 **B** 400 **3. A** $\frac{1}{500}$ **B** 1000 **4. A** 977 040 **B** 722 160 **C** 424 800 **5.** 28 800 **6.** 150

D-42 REVIEWING THE IDEAS

1. *B* **2.** {(H,H), (H,T), (T,H), (T,T)} **3. A** 25 **B** $\frac{1}{2}$ **4.** $\frac{1}{2}$ **5.** $\frac{2}{3}$ **6. A** $\frac{1}{3}$ **B** $\frac{2}{3}$ **C** $\frac{3}{5}$ **7. A** {(*H*,*A*), (*H*,*B*), (*T*,*A*), (*T*,*B*)} **B** $\frac{1}{4}$ **8. A** 1 : $\frac{1}{6}$; 2 : $\frac{1}{3}$; 3 : $\frac{1}{2}$; 4 : $\frac{1}{6}$ **B** 20 **9. A** 85 **B** 50 **C** 25 **10.** 4 500 000

D-46 DISCUSSING THE IDEAS

1. yes **2.** alphabetically, numerically by lows (or) highs (or) by greater or lesser differences

D-47 USING THE IDEAS

1. A tungsten **B** lead **C** aluminum **D** gold and copper **2. A** General Motors **B** Hitachi **C** Exxon **D** Royal Dutch Shell **3.** Shanghai, 10.7 million; Peking, 10 million; Tokyo, 8.85 million, New York, 7.9 million; London, 7.4 million **4. A** Sears Tower, Chicago 442 m; World Trade Center, N.Y.C. 412 m; Empire State, N.Y.C. 381 m; Standard Oil, Chicago 346 m; John Hancock Center, Chicago 337 m **B** Caspian Sea 365 000 km²; Superior 82 000 km²; Aral Sea 62 500 km²; Victoria 62 000 km²; Huron 61 000 km² **C** Everest 8850 m; K2 8610 m; Kangchenjunga 8590; Lhotse 8500 m; Kangchenjunoa Speak 8470 **5.** NATIONAL LEAGUE: Hack Wilson, Chicago 56 in 1930; Ralph Kiner, Pittsburgh and John Mize, N.Y. 51, in 1947; Ralph Kiner, Pittsburgh 54 in 1949; Willie Mays, N.Y. 51 in 1955; Willie Mays, San Francisco 52 in 1965 — AMERICAN LEAGUE: Babe Ruth, N.Y. 54 in 1920, 59 in 1921, 60 in 1927, 54 in 1928; Jimmy Fox, Philadelphia 58 in 1932; Hank Greenberg, Detroit 58 in 1938; Mickey Mantle, N.Y. 52 in 1956; Roger Maris, N.Y. 61 in 1961

Think yes

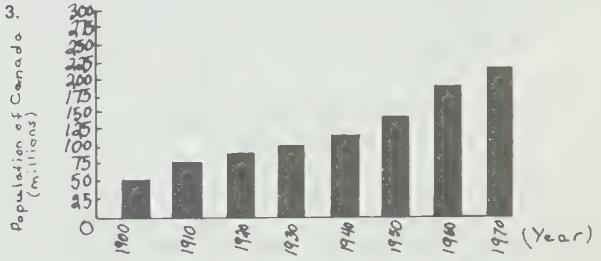
D-48 DISCUSSING THE IDEAS

1. bar graph **2.** no **3.** Bill, $1.25; Fred, $2.00; Bob, $1.50; Paul, $1.75 **4.** Bill's graph because it has a larger scale

D-49 USING THE IDEAS

1. Nile: 6600; Amazon: 6400; Ob-Irtysh: 5500; Yangtze: 5400; Huang: 4750 **2. A** 1900 = 900 000; 1910 = 1 100 000; 1920 = 1 700 000; 1930 = 2 100 000; 1940 = 2 200 000; 1950 = 2 000 000; 1960 = 3 700 000; 1970 = 5 500 000 **B** 4 600 000 **C** 1960–1970 **D** 1940–1950

3.

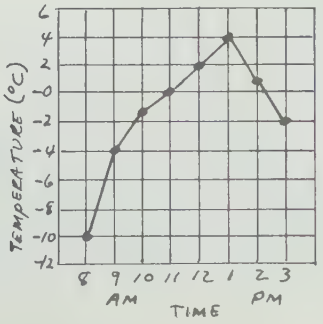


4. 37 000 000; 387 000 000

D-50 DISCUSSING THE IDEAS

1. A 8 : 16°; 9 : 19°; 10 : 22°; 11 : 24°; 12 : 25°; 1 : 26.5°; 2 : 26°; 3 : 24° **B** 23° **2. A** 10.5° **B** You must include at least the highest and lowest temperatures.

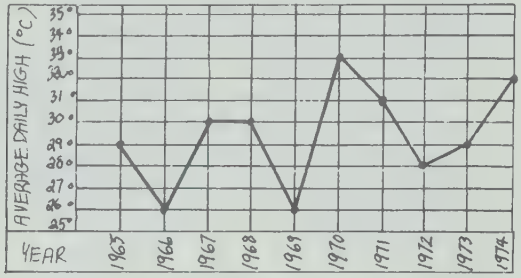
3.



D-51 USING THE IDEAS

1. Jan: 6; Feb: 7; Mar: 9; Apr: 9; May: 8; June: 14; July: 16; Aug: 15; Sept: 15; Oct: 13; Nov: 4; Dec: 5 **2. A** Nov; Dec **B** June; July; Aug; Sept; Oct **3.** 121 **4. A** 28° **B** 6°; 3 AM, 5 AM, 6 AM **C** 18°

5.



D-52 PICTURE GRAPHS

1. A 100 million people **B** German: 110 million; Hindu: 150 million; Spanish: 200 million; Russian: 200 million; English: 320 million; Chinese: 600 million **2. A** 1965 **B** 1970 **C** 1970: 6.5 million; 1965: 11 million; 1960: 7 million; 1955: 7 million **3. A** Angel: 1000 m; Tugela: 900 m; Yosemite: 700 m; Cuquenan 600 m; Sutherland: 600 m **B** 300 m **C** 300 m

D-53 CIRCLE GRAPHS

1. personal income tax **2.** same as #1 **3.** $4 694 778 000 **4.** $2 133 990 000; $569 064 000 **5.** 0 to 23 years **6. A** 4 098 110 **B** 9 706 050 **7.** 1 941 210 **8.** 1 401 985

D-54 DISCUSSING THE IDEAS

2. 61 **3.** probably about 86

D-55 USING THE IDEAS

1. A 306.4 **B** 18.6 **C** 23.5 **D** 2.3 **2.** 0.66 **3.** 39 **4.** 85 **5.** $10 036.00 **6.** 28¢(2¢ left over) **7.** 94 **8.** 2124.8 **9.** 369 997

Think Arnold: 1; Barbara: 3; Chris: 2

D-56 DISCUSSING THE IDEAS

1. List students and scores in order of scores.
2. A 17 **B** 32 **C** 93 **3.** yes

D-57 USING THE IDEAS

1. A 46 **B** 15 **C** 9 **D** 363.5 **E** 27 **F** 5.5 **2.** 89.5 **3.** 16 **4.** 8.3; 6 **5.** 1.57 **6.** 49

Think

Cut link Number 3 so result is: 1st day — pay one link; 2nd day — pay two links, take back one; 3rd day — pay one link; 4th day — pay four links, take back one and two; 5th day — pay one link; 6th day — pay two links, take back one; 7th day — pay one link.

D-58 DISCUSSING THE IDEAS

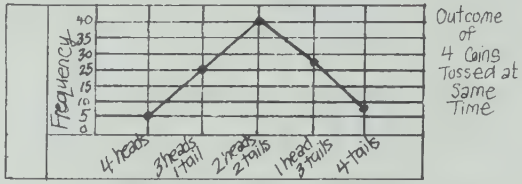
2.

Sums	Frequency	Sums	Frequency
2	2	7	19
3	4	8	16
4	5	9	12
5	13	10	9
6	15	11	3
		12	2

D-59 USING THE IDEAS

1. A 88 **B** 1.60 **C** 1

2.



3. 4

D-60 REVIEWING THE IDEAS

1. A 2654 **B** Whitney **2.** Baffin: 475 000; Ellesmere: 215 000; Victoria: 210 000; Newfoundland: 110 000; Banks: 60 000 **3.** June: 13 cm; July: 10 cm; Aug.: 5 cm; Sept.: 7 cm; Oct.: 10 cm; Nov.: 10 cm; Dec.: 15 cm **4.** 10 **5. A** $\frac{1}{4}$ **B** $\frac{3}{8}$ **C** $\frac{1}{2}$ **6.** 72¢ **7. A** 38 **B** 82.5 **8.** 84

SUPPLEMENTARY EXERCISES

S-2 Set 1

1. A $\{a, b, c, d, e, f\}$ **B** $\{a, c\}$ **C** $\{a, b, c, d, e\}$ **D** $\{a, b, d\}$ **E** $\{a, c, t, b, d\}$ **F** $\{a\}$ **G** $\{a, b, c, d, e\}$ **H** $\varnothing$ **2. A** $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ **B** $\varnothing$ **C** $\{0, 2, 3, 4, 6, 8, 9\}$ **D** $\{0, 6\}$ **E** $\{0, 1, 3, 5, 6, 7, 9\}$ **F** $\{3, 9\}$ **G** $\{0, 2, 4, 6, 8\}$ **H** $\{0, 2, 4, 6, 8\}$ **I** $\{1, 3, 5, 7, 9\}$ **J** $\varnothing$ **K** $\{0, 1, 3, 4, 5, 6, 7, 8, 9\}$ **L** $\{0, 3, 6, 9\}$ **M** $\{0, 3, 6, 9\}$ **N** $\{0, 6\}$ **O** $\{0, 2, 3, 4, 6, 8, 9\}$ **P** $\{0, 6\}$

S-3 Set 2

1. A 10^4 **B** 10^6 **C** 10^7 **D** 10^9 **2. A** 1000 **B** 100 000 **C** 10 **D** 1 000 000 000 **E** 10 000 000 **F** 10 000 000 000 **G** 1 000 000 **H** 1 **I** 1 000 000 000 000 000 **J** 100 000 000 000 000 000 000 000 **K** 1 000 000 000 000 **L** 100 000 000 **3. A** 10^4 **B** 10^8 **C** 10^6 **D** 10^6 **E** 10^8 **F** 10^8 **G** 10^6 **H** 1 **4. A** 10^2 **B** 10^2 **C** 10^2 **D** 1 **E** 10^5 **F** 10 **G** 10^3 **H** 10^2

S-3 Set 3

1. A 300 **B** 800 000 **C** 60 000 **D** 9 000 000 **E** 40 000 000 **F** 20 000 000 000 **G** 5 **H** 6 000 000 000 **I** 80 **J** 40 000 000 000 000 **2. A** $6 \cdot 10^2$ **B** $5 \cdot 10^1$ **C** $4 \cdot 10^3$ **D** $8 \cdot 10^4$ **E** $5 \cdot 10^5$ **F** $9 \cdot 10^6$ **G** $3 \cdot 10^7$ **H** $5 \cdot 10^{13}$ **3. A** $4 \cdot 10^1$ **B** $3 \cdot 10^2$ **C** $2 \cdot 10^4$ **D** $9 \cdot 10^3$ **E** $5 \cdot 10^6$ **F** $8 \cdot 10^7$ **G** $6 \cdot 10^9$ **H** $3 \cdot 10^{10}$ **I** $2 \cdot 10^5$ **J** $4 \cdot 10^8$ **K** $9 \cdot 10^{11}$ **L** $5 \cdot 10^{12}$

S-4 Set 4

1. A < **B** < **C** > **D** < **E** > **F** < **G** > **H** < **I** < **J** > **K** < **L** < **2. A** 99 **B** 90 **C** 900 **D** 999 **E** 9000 **F** 9901 **G** 99 000 **H** 99 099 **I** 8889 **J** 900 000 **K** 989 999 **L** 899 999

S-4 Set 5

1. A $6_{(8)}$ **B** $16_{(8)}$ **C** $30_{(8)}$ **D** $40_{(8)}$ **E** $22_{(8)}$ **F** $47_{(8)}$ **G** $70_{(8)}$ **H** $72_{(8)}$ **I** $42_{(8)}$ **J** $105_{(8)}$ **K** $100_{(8)}$ **L** $103_{(8)}$ **M** $337_{(8)}$ **N** $561_{(8)}$ **O** $526_{(8)}$ **P** $700_{(8)}$ **Q** $700_{(8)}$ **R** $1020_{(8)}$ **2. A** $4_{(8)}$ **B** $7_{(8)}$ **C** $11_{(8)}$ **C** $11_{(8)}$ **D** $7_{(8)}$ **E** $14_{(8)}$ **F** $12_{(8)}$ **G** $56_{(8)}$ **H** $41_{(8)}$ **I** $15_{(8)}$ **J** $14_{(8)}$ **K** $71_{(8)}$ **L** $33_{(8)}$

S-4 Set 6

1. A $14_{(8)}$ **B** $50_{(8)}$ **C** $46_{(8)}$ **D** $74_{(8)}$ **E** $70_{(8)}$ **F** $67_{(8)}$ **G** $100_{(8)}$ **H** $116_{(8)}$ **I** $144_{(8)}$ **J** $334_{(8)}$ **K** $414_{(8)}$ **L** $647_{(8)}$ **2. A** $3_{(8)}$ **B** $3_{(8)}$ **C** $3_{(8)}$ **D** $6_{(8)}$ **E** $7_{(8)}$ **F** $7_{(8)}$ **G** $25_{(8)}$ **H** $23_{(8)}$

S-5 Set 7

1. A 35 **B** 23 **C** 7 **D** 5 **E** 52 **F** 24 **G** 48 **H** 138 **I** 22 **J** 42 **K** 26 **L** 21 **M** 18 **N** 26 **O** 6 **2. A** 3 **B** 6 **C** 5 **D** 9 **E** 6 **F** 4 **G** 2 **H** 4 **I** 9 **J** 1 **K** 6 **L** 5 **M** 100 **N** 0

S-5 Set 8

1. A 94 **B** 130 **C** 555 **D** 1593 **E** 3000 **F** 33 182 **G** 78 **H** 674 **I** 3357 **J** 21 381 **K** 134 552 **L** 11 880 **2. A** 240 **B** 1672 **C** 189 **D** 2934 **E** 776 **F** 30 246 **G** 4995 **H** 2898 **I** 188 723 **J** 24 290 **K** 22 327 **L** 876 900

S-6 Set 9

1. A 14 **B** 13 **C** 24 **D** 188 **E** 229 **F** 72 **G** 3391 **H** 2759 **I** 3389 **J** 2096 **K** 199 **L** 8453 **M** 16 109 **N** 53 302 **O** 36 564 **P** 1048 **Q** 199 101 **R** 386 934 **2. A** 26 **B** 149 **C** 677 **D** 629 **E** 8952 **F** 9050 **G** 987 **H** 310 180 **I** 798 999

S-6 Set 10

1. A 70 **B** 900 **C** 6000 **D** 50 000 **E** 300 000 **F** 4 000 000 **G** 300 **H** 3000 **I** 30 000 **J** 30 000 **K** 300 000 **L** 300 000 **M** 1200 **N** 15 000 **O** 3600 **P** 8000 **Q** 3500 **R** 4000 **S** 12 000 **T** 10 000 **U** 60 000 **V** 320 000 **W** 120 000 **Y** 3 000 000 **2. A** 40 **B** 200 **C** 70 **D** 60 **E** 30 **F** 2 **G** 5 **H** 7 **I** 80 **J** 70 **K** 70 **L** 9 **M** 20 **N** 5 **O** 30 **P** 80 **Q** 800 **R** 400 **S** 30 **T** 300 **U** 300 **V** 6000 **W** 2000 **X** 90 000

S-7 Set 11

1. A 114 **B** 371 **C** 702 **D** 948 **E** 1780 **F** 5920 **G** 7677 **H** 2032 **I** 6615 **J** 67 466 **K** 141 624 **L** 1 292 685 **2. A** 391 **B** 1127 **C** 2268 **D** 3536 **E** 8342 **F** 7068 **G** 5229 **H** 59 057 **I** 35 742 **J** 230 615 **K** 810 823 **L** 2 745 280 **3. A** 37 975 **B** 153 180 **C** 134 325 **D** 73 440 **E** 49 113 **F** 811 801 **G** 380 492 **H** 327 448 **I** 605 394 **J** 1 367 782 **K** 806 400 **L** 3 824 557

S-7 Set 12

1. A 28 **B** 32 **C** 166 **D** 54 **E** 48 **F** 187 **G** 67 **H** 137 **I** 56 **J** 295 **K** 495 **L** 534 **M** 361 **N** 689 **O** 1554 **2. A** 16 R3 **B** 14 R1 **C** 30 R3 **D** 39 R4 **E** 38 R6 **F** 66 R1 **G** 23 R3 **H** 66 R3 **I** 87 R5 **J** 2157 R3 **K** 349 R1 **L** 860 R1 **M** 834 R1 **N** 1519 R1 **O** 2887 R1

S-8 Set 13

1. A 22 R11 **B** 31 R11 **C** 26 R7 **D** 11 R36 **E** 25 R20 **F** 63 R56 **G** 81 R22 **H** 76 R38 **I** 65 R67 **J** 413 R46 **K** 525 R70 **L** 1538 R38 **2. A** 3 R5 **B** 2 R51 **C** 2 R48 **D** 6 R172 **E** 4 R81 **F** 8 R146 **G** 37 R718 **H** 20 R40 **I** 147 R177 **J** 474 R86 **K** 1302 R584 **L** 429

S-8 Set 14

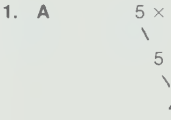
1. A $n - 16 = 57$ **B** $6 \cdot n = 96$ **C** $n \div 8 = 13$ **D** $n \cdot 2 + 3 = 25$ **E** $n \div 5 + 5 = 15$ **F** $(7 + n) \cdot 3 = 36$ **2. A** $45 - n = 31$ **B** $9 : n = 108$ **C** $n \div 8 = 25$ **D** $5 \cdot n + 3 = 18$ **E** $(n \div 6) - 9 = 0$ **F** $(n + 8) \cdot 3 = 60$

S-9 Set 15

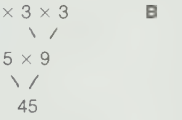
1. $n + 5 = 27$; 22 **2.** $3 \cdot n - 7 = 35$; 14 **3.** $165 - 18 = n$; 147 cm **4.** $371 \div 7 \cdot n$; 53 kilometers hr **5.** $(n + 8) \div 5 = 10$; 42 **6.** $3 \cdot n = 69$; 23 cm **7.** $96 \cdot 3 \cdot n$; 288 km **8.** $2 \cdot 8 + 3 = n$; 19 **9.** $3 \cdot n + 7 = 199$; 64 **10.** $2n + 2(n + 3) = 30$; 6 m, 9 m **11.** $2n - 18 = 2754$; 1386 **12.** $2 \cdot 30 + 2 \cdot 6 = n$; 72 m **13.** $3n = 4.52 - .26$; $1.42 **14.** $.79n + .63 = 3.79$; 4

S-10 Set 16

1. A

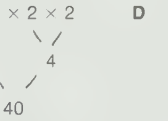


B



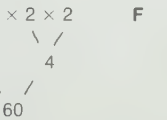
C

D

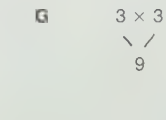


E

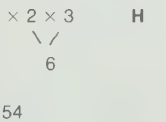
F



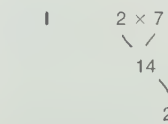
G



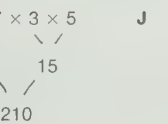
H



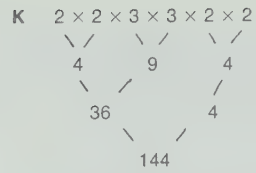
I



J



K $2 \times 2 \times 3 \times 3 \times 2 \times 2$



2. A 2^3 **B** $3 \cdot 5$ **C** $2 \cdot 3^2$ **D** $2^3 \cdot 3$ **E** 3^3 **F** $2 \cdot 3 \cdot 5$
G 31 **H** 2^5 **I** $5 \cdot 7$ **J** $3 \cdot 13$ **K** $2^3 \cdot 5$ **L** 41 **M** $3 \cdot 17$
N $2^4 \cdot 5$ **O** $2 \cdot 3^2 \cdot 5$ **P** 103 **Q** $2 \cdot 3^3$ **R** $2^3 \cdot 3^2$ **S** $2^5 \cdot 3$
T $2^2 \cdot 5^2$ **U** $2 \cdot 3 \cdot 5^2$ **V** $2 \cdot 3 \cdot 5 \cdot 7$ **W** $2^2 \cdot 5^3$ **X** 2^9

S-11 Set 17

1. A 1, 3, 5, 15 **B** 1, 3, 9 **C** 3, 1 **D** 3
2. A 1, 2, 3, 4, 6, 8, 12, 24 **B** 1, 2, 4, 5, 8, 10, 20, 40
C 1, 2, 4, 8 **D** 8 **3. A** 2 **B** 1 **C** 4 **D** 3 **E** 7 **F** 5
G 9 **H** 12 **I** 1 **J** 5 **K** 5 **L** 9 **M** 5 **N** 25 **O** 12
P 3 **Q** 5 **R** 40 **S** 30 **T** 1

S-11 Set 18

1. A 0, 4, 8, 12, 16, 20, 24, 28
B 0, 12, 24, 36, 48, 60, 72, 96 **C** 12, 24 **D** 12
2. A 0, 9, 18, 27, 36, 45, 54, 63
B 0, 15, 30, 45, 60, 75, 90, 105 **C** 45 **D** 45 **3. A** 21
B 22 **C** 15 **D** 36 **E** 40 **F** 60 **G** 50 **H** 66 **I** 90
J 200 **K** 20 **L** 18 **M** 10 **N** 60 **O** 56 **P** 24
Q 48 **R** 80 **S** 30 **T** 150

S-12 Set 19

1. A $\frac{3}{4}$ **B** $\frac{3}{5}$ **C** $\frac{5}{6}$ **D** $\frac{1}{3}$ **E** $\frac{3}{8}$ **F** $\frac{5}{6}$ **2. A** $\frac{4}{12}, \frac{5}{15}, \frac{6}{18}$ **B** $\frac{12}{16}, \frac{15}{20}, \frac{18}{24}$ **C** $\frac{12}{20}, \frac{15}{25}, \frac{18}{30}$ **D** $\frac{12}{28}, \frac{15}{35}, \frac{18}{42}$ **E** $\frac{4}{32}, \frac{5}{40}, \frac{6}{48}$ **F** $\frac{12}{4}, \frac{15}{5}, \frac{18}{6}$
G $\frac{12}{40}, \frac{15}{50}, \frac{18}{60}$ **H** $\frac{24}{20}, \frac{30}{25}, \frac{36}{30}$ **I** $\frac{4}{4}, \frac{5}{5}, \frac{6}{6}$ **J** $\frac{29}{32}, \frac{25}{40}, \frac{30}{48}$ **K** $\frac{21}{27}, \frac{28}{36}, \frac{35}{45}$
L $\frac{33}{12}, \frac{44}{16}, \frac{55}{20}$ **M** $\frac{18}{3}, \frac{24}{4}, \frac{30}{5}$ **N** $\frac{0}{3}, \frac{0}{4}, \frac{0}{5}$ **O** $\frac{3}{75}, \frac{4}{100}, \frac{5}{125}$

S-12 Set 20

1. A $\frac{1}{2}$ **B** $\frac{1}{3}$ **C** $\frac{3}{5}$ **D** $\frac{7}{10}$ **E** $\frac{1}{4}$ **F** $\frac{3}{10}$ **G** $\frac{5}{3}$ **H** $\frac{3}{4}$ **I** $\frac{3}{11}$
2. A $\frac{3}{8}$ **B** $\frac{2}{3}$ **C** $\frac{2}{3}$ **D** $\frac{3}{4}$ **E** $\frac{2}{5}$ **F** $\frac{13}{18}$ **G** $\frac{2}{5}$ **H** $\frac{3}{7}$ **I** $\frac{8}{8}$
J $\frac{14}{27}$ **K** $\frac{4}{7}$ **L** $\frac{3}{8}$ **M** $\frac{3}{5}$ **N** $\frac{5}{8}$ **O** $\frac{3}{5}$ **P** $\frac{5}{12}$ **Q** $\frac{3}{4}$ **R** $\frac{2}{5}$ **S** $\frac{2}{3}$
T $\frac{4}{5}$ **U** $\frac{3}{10}$ **V** $\frac{3}{8}$ **W** $\frac{27}{37}$ **X** $\frac{3}{50}$

S-13 Set 21

1. A = **B** = **C** ≠ **D** ≠ **E** = **F** ≠ **G** = **H** =
I ≠ **J** ≠ **K** ≠ **L** = **M** ≠ **N** = **O** = **P** ≠
2. A 6 **B** 12 **C** 2 **D** 60 **E** 32 **F** 24 **G** 12 **H** 9
I 25 **J** 3 **K** 4 **L** 4 **M** 20 **N** 75 **O** 60 **P** 0

S-13 Set 22

1. A > **B** > **C** < **D** > **E** > **F** < **G** < **H** >
I > **J** > **K** > **L** > **M** < **N** > **O** > **P** >
Q > **R** < **S** > **T** > **2. A** $\frac{1}{2}$ **B** $\frac{4}{5}$ **C** $\frac{3}{5}$ **D** $\frac{7}{2}$ **E** $\frac{6}{11}$
F $\frac{7}{8}$ **G** $\frac{5}{4}$ **H** $\frac{3}{4}$ **I** $\frac{7}{3}$ **J** $\frac{4}{3}$ **K** $\frac{10}{10}$ **L** $\frac{1}{3}$

S-14 Set 23

1. A $\frac{5}{6}$ **B** $\frac{7}{10}$ **C** $\frac{5}{6}$ **D** $\frac{8}{9}$ **E** $\frac{7}{8}$ **F** $\frac{14}{15}$ **G** $\frac{13}{21}$ **H** $1\frac{1}{18}$
I $1\frac{5}{12}$ **J** $1\frac{7}{24}$ **K** $1\frac{1}{35}$ **L** $\frac{37}{40}$ **M** $1\frac{1}{30}$ **N** $\frac{13}{27}$ **O** $1\frac{2}{15}$
2. A $1\frac{1}{12}$ **B** 2 **C** $1\frac{7}{8}$ **D** $\frac{15}{16}$ **E** $\frac{19}{20}$ **F** $1\frac{7}{18}$ **G** $1\frac{29}{30}$
H $1\frac{13}{15}$ **I** $2\frac{11}{24}$ **J** $\frac{4}{25}$ **K** $2\frac{1}{24}$ **L** $\frac{21}{25}$ **3. A** $\frac{1}{3}$ **B** $\frac{1}{8}$ **C** $\frac{4}{9}$
D $\frac{1}{4}$ **E** $\frac{4}{21}$ **F** $\frac{1}{20}$ **G** $\frac{3}{10}$ **H** $\frac{5}{24}$ **I** $\frac{9}{20}$ **J** $\frac{13}{40}$ **K** $\frac{13}{48}$ **L** $\frac{11}{35}$
M $\frac{23}{55}$ **N** $\frac{1}{18}$ **O** $\frac{29}{60}$

S-14 Set 24

1. A $1\frac{1}{3}$ **B** $1\frac{3}{5}$ **C** $2\frac{1}{10}$ **D** $5\frac{2}{3}$ **E** $6\frac{1}{2}$ **F** $4\frac{4}{7}$ **G** $2\frac{2}{7}$
H $1\frac{30}{31}$ **I** $1\frac{1}{2}$ **J** $2\frac{1}{2}$ **K** $2\frac{1}{5}$ **L** $1\frac{2}{3}$ **M** $1\frac{1}{6}$ **N** $1\frac{2}{3}$ **O** $2\frac{1}{2}$
P $2\frac{1}{10}$ **Q** $6\frac{1}{4}$ **R** $4\frac{1}{3}$ **S** $2\frac{1}{2}$ **T** $2\frac{1}{3}$ **U** $2\frac{1}{4}$ **V** $2\frac{2}{3}$ **W** $3\frac{3}{4}$
X $1\frac{2}{3}$ **2. A** $\frac{8}{7}$ **B** $\frac{8}{5}$ **C** $\frac{7}{3}$ **D** $\frac{41}{10}$ **E** $\frac{11}{3}$ **F** $\frac{9}{2}$ **G** $\frac{53}{6}$ **H** $\frac{41}{12}$
I $\frac{50}{3}$ **J** $\frac{65}{6}$ **K** $\frac{101}{4}$ **L** $\frac{100}{9}$ **M** $\frac{115}{2}$ **N** $\frac{100}{3}$ **O** $\frac{94}{15}$ **P** $\frac{99}{8}$
Q $\frac{199}{10}$ **R** $\frac{83}{25}$ **S** $\frac{162}{20}$ **T** $\frac{63}{16}$ **U** $\frac{191}{4}$ **V** $\frac{73}{3}$ **W** $\frac{88}{45}$ **X** $\frac{165}{19}$

S-15 Set 25

1. A $2\frac{1}{6}$ **B** $8\frac{7}{12}$ **C** $8\frac{1}{5}$ **D** $9\frac{11}{12}$ **E** $6\frac{1}{18}$ **F** $10\frac{7}{15}$
G $11\frac{1}{12}$ **H** $14\frac{19}{21}$ **I** $30\frac{19}{24}$ **J** $22\frac{18}{30}$ **K** $40\frac{7}{36}$ **L** $78\frac{59}{60}$
2. A $2\frac{1}{3}$ **B** $3\frac{5}{12}$ **C** $7\frac{5}{8}$ **D** $8\frac{1}{15}$ **E** $3\frac{11}{18}$ **F** $7\frac{11}{20}$ **G** $7\frac{1}{8}$
H $22\frac{13}{21}$ **I** $4\frac{91}{100}$ **J** $19\frac{13}{24}$ **K** $131\frac{3}{40}$ **L** $331\frac{41}{42}$

S-15 Set 26

1. A $\frac{1}{8}$ **B** $\frac{1}{12}$ **C** $\frac{1}{4}$ **D** $\frac{1}{15}$ **E** $\frac{1}{12}$ **F** $\frac{1}{16}$ **G** $\frac{1}{30}$ **H** $\frac{1}{70}$
I $\frac{1}{10}$ **J** $\frac{1}{24}$ **K** $\frac{1}{16}$ **L** $\frac{1}{50}$ **M** $\frac{1}{400}$ **N** $\frac{1}{60}$ **O** $\frac{1}{3000}$ **2. A** $\frac{2}{3}$
B $\frac{4}{7}$ **C** $1\frac{1}{5}$ **D** $4\frac{1}{2}$ **E** $\frac{1}{2}$ **F** $1\frac{1}{2}$ **G** $\frac{2}{3}$ **H** 2 **I** $1\frac{2}{3}$ **J** $\frac{1}{16}$
K 1 **L** $2\frac{2}{3}$ **M** 0 **N** $4\frac{2}{3}$ **O** 25 **P** $2\frac{1}{4}$ **Q** 2 **R** $8\frac{1}{3}$
S $7\frac{1}{2}$ **T** $4\frac{3}{5}$

S-16 Set 27

1. A $\frac{3}{10}$ **B** $\frac{9}{16}$ **C** $\frac{25}{42}$ **D** $\frac{21}{80}$ **E** $\frac{1}{3}$ **F** $\frac{3}{10}$ **G** $\frac{5}{14}$ **H** $\frac{1}{10}$
I $\frac{5}{12}$ **J** $\frac{9}{70}$ **K** $\frac{1}{2}$ **L** $\frac{21}{44}$ **M** $\frac{5}{6}$ **N** $\frac{17}{21}$ **O** $\frac{5}{8}$ **P** $\frac{4}{5}$ **Q** $\frac{3}{64}$
R $\frac{13}{30}$ **S** $\frac{1}{10}$ **T** $\frac{3}{7}$ **2. A** $2\frac{2}{3}$ **B** $8\frac{1}{3}$ **C** $10\frac{1}{2}$ **D** $5\frac{1}{2}$ **E** 27
F 62 **G** $\frac{1}{2}$ **H** $1\frac{1}{3}$ **I** $4\frac{4}{5}$ **J** $2\frac{11}{16}$ **K** $\frac{16}{27}$ **L** $4\frac{1}{5}$ **M** 5
N $2\frac{3}{4}$ **O** $14\frac{1}{6}$ **P** $7\frac{7}{10}$ **Q** $11\frac{1}{3}$ **R** 30 **S** 28 **T** $19\frac{3}{7}$

S-16 Set 28

1. A 34 **B** 20 **C** $35\frac{1}{2}$ **D** 24 **E** $12\frac{1}{2}$ **F** $36\frac{1}{6}$
G $254\frac{1}{10}$ **H** $52\frac{4}{15}$ **I** $293\frac{3}{20}$ **J** 380 **2. A** 22 **B** $2\frac{1}{8}$
C 125 **D** 70 **E** $40\frac{1}{3}$ **F** $24\frac{1}{2}$ **G** $1\frac{1}{21}$ **H** $50\frac{1}{10}$ **I** $5\frac{1}{24}$
J $6\frac{1}{3}$ **K** 95 **L** $50\frac{1}{2}$ **M** $2\frac{3}{10}$ **N** $6\frac{3}{20}$ **O** $20\frac{1}{4}$ **P** $10\frac{1}{14}$
Q $22\frac{4}{9}$ **R** $3\frac{1}{42}$ **S** 465 **T** $40\frac{1}{3}$

S-17 Set 29

1. A $\frac{5}{6}$ **B** 1 **C** $\frac{4}{6}$ **D** 24 **E** $6\frac{2}{3}$ **F** $1\frac{1}{2}$ **G** $\frac{2}{3}$ **H** $\frac{36}{49}$
I 2 **J** $\frac{1}{3}$ **K** 20 **L** $\frac{2}{9}$ **M** $\frac{7}{36}$ **N** $\frac{1}{3}$ **O** 26 **P** 3 **Q** $\frac{1}{5}$
R $\frac{2}{15}$ **S** $1\frac{1}{2}$ **T** $\frac{1}{10}$ **2. A** $1\frac{7}{8}$ **B** $\frac{5}{6}$ **C** $1\frac{5}{8}$ **D** $12\frac{1}{2}$ **E** $1\frac{1}{3}$
F $16\frac{1}{2}$ **G** $2\frac{1}{2}$ **H** 3 **I** $\frac{6}{29}$ **J** $\frac{8}{9}$ **K** $2\frac{2}{5}$ **L** $\frac{11}{21}$ **M** $\frac{1}{4}$
N $1\frac{7}{43}$ **O** 1 **P** $1\frac{13}{33}$ **Q** $9\frac{1}{3}$ **R** $\frac{1}{5}$ **S** $\frac{9}{10}$ **T** $\frac{77}{104}$

S-17 Set 30

1. A $\frac{4}{1} \div \frac{3}{5}$ **B** $\frac{5}{6} \div \frac{10}{1}$ **C** $\frac{4}{3} \div \frac{1}{2}$ **D** $\frac{1}{2} \div \frac{7}{2}$ **E** $\frac{13}{3} \div \frac{9}{10}$
F $\frac{64}{3} \div \frac{87}{10}$ **G** $\frac{8}{5} \div \frac{6}{1}$ **H** $\frac{8}{1} \div \frac{5}{2}$ **I** $\frac{16}{3} \div \frac{1}{4}$ **J** $\frac{32}{3} \div \frac{39}{8}$
K $\frac{1}{1} \div \frac{2}{5}$ **L** $\frac{33}{5} \div \frac{1}{100}$ **2. A** $\frac{1}{8}$ **B** $2\frac{2}{3}$ **C** $\frac{1}{6}$ **D** $1\frac{2}{3}$ **E** $4\frac{1}{2}$
F $\frac{9}{25}$ **G** $\frac{3}{16}$ **H** $3\frac{4}{5}$ **I** $1\frac{1}{6}$ **J** 26 **K** $2\frac{3}{5}$ **L** 2

S-18 Set 31

1. A 5.89 **B** 40.2 **C** 1.04 **D** 2.257 **E** 13.2
F 23.91 **G** 22.79 **H** 1.087 **I** 871.567 **J** 17.956
K 254.16 **L** 32.393 **M** 2.1516 **N** 5806.93
O 100.122 **P** 6851.958 **Q** 5980.632 **R** 1.917
2. A 2.33 **B** 14.6 **C** 2.36 **D** 0.634 **E** 0.45 **F** 15.27
G 15.25 **H** 3.7 **I** 0.059 **J** 5.138 **K** 44.706 **L** 4.021
M 75.4 **N** 98.1 **O** 9.09 **P** 1.625 **Q** 24.985
R 8.9991 **S** 39.999 **T** 999.999

S-18 Set 32

1. A 2.7 **B** 0.35 **C** 0.018 **D** 0.0042 **E** 0.0045
F 0.00016 **G** 0.0024 **H** 0.00004 **2. A** 45.5 **B** 2.96
C 1.104 **D** 21.8 **E** 0.3222 **F** 90.1 **G** 335.8

H 125.46 **I** 67.31 **J** 555.1 **K** 1.4994 **L** 1.2936
M 74.8 **N** 31.135 **O** 0.0018 **P** 3.045 **Q** 146.041
R 34.4565 **S** 6.74328 **T** 1.010086

S-19 Set 33

1. A 71 **B** 3.8 **C** 12.5 **D** 9.04 **E** 0.03 **F** 310
G 873 **H** 1390 **I** 37.5 **J** 0.4 **K** 32 000 **L** 8700
M 1390 **N** 765 **O** 69 **P** 0.036 **Q** 80.9 **R** 0.23
S 38 **T** 700.04 **2. A** 2.6 **B** 0.49 **C** 12.4 **D** 0.169
E 0.043 **F** 4.12 **G** 0.119 **H** 0.01635 **I** 0.9704
J 0.0067 **K** 2.567 **L** 0.4531 **M** 0.0857 **N** 0.00376
O 0.00096 **P** 0.0097 **Q** 0.00033 **R** 0.005503
S 0.000018 **T** 0.002345

S-19 Set 34

1. A 8.53 **B** 0.532 **C** 0.48 **D** 0.039 **E** 6.6 **F** 94
G 8.6 **H** 1.3 **I** 0.89 **J** 0.063 **K** 64 **L** 0.43
2. A 0.147 **B** 23.1 **C** 5.14 **D** 6.3 **E** 0.35 **F** 1.275
G 3.25 **H** 12 **I** 55 **J** 25 **K** 3.35 **L** 305 **M** 400
N 304 **O** 4.375 **P** 0.075 **Q** 0.9375 **R** 4600
S 3775 **T** 45

S-20 Set 35

1. A 4, 0.4 **B** 45, 0.45 **C** 74, 0.74 **D** 48, 0.48
E 185, 0.185 **F** 98, 0.098 **G** 372, 0.372 **H** 8, 0.008
I 175, 0.175 **J** 992, 0.992 **K** 625, 0.625 **L** 25, 0.0025
2. A 0.5 **B** 0.25 **C** 0.2 **D** 0.1 **E** 0.05 **F** 0.04
G 0.02 **H** 0.01 **I** 0.6 **J** 0.75 **K** 0.3 **L** 0.85
M 0.36 **N** 0.74 **O** 0.93 **P** 0.395 **Q** 0.025 **R** 0.005
S 0.004 **T** 0.002 **3. A** 0.75 **B** 0.8 **C** 0.125 **D** 0.85
E 0.76 **F** 0.275 **G** 0.3125 **H** 0.2125 **I** 0.5625
J 0.925 **K** 0.7875 **L** 0.15625 **4. A** 0.8 **B** 0.35
C 0.32 **D** 0.34 **E** 0.625 **F** 0.1875 **G** 0.172
H 0.275 **I** 3.75 **J** 1.86 **K** 1.08 **L** 0.174 **M** 1.625
N 0.325 **O** 0.168 **P** 0.6 **Q** 0.035 **R** 0.9375
S 0.53125 **T** 0.015625

S-21 Set 36

1. A $0.6\frac{2}{3}$ **B** $0.3\frac{3}{4}$ **C** $0.1\frac{2}{3}$ **D** $0.7\frac{1}{3}$ **E** $0.1\frac{3}{4}$ **F** $0.8\frac{2}{3}$
2. A $0.62\frac{1}{2}$ **B** $0.33\frac{1}{3}$ **C** $0.43\frac{3}{4}$ **D** $0.58\frac{1}{3}$ **E** $0.06\frac{2}{3}$
F $0.07\frac{1}{2}$ **3. A** $0.166\frac{2}{3}$ **B** $0.636\frac{4}{11}$ **C** $0.91\frac{2}{3}$ **D** $0.337\frac{1}{2}$
E $0.266\frac{2}{3}$ **F** $0.001\frac{2}{3}$ **4. A** $\frac{3}{20}$ **B** $\frac{9}{40}$ **C** $\frac{51}{200}$ **D** $\frac{1}{15}$ **E** $\frac{1}{3}$
F $\frac{3}{400}$

S-21 Set 37

1. A $6 \cdot 10^2$ **B** $2 \cdot 10^3$ **C** $4 \cdot 10^4$ **D** $8.1 \cdot 10^3$
E $6.2 \cdot 10^2$ **F** $4.37 \cdot 10^3$ **G** $7.2 \cdot 10^4$ **H** $1.05 \cdot 10^4$
I $6 \cdot 10^5$ **J** $2.1 \cdot 10^5$ **K** $8.3 \cdot 10^6$ **L** $2.3 \cdot 10^7$
M $4.735 \cdot 10^6$ **N** $1.25 \cdot 10^8$ **O** $5.1 \cdot 10^6$ **P** $9.75 \cdot 10^6$
Q $2.603 \cdot 10^6$ **R** $4.89 \cdot 10^7$ **S** $1.05 \cdot 10^6$ **T** $9.48 \cdot 10^8$
U $9.15 \cdot 10^8$ **V** $4.069 \cdot 10^9$ **W** $6.705 \cdot 10^9$ **X** $4.008 \cdot 10^{11}$
2. A 300 **B** 90 000 **C** 600 000 **D** 120 **E** 3800
F 47 000 **G** 1 100 000 **H** 709 000 **I** 375 **J** 6030
K 999 000 **L** 110 600 000 **M** 15 300 000
N 4 006 000 000 **O** 349 000 000 **P** 8 300 000 000 000

S-22 Set 38

1. A 12 **B** 25 **C** 42 **D** 5 **E** 85 **F** 9 **G** 16 **H** 8
I 27 **J** 24 **K** 7 **L** 78 **M** 6 **N** 200 **O** 8 **P** 135
3. A $\frac{3}{8} = n/96$; 36 **B** $\frac{6}{1} = n/37$; 222 **C** $\frac{7}{6} = n/135$; 105
D $\frac{2}{1} = n/126$; 252 **E** $\frac{7}{9} = 56/n$; 72 **F** $\frac{6}{5} = n/3000$; 3600
G $\frac{1}{35} = 4 \cdot 9/n$; 171.5 **H** $\frac{5}{1} = n/1000$; 5000

S-23 Set 39

1. A 0.17 **B** 0.28 **C** 0.85 **D** 0.31 **E** 0.34 **F** 0.40
G 0.11 **H** 0.99 **I** 0.10 **J** 0.09 **K** 0.02 **L** 0.01
M 1.00 **N** 2.00 **O** 0.20 **P** 0.02 **Q** 1.05 **R** 0.05
S 0.50 **T** 0.15 **U** 7.00 **V** 0.70 **W** 0.07 **X** 0.007
2. A 29% **B** 66% **C** 13% **D** 47% **E** 12% **F** 8%
G 2% **H** 1% **I** 60% **J** 120% **K** 100% **L** 200%
M 30% **N** 3% **O** 300% **P** 3000% **Q** 18% **R** 180%
S 1800% **T** 108% **U** 20% **V** 1000% **W** 43%
X 5% **3. A** $\frac{57}{100}$ **B** $\frac{1}{2}$ **C** $\frac{3}{10}$ **D** $\frac{31}{100}$ **E** $\frac{1}{4}$ **F** $\frac{1}{5}$ **G** $\frac{1}{10}$
H $\frac{3}{50}$ **I** $\frac{1}{20}$ **J** $\frac{1}{25}$ **K** $\frac{1}{50}$ **L** $\frac{1}{100}$ **M** $\frac{3}{20}$ **N** $\frac{4}{25}$ **O** $\frac{17}{100}$
P $\frac{9}{50}$ **Q** $\frac{4}{5}$ **R** $\frac{2}{25}$ **S** $\frac{8}{1}$ **T** $\frac{22}{25}$ **U** $1\frac{2}{5}$ **V** $1\frac{3}{4}$ **W** $\frac{11}{25}$
X $1\frac{7}{20}$ **4. A** 37% **B** 9% **C** 7% **D** 1% **E** 70%
F 10% **G** 50% **H** 25% **I** 20% **J** 2% **K** 75%
L 5% **M** 60% **N** 4% **O** 14% **P** 98% **Q** 35%
R 36% **S** 26% **T** 96% **U** 105% **V** 16% **W** 140%
X 275%

S-24 Set 40

1. A 45 **B** 32 **C** 20 **D** 9 **E** 34 **F** 58 **G** 232.2
H 104.5 **I** 0.43 **J** 42 **K** 2.7 **L** 1045 **M** 140
N 300 **O** 634.26 **P** 979.29 **2. A** 126 **B** 41.5
C 237 **D** 84 **E** 250 **F** 25 **G** 2.5 **H** 0.25 **I** 174
J 5.655 **K** 10.62 **L** 14.79 **M** 30 **N** 54.36 **O** 12.5
P 34.5 **3. A** \$78 **B** \$97.20 **C** \$22.75 **D** \$46.79

S-24 Set 41

1. A 70% **B** 25% **C** 38% **D** 75% **E** 50% **F** 80%
G 60% **H** 30% **I** 20% **J** 25% **K** 33.3% **L** 62.5%
M 62.5% **N** 150% **O** 4% **2. A** 12.1% **B** 34.6%
C 90.5% **D** 55.6% **E** 138.5% **F** 89.3% **G** 24%
H 65.3% **I** 83.8% **J** 91.1%

S-25 Set 42

1. A 108 cm^2 **B** 66.15 cm^2 **C** 57.35 m^2
D 90 000 mm^2 **E** 39.95 km^2 **2. A** 108 cm^2 **B** 102 cm^2
C 5300 mm^2 **D** 63 m^2 **3. A** 88 cm^2 **B** 38.95 m^2
C 24.96 cm^2 **D** 63 m^2 **E** 125.46 m^2

S-25 Set 43

1. A 297 mm^2 **B** 858 cm^2 **C** 138.7 m^2
D 49 280 mm^2 **E** 2323 cm^2 **F** 9.88 m^2 **G** 27.5 cm^2
H 193 500 mm^2 **2. A** 32.5 cm^2 **B** 425 m^2 **C** 80 cm^2
D 10 625 mm^2

S-26 Set 44

1. A 640 m^3 **B** 90 cm^3 **C** 16.25 m^3 **D** 4212 mm^3
E 540 cm^3 **F** 40 000 m^3 **G** 237.86 cm^3 **H** 365.4 m^3
2. A 9 m^3 **B** 35 m^3 **C** 6.381 m^3 **D** 486 cm^3

S-26 Set 45

1. A 216 cm^2 **B** 240 cm^2 **C** 6.94 m^2 **D** 489.12 cm^2
E 15.18 m^2 **F** 15.38 m^2 **G** 25 800 mm^2
H 619 200 cm^2 **2. A** 32 m^2 **B** 74 m^2 **C** 24.44 m^2
D 432 cm^2

S-27 Set 46

1. A 37.68 mm **B** 96.08 m **C** 329.7 mm
D 282.6 cm **E** 26.22 m **F** 1042.48 mm **G** 5.65 km
H 13.38 m **2. A** 66 cm **B** 99 m **C** $31\frac{3}{7}$ cm
D 220 mm **E** 55 m **F** 1540 mm **G** $15\frac{2}{5}$ m **H** $19\frac{4}{5}$ cm

S-27 Set 47

1. A 12.56 m^2 **B** 254.3 cm^2 **C** 5024 mm^2
D 2826 mm^2 **E** 2.0096 m^2 **F** 5278.3 cm^2
G 34 618.5 mm^2 **2. A** 113.1 cm^2 **B** 154 cm^2
C 15 400 mm^2 **D** 50.3 cm^2 **E** 314.3 m^2 **F** 1.13 km^2
G 95.1 m^2 **3. A** 692.4 cm^2 **B** 615.4 mm^2 **C** 30.96 m^2
D 18.8 cm^2

S-28 Set 48

1. A 120 cm^3 **B** 196 cm^3 **C** 1045 mm^2 **D** 0.98 m^3
E 12 190 cm^3 **F** 4725 mm^3 **G** 2549.3 cm^3
H 12 826.5 m^3 **2. A** 171 m^3 **B** 10.98 cm^3 **C** 9100 cm^3
D 560 mm^3 **E** 2498.6 mm^3

S-28 Set 49

1. A 226.1 m^3 **B** 52.5 m^3 **C** 339 120 mm^3
D 290 450 **E** 0.019 **F** 2.31 **2. A** 65.94 cm^3
B 4.71 cm^3 **C** 105 975 mm^3 **D** 25.12 m^3

S-29 Set 50

1. A 2 **B** −14 **C** −6 **D** −3 **E** −6 **F** −13 **G** 5
H 37 **I** −5 **J** −37 **K** 21 **L** −37 **M** 22 **N** −4
O 26 **P** −22 **Q** −4 **R** 4 **S** 26 **T** −64 **2. A** 15
B 5 **C** −5 **D** −15 **E** 21 **F** 120 **G** 2 **H** 43
I −27 **J** −43 **K** −73 **L** −3 **M** 43 **N** −9 **O** 140
P −270 **Q** 132 **R** 13

S-29 Set 51

1. A 11 **B** −8 **C** −14 **D** 5 **E** −8 **F** −13
2. A 20 **B** 14 **C** −12 **D** −3 **E** −16 **F** 6 **G** −6
H −18 **I** 18 **J** 6 **K** −5 **L** 5 **M** 17 **N** −3
O −21 **P** 79 **Q** −89 **R** −58 **S** 155 **T** −86

S-30 Set 52

1. A −5 **B** −2 **C** 3 **D** 5 **E** 8 **F** 13 **G** 2
H −3 **I** 0 **J** 10 **K** 0 **L** −10 **2. A** 7 **B** −7 **C** 23
D −23 **E** −7 **F** 23 **G** −27 **H** 13 **I** −34 **J** 48
K −48 **L** 0 **M** 80 **N** −60 **O** −46 **P** 89 **Q** −68
R −100 **S** 212 **T** 654 **U** −55 **V** 280 **W** −306 **X**

S-30 Set 53

1. A −45 **B** −45 **C** 0 **D** −9 **E** 9 **F** −33
G −90 **H** −170 **I** 430 **J** −670 **K** −600 **L** −1400
M −441 **N** 441 **O** −2300 **P** 5600 **Q** −125 000
R −1001 **S** −1044 **T** 954 **2. A** −40 **B** 40 **C** −40
D −45 **E** 48 **F** −120 **G** −100 **H** 100 **I** −1000
J −1000 **K** 1000 **L** −10 000 **M** 96 **N** −77
O −250 **P** 430 **Q** 144 **R** −189 **S** 0 **T** 400
U −126 **V** 720 **W** 315 **X** 120

S-31 Set 54

1. A 4 **B** −7 **C** −4 **D** −9 **E** 11 **F** −9 **G** −9
H 20 **I** −20 **J** −20 **K** −1 **L** 13 **M** −14 **N** −15
O −8 **P** −5 **Q** −5 **R** 19 **S** −5 **T** 7, 7; −7, −7
2. A −3 **B** 9 **C** 7 **D** −6 **E** 3 **F** 1 **G** 0 **H** 15
I 25 **J** −9 **K** −12 **L** −1 **M** −9 **N** −15 **O** −4
P 8 **Q** 3 **R** −5 **S** −10 **T** −20 **U** 24 **V** −32
W −8 **X** −1 **Y** −193

S-31 Set 55

1. A 26 **B** 64 **C** 84 **D** 66 **E** 905 **F** 120
G 4125 **H** 218 **I** 9.13 **J** 1.61 **2. A** 39 **B** 54
C 95 **D** 212 **E** 408 **F** 1431 **G** 1004 **H** 248
I 627 **J** 5074

S-32 Set 56

1. A 35 **B** 32 **C** 51 **D** 400 **E** 90.5 **F** 9 **G** 53
H 20 **I** 42 **J** 4.02 **2. A** 88 **B** $9100 **C** 1.54, 1.68, 1.585

S-32 Set 57

1. A 4 **B** 86, 93 **C** 14 **2. A** 74.9; 79.4; 81.1

MAKING METRIC MEANINGFUL

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MAKING THE METRIC SYSTEM MEANINGFUL

To make the metric system meaningful it is useful to understand something of the origin of weights and measures, the emergence of early decimal measurement systems, and the development of the modern metric system. Although the metric system is simply the application of the decimal number system to measurement based on the metre and other standard units, its usefulness has not been readily accepted universally. The struggle to develop a universal language of measurement has been long and difficult.

It is a little known fact that even though our *old* measurement system is called the *British Imperial* or *English* measurement system, the British neither originated nor advocated this system in earlier days. The British used a system similar to metric nearly 1500 years ago, abandoning it about 700 years ago only to go metric again in the 1970's.

How did measurement begin? Probably back in pre-history, one cave man showed another just how big the fish was which he caught (or which got away) by using his out-stretched arms: since then, man has used parts of his body for measuring.

Ancient Egyptian history of measurement was recorded as a readable story with markings cut into rock used to keep track of the rise and fall of the Nile waters at floodtime. Every *tenth* line was emphasized, a measure roughly corresponding to the span of the hand. The units used in land surveying were as follows:

10 zebos (fingerwidths)	= 1 span
10 spans (100 zebos)	= 1 nent (armstretch)
10 nents	= 1 khet

The builders of pyramids and temples used the *cubit* (length of the forearm).

With the domestication of the Arabian camel, two Phoenician ports, Tyre and Sidon, became thriving commercial centres where trade flourished for centuries. Camels made possible caravan routes from Babylon to Egypt. Since Egypt was an important customer, the Phoenicians adopted the Egyptian *decimal* system. One small modification was made in order to measure cloth–half a nent (armstretch) was used, a unit close to a yard.

The Phoenicians had established trading centres along the Mediterranean shores including Greece. Around the 3rd to 4th century contact was made with the northern tribes, particularly with the Goths who had moved south along the Black Sea into Greece. Odin whose name survives in our word "Wednesday", was a mighty chieftain in Teutonic mythology, and guided his people from a semi-nomadic to a more settled way of life. It is said that he accepted the Phoenician way of measurement–only his "armstretch" unit was larger. And so the *decimal* measurement spread among the peoples known as Norse or German. Around this time portions of Britain were occupied by Roman legions who confined themselves mainly to the fortresses built along the defending walls. With the recall of the Roman soldiers Britan was left defenceless.

This situation allowed the migrations to Britain to take place from the 5th to the 8th century A.D. Across the North Sea came the Angles, Saxons, Jutes, and Danes. They brought with them the *decimal* system of measurement.

10 fingerwidths	= 1 span-of-length
10 spans	= 1 armstretch
10 armstretches	= 1 chain
10 chains	= 1 furlong
10 chains × 1 chain	= 1 acre

These measurements were suitable for surveying and building. The housewife used smaller units based on 10.

10 one-half fingerwidths	= 1 handbreadth (across knuckles)
10 handbreadths	= 1 wand (half an armstretch)

In most measurement systems, it is noteworthy that from the selected unit of length a cube is described which determines the unit of capacity or volume; the unit of mass is derived from the mass of cool water in this cube. For example, the capacity of a cubic handbreadth was called a *measure-full.* The mass of cool water in this cube gave the early Britons the basic mass unit called a *measure-weight;* the mass of a cubic wand of cool water was a *tun-weight.* Length received greater emphasis in this historical development because the derived units were determined by the unit of length and limited space.

However, about 700 years ago the foot was introduced to England from continental Europe. The foot did not originate with the English. The Romans used such a unit and they in turn had borrowed it from the Greeks who devised it in the first place.

The Greeks worked out a system for fighting men using paces (double steps) as units. Romans used *mille* for a thousand paces, giving us the word *mile*. Some areas affected by Roman units were Italy, Spain and France while those retaining the units of the Gothic tribe were England, Germany and the Scandinavian countries.

Around 1200 A.D., about 700 years after the fall of the Roman Empire, a new industrial era began in Europe. Trade increased, trade organizations were formed, fairs were established, and cathedrals were built. Unfortunately the measurement plan used was the "foot-length". With the introduction of the industrial era in England, there too the foot-length and other related units were adopted. Since it was non-decimal it did not fit the simple Gothic system. No doubt many in Britain at the time were dismayed by this decision to allow the Roman units to be used.

At the beginning of the 17th century, Simon Stevin, an inspector of dikes in the Low Countries, proposed a *decimal* measurement system, and shortly thereafter Father Gabriel Mouton suggested to the French Academy of Sciences a *decimal* system based on the circumference of the earth.

By the 18th century in Europe, there were *280* measures called "foot", all of different lengths. Imagine the quarrels and squabbles that this could cause. In the midst of this confusion people were clamouring for a simple system. One very important development in the U.S.A. captured the people's attention: namely, the *decimal* coinage plan approved by Congress in 1786.

10 mills = 1 cent
10 cents = 1 dime
10 dimes = 1 dollar

At this time Thomas Jefferson, as U.S. Secretary of State, prepared a unified system of weights and measures which was *decimal* in nature. He sub-divided the foot into 10 parts, larger units being formed by multiplying by 10.

Meanwhile the scientists at the Academy of Sciences of Paris began to devise a fully *decimalized* measurement system to dovetail with the decimal number system. At this time certain mathematicians were proposing a duodecimal number system and advocating a duodecimalized measurement system to match. This idea failed; instead the idea of linking the measurement system to the place-value number system based on 10, flourished.

In the 1790's the French instructed a small group of outstanding scientists to measure the distance from Dunkirk to Barcelona and this information was used to calculate the distance from the North Pole to the Equator. This land measurement was chosen because it was the longest segment of land along a meridian in Europe. Assuming the earth to be perfectly round, one ten-millionth of this arc was designated as the *metre* (greek metron—*a measure*).

Beginning with the metre, larger and smaller units were formed based on the decimal number system, i.e. multiples and sub-multiples of ten with the appropriate prefix.

PREFIX		SYMBOL	MULTIPLYING FACTOR
Greek	kilo	k	$1000 = 10^3$
	hecto	h	$100 = 10^2$
	deca	da	$10 = 10^1$
Latin	deci	d	$0.1 = 10^{-1}$
	centi	c	$0.01 = 10^{-2}$
	milli	m	$0.001 = 10^{-3}$

Some of the common metric units used in most daily situations are derived from the metre, gram and litre and these are:

QUANTITY	UNIT	SYMBOL
Length	millimetre	mm
	centimetre	cm
	metre	m
	kilometre	km
Capacity	millilitre	ml
	litre	l or ℓ
	kilolitre	kl
Mass	gram	g
	kilogram	kg
	tonne	t
Volume	cubic centimetre	cm^3
	cubic metre	m^3

It is remarkable how near the modern metric system is to the early decimal system. The wand of old was only 0.5 cm longer than the metre. Had the early system remained, little change would have been necessary in order to bring the *old* into the *new*. However, the metric system as proposed by France gained ready acceptance in certain Latin American and European countries. Germany adopted the metric system after defeating the French in the war of 1870-1871. The British however, in 1871, rejected the proposal to convert to the metric system by defeating the Bill in the House of Commons by a mere five votes.

Since Napoleonic times the original metric system has undergone changes and in 1960 the General Conference on Weights and Measures (Conférence Générale des Poids et Mesures) adopted the name *Systéme International d'Unites (International System of Units)* abbreviated **SI**. The SI system is an extension and refinement of the traditional systems (*CGS* and *MKS*) and superior because it is a rationalized, coherent, and simplified metric system.

The base units of SI are seven independent units: *metre* (length), *kilogram* (mass), *second* (time), *ampere* (electric current), *kelvin* (temperature), *candela* (luminous intensity), and *mole* (amount of substance). All other derived units are defined in terms of these units.

QUANTITY	UNIT	SYMBOL
Length	metre	m
Mass	kilogram*	kg
Time	second†	s
Electric current	ampere	A
Thermodynamic temperature	kelvin#	K
Luminous intensity	candela	cd
Amount of substance	mole[xx]	mol

In addition to the seven base units are the following SI units—supplementary units and derived units having special names.

	QUANTITY	UNIT	SYMBOL
Supplementary Units	Plane angle	radian	rad
	Solid angle	steradian	sr
Derived Units having Special Names	Frequency (cycle per second)	hertz	Hz
	Force	newton	N
	Energy	joule	J
	Power	watt	W
	Electric charge	coulomb	C
	Electric potential	volt	V
	Electric resistance	ohm	Ω
	Electric capacitance	farad	F
	Magnetic flux	weber	WB
	Inductance	henry	H
	Magnetic induction	tesla	T
	Luminous flux	lumen	lm

*The *Kilogram* is the only base unit containing a prefix and is used because the gram is too small.

† As early as 1670, it was suggested by C. Wren that the length of the pendulum whose swing takes one second be the unit of length. By coincidence the length of this pendulum was very nearly 1 m long.

The *Celsius* temperature scale (previously called Centigrade) is common in everyday usage and even though it is a non-SI unit it is permissible to use it for temperature measurements. One degree Celsius equals one kelvin ($1^{\circ}C = 1$ K).

xx The mole is defined in chemistry in terms of a number of elementary particles (Avogadro's number).

THE RULES GOVERNING THE USE OF SI UNITS

In order to preserve uniformity in the writing of the units, the symbols, the prefixes, and the numerals, the following list of rules should be followed. For a more complete treatment write to: *Canadian Standards Association,* 178 Rexdale Blvd., Rexdale, Ontario and request a copy of a CSA Standard Z234.1–1973, Metric Practice Guide.

A. SI Units when written out in full:

(1) Units are not capitalized except at the beginning of a sentence. Example: ampere, newton, kelvin. Only Celsius begins with a capital letter.
(2) The word *per* is used to denote division between the units. Example: kilometres per hour not kilometres/hour.

B. Symbols for SI Units:

(1) A symbol represents a unit and is the same in all languages.
(2) The symbols do not change in the plural.
Example: 10 metres 10 m
(3) Because the symbols are not abbreviations no periods are written after the symbol except at the end of a sentence: The symbols can be manipulated like algebraic variables i.e. multiplied, divided, squared, cubed, etc.
(4) Symbols are usually not capitalized except for a unit named after an individual. Example: A for ampere, N for newton, and K for kelvin. (Note: Hz for hertz, Pa for pascal, and Wb for weber).
(5) Symbols should not begin a sentence.
(6) The symbol of a derived unit formed by division is shown by using a solidus (/) or a negative exponent.
Example: km/h or $km \cdot h^{-1}$, not kmph or k.p.h. m/s^2 or $m \cdot s^{-2}$, not m/s/s/.
(7) The symbol must be separated from the numeral by a space. Example: 75 cm
(8) The symbol for degree ° is used to denote degree Celsius (°C) to differentiate it from a coulomb (C), a unit of electrical charge.

C. Prefixes for SI Units:

The prefixes larger than kilo and smaller than milli are listed as additional information to teachers and interested students.

PREFIX	SYMBOL	MULTIPLYING FACTOR
tera	T	10^{12}
giga	G	10^{9}
mega	M	10^{6}
micro	μ	10^{-6}
nano	n	10^{-9}
pico	p	10^{-12}
femto	f	10^{-15}
atto	a	10^{-18}

(1) The only prefix symbols capitalized are T for tera, G for giga, M for mega.
(2) The prefix symbol and the unit symbol are printed in roman (upright) type without spacing between them. Examples: kPa for kilopascal, mN for millinewton, MN for meganewton.
(3) Only one prefix is applied at one time to a given unit. Example: picometre (pm), not millinanometre (mnm).
(4) The prefix symbol combines with the unit symbol to form a new symbol which can be raised to a positive or negative power.
Example: $1\ cm^2 = 1\ (cm)^2 = (10^{-2}m)^2 = 10^{-4}m^2$
(5) The prefixes denoting multiples or sub-multiples of 1000 are preferred where it is practical. This implies that the prefixes h, da, d, c, are to be avoided; however, in measuring length the cm is acceptable.

D. Numerals used with SI Units:

(1) Instead of using commas to group blocks of three digits to the left and right of the decimal point, a space is used as a separator.
Examples: 654 321.234 56
(2) A prefix should be chosen so that the numerical values fall between 0.1 and 1000 if it is suitable.
Example: 23 000 J becomes 23 kJ
0.005 81g becomes 5.81 mg
(3) A zero is written in the units column for a number less than one.
Example: 0.3 in place of .3
(4) Decimal notation should be used where it is practical.
Example: 4/5 should be written 0.8

E. General comments:

(1) The following should *not* be used:
amp to mean ampere;
kilo to mean kilogram;
sq. cm to mean square centimetre (use cm^2);
cc to mean cubic centimetre (use cm^3).

(2) Metric (but non-SI) units are to be avoided.
Example: calorie, micron, dyne, erg.

(3) Some *NON-SI* units are permissible with the SI.
Example: minute, hour, day, litre, tonne, degree Celsius, nautical mile, knot, ängström, hectare.

THE ADVANTAGES OF SI

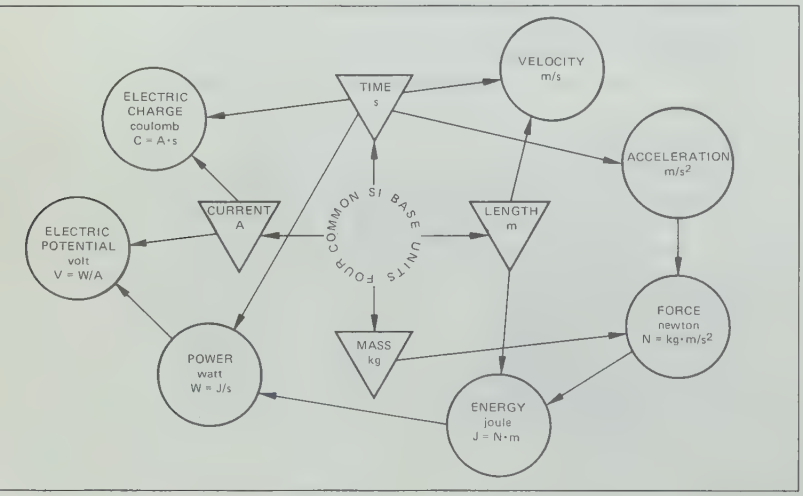
UNIQUE: The SI system is convenient in measuring in that there is only *one* unit for each physical quantity as illustrated by the seven base units such as the metre for length, second for time, etc. There is absolutely only *one* version of all the base units all over the world.

COHERENT: This can be defined
(a) as the act of sticking together or
(b) as a natural or logical connection.

Starting with a unique set seven base units all other derived units can be obtained by multiplying or dividing two units either basic or derived.

Example: velocity unit is the metre per second. i.e. m/s

The coherence can be better illustrated by the following schematic diagram.



Because of its coherence, complicated conversion factors are eliminated. When measuring physical quantities one unit applies to the mechanical, electrical and thermal fields.

Example: 1 watt = 1 newton × 1 metre per second
1 watt = 1 ampere × 1 volt
1 watt = 1 joule per second.

INTERRELATIONSHIP:

1 cubic centimetre (cm) container holds 1 millilitre of water of mass 1 gram (g) at 4° Celsius. Consequently a 1000 cubic centimetres (cm^3) or 1 cubic decimetre (dm^3) container holds 1000 millilitres (ml) or 1 litre (ℓ) of water of mass 1000 grams (g) of 1 kilogram (kg).

UNIVERSAL: The metric system (more specifically, SI units) has become the "lingua franca" of the world. The symbols are identical whenever and wherever SI units are used and thereby form part of a worldwide language of measurement.

THINK METRIC

The SI units are distinct and unique in that they are not derived by converting units from other systems. In order to preserve this "purity", conversions from one system of measurement to another should be avoided. Education research strongly supports the theory that the best way to learn the metric (SI) system is to *think metric.* Languages are best understood if the person thinks in the language spoken.

However, it is not always possible to strictly enforce this way of thinking. If conversion is necessary it is better to use the form 1 m = 1.094 yd. instead of 1 yd. = 0.9144 m.

It is easier to relate units of measurement to parts of the body.

1 fingerwidth–2 cm
1 small fingernail width–1 cm
1 handbreadth–10 cm
1 cubic handbreadth–1 dm^3 or 1 ℓ
(remember the mass of 1 ℓ water is 1 kg)

In this fashion the everyday units can be readily visualized. It is more difficult to visualize distances in kilometres. For this reason the more common relationships can be applied. For example:

60 miles is approximately 100 km
30 miles is approximately 50 km
20 miles is approximately 30 km
50 miles is approximately 80 km
100 miles is approximately 160 km.

It is convenient to think of a common paper clip to have a mass of 1 g. A litre is less than the Imperial or Canadian quart but greater than the U.S. quart.

In developing a *feel* for the temperatures expressed in degrees Celsius, it may be useful to relate it to some common experiences.

100°C boiling point of water
37°C normal body temperature
30°C hot summer day
20°C comfortable room temperature
10°C cool summer day
0°C freezing
-20°C cold winter day
-40°C extremely cold winter day (equivalent to -40°F)

CONCEPT OF MEASUREMENT

To measure is to count or mark off using a selected unit as the standard. In other words, to measure is to determine a number which indicates the number of times the standard unit has been applied.

Concepts of measurement are best learned if students are meaningfully involved in activity. The Chinese proverb "I do, I understand" applies very well to the development of measurement concepts. The U.S. Metric Study Interim Report states: "To begin with, measurement should not and really cannot be "taught" through a series of planned lessons. Learning to measure is a gradual process related to each child's experiences . . . children will learn best if it is not *taught* but *experienced* and used in some activity in the context of situations in which a child is actively involved. . . ."

The variety and extent of the hands-on approach is limited only to the imagination of the students and teacher. Although excellent commercial material is available, it is at times less expensive and more beneficial to the learner to use materials in the school or home. Cuisenaire rods and cubes are excellent to demonstrate the cm, cm^2, and cm^3. The container of a 2 quart milk carton is 10 cm × 10 cm. This would readily provide the student with a cubic decimetre, or 1 litre, if 10 cm high. This type of activity reinforces the *think metric* theme.

Teachers are strongly urged to allow students the opportunity to develop the skill of estimation. It is fun and very useful. Like any other skill, perfection is attained through meaningful practice.

When industry in Great Britain went metric the companies streamlined their operations. Forty different sizes of screws were reduced to seven. Going metric gives teachers an opportunity to streamline mathematical operations and methods of instruction of measurements concepts.

The adoption of the metric (SI) system will simplify and unify the teaching of mathematics and science, reduce calculation errors, save teaching time, and give a better understanding of basic physical principles.

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